The Elements of Navigation and Nautical Astronomy

The Elements of Navigation and Nautical Astronomy

A TEXT-BOOK OF NAVIGATION AND NAUTICAL ASTRONOMY

First revison
BY

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What is beyond the horizon?
Let us discover

PREFACE

THIS new edition includes numerous changes and additions. Various steps involving calculations and procedure have been included in worked out examples particularly in navigational sight calculations.

All calculations are worked out into metric (S.I.) units.

Although the plan of this present book follows closely of the original, the new features include the latest methods of Tidal calculation using 1987 Tide Tables. All information relating to chartwork has been updated. Other new features are the inclusion of a chapter in Satellite Navigation System and Omega. Many new diagrams have been added.

This revised edition and enlarged version provide the basic groundwork for the professional examination for Class I, Class II and Class III Certificate of Competency in Navigation and Nautical Astronomy and aimed specially to cadets preparing for B/TEC National Diploma and Higher Diploma in Nautical Science.

This book brings a fresh approach to the study of Navigation, and the emphasis is on the understanding of principles as well as on practical applications. It provides a thoroughly comprehensive and logically arranged scheme of studies in all fields of Navigation and Nautical Astronomy. The book is splendidly produced in large format, and among its attractive features are numerous clear line drawings, a wide range of worked examples and a large number of exercises, making it ideally suitable for classroom tution as well as for students working independently.

My share of the labour involved in producing the revised version was greatly reduced by Mrs. Manjusha Lahiry, my wife, for her patient and invaluable assistance with proof reading. It is a pleasure for me to record my sincere appreciation.

I have great faith and hope that the students and officers in Merchant Navy from all parts of the world will benefit from this book.

HIMADRI K. LAHIRY
FLEETWOOD, 1987

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PART 1

THE MATHEMATICS OF NAVIGATION AND NAUTICAL ASTRONOMY

Navigation embraces the techniques used by a navigator for finding the distance to travel and the direction to steer from one given position to another. The basic instruments of navigation are the chart, on which positions and course-lines are plotted; the log, by which distances travelled through the water are measured; and the compass, by which horizontal directions are indicated or ascertained. Navigating by log and compass—a technique known as dead reckoning—yields approximate and usually unreliable results, so that the prudent navigator loses no opportunity of checking his ship's position to reassure himself that his vessel is never on a dangerous course.

To determine the position of his vessel when the land is in sight observations of charted land- and sea-marks, such as lighthouses, buoys and beacons, and the application of simple geometrical principles, permit the navigator to fix his ship's position with relative ease. In recent decades a wide range of electronic and radio instruments is available for determining a ship's position even when the land is not in sight. In the absence of such aids the traditional methods of finding latitude and longitude are those which employ astronomical principles: these methods collectively form the science of nautical astronomy.

To understand navigation and nautical astronomy a thorough knowledge of elementary mathematics—particularly geometry and trigonometry—is essential. It is upon these branches of mathematics that the whole structure of navigation and nautical astronomy has been built. Part 1 deals essentially with elementary trigonometry, the application of which to the work of a navigator is easy and interesting only if it is properly comprehended.

CHAPTER 1

PLANE TRIGONOMETRY

1. Introduction

The branch of mathematics which deals with the calculation of the unknown parts of a triangle is known as trigonometry. The word *trigonometry*—usually frightening to a beginner—means nothing more than triangle measurement. An important part of a navigator's work involves solving triangles, so that it is essential for a student of navigation to have a full knowledge of the elements of this important branch of mathematics.

Trigonometry is divided into two parts. The part which deals with the mathematics of plane triangles is known as plane trigonometry: that which deals with spherical triangles is known as spherical trigonometry.

Of the six parts—three angles and three sides—of a plane triangle, provided that at least one of the three known parts is a side, the unknown parts may be found geometrically by construction, or trigonometrically by calculation. The calculations of the unknown parts of a triangle is facilitated by using tables of trigonometrical functions. A function in mathematics is a quantity whose value depends upon the value of some other quantity. We say, for example, that the speed of a ship through the water is dependent upon her displacement, so that speed is a function of displacement. It is also a function of any of a variety of other factors, such as the rate at which the propeller is revolving, and the rate at which fuel is being consumed.

A trigonometrical ratio is a function of an angle. In a right-angled triangle which contains an acute angle θ, the trigonometrical ratios of the angle θ are the numerical comparisons between the lengths of pairs of sides of the triangle. Because a triangle has three sides there are, accordingly, six trigonometrical ratios. These are known, respectively, as the sine, cosine, tangent, cotangent, secant and cosecant.

In the right-angled triangle illustrated in fig. 1·1, the trigonometrical ratios of the angle θ are:

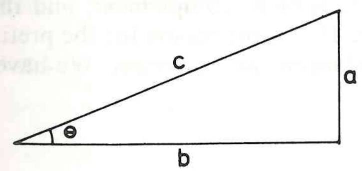


Fig. 1·1

The ratio $a{:}c$, that is a/c, is the sine of θ, or sin θ
The ratio $b{:}c$, that is b/c, is the cosine of θ, or cos θ
The ratio $a{:}b$, that is a/b, is the tangent of θ, or tan θ
The ratio $b{:}a$, that is b/a, is the cotangent of θ, or cot θ
The ratio $c{:}b$, that is c/b, is the secant of θ, or sec θ
The ratio $c{:}a$, that is c/a, is the cosecant of θ, or cosec θ.

In addition to these six trigonometrical ratios there are two other trigonometrical functions known, respectively, as the versine and the haversine. These are of considerable use in the practice of navigation and nautical astronomy.

versine (or vers) $\theta = 1 - \cos\ \theta)$
haversine (or hav) $\theta = \text{half vers}\ \theta)$
$= \frac{1}{2}(1 - \cos\ \theta)$

The trigonometrical functions of an angle are dependent solely upon the magnitude of the angle.

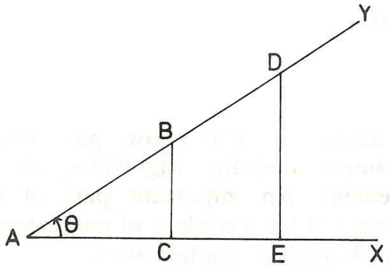


Fig. 1·2

Referring to fig. 1·2, suppose the line AX to be rotated about the point A so that any acute angle θ is swept out. Perpendiculars dropped from any points B and D on AY onto AX will form two similar or equiangular triangles ABC and ADE respectively. Now the ratio between any two corresponding sides of similar triangles is a constant amount. It follows that:

$$BC/AB = DE/AD = \sin\ \theta$$
$$AC/AB = AE/AD = \cos\ \theta$$
$$BC/AC = DE/AE = \tan\ \theta$$

2. Complementary Angles

Two angles are said to be complimentary when their sum is 90°. Angles of 30° and 60° are complementary, each being the complement of the other. Because the sum of the three angles of a plane triangle is 180°, it follows that the two non-90° angles of a right-angled triangle are complementary.

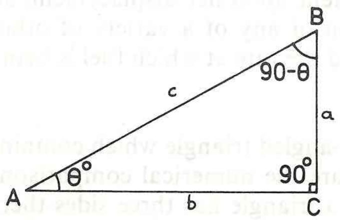


Fig. 1·3

In the right-angled triangle illustrated in fig. 1·3, the angle BAC is denoted by θ. The angle ABC, therefore, is $(90° - \theta)$.

It is readily seen that:

$$\sin\ \theta = a/c = \cos\ (90° - \theta)$$
$$\tan\ \theta = a/b = \cot\ (90° - \theta)$$
$$\sec\ \theta = c/b = \text{cosec}\ (90° - \theta)$$

Thus, the sine of an angle is equal to the cosine of its complement; the tangent of an angle is equal to the cotangent of its complement; and the secant of an angle is equal to the cosecant of its complement. Similarly, the cosine of an angle is equal to the sine of its complement; the cotangent of an angle is equal to the tangent of its complement; and the secant of an angle is equal to the cosecant of its complement. This is the reason for the prefix "co" which stands for complement, in the names cosine, cotangent and cosecant. We have, for example:

sin 55° = cos 35°
tan 60° = cot 30°
sec 24° = cosec 66°
cos 67° = sin 23°
cot 86° = tan 4°
cosec 14° = sec 76°

3. Trigonometrical Functions as Straight Lines

Sine and Cosine. Suppose the radius of the circle illustrated in fig. 1·4 to be of any unit length. Let the radius AB sweep out any acute angle θ. Because the radius of the circle is unity the length of the side DC in the right-angled triangle ACD is the sine of the angle θ. That is: $\sin\ \theta = DC$

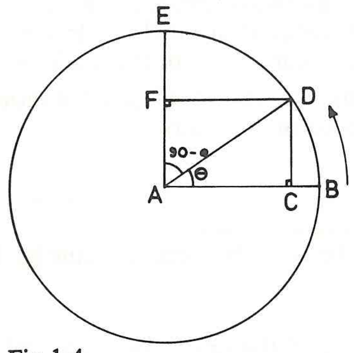


Fig. 1·4

The sine of an arc or angle may, therefore, be defined as the length of a perpendicular dropped from one extremity of an arc of unit radius onto the diameter of the circle of which the arc forms part drawn through the other extremity.

If the angle BAE infig. 1·4 is 90°, the angle EAD is the complement of θ. The sine of the complement of θ, that is to say, cos θ, is denoted in fig. 1·4 by the line DF. But DF is equal in length to AC, so that:

$$\cos\ \theta = AC$$

The sine of 0° is zero because when θ is 0° the length of the perpendicular DC is zero. The cosine of 0° is unity because when θ is 0° the length of AC is equal to the radius of the circle, which is unity.

The value of the sine of an angle increases from 0 to 1 as the angle increases from 0 to 90°, but the value of the cosine of an angle decreases from 1 to 0 as the angle increases from 0 to 90°.

If, in fig. 1·4, distances measured to the right of A in the direction of B are designated positive, then distances measured to the left of A will be designated negative. Again, if distances measured from A in the direction of E are designated positive, distances measured in the opposite direction will be designated negative. It follows that the sines and cosines of all acute angles are positive. But consider the situation for angles greater than 90°.

The sine of an angle decreases from +1 to 0 as the angle increases from 90° to 180°, and the cosine decreases from 0 to − 1 as the angle increases from 90° to 180°. As the angle increases from 180° to 270° the sine decreases from 0 to − 1 and the cosine increases from − 1 to 0. As the angle increases from 270° to 360°, to complete the circle, the sine increases from − 1 to 0 and the cosine increases from 0 to +1.

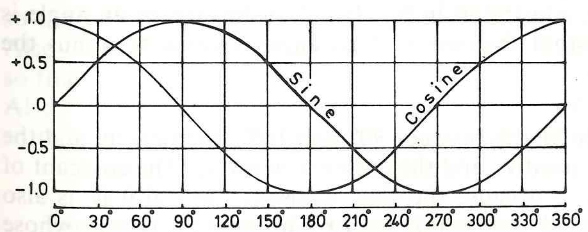


Fig. 1·5

The graphs of the sine and cosine of angles between 0° and 360° are illustrated in fig. 1·5. The two curves are said to be out of phase with each other to the extent of 90°. In other words if the cosine curve is moved 90° to the right it will coincide with the sine curve.

Tangent and Cotangent. Suppose the radius of the arc illustrated in fig. 1·6 is of unit length. Let the radius AB rotate about A to form the acute angle θ. Let a straight line drawn tangentially to the arc at B cut the radius AD produced at G. The radius of the arc is unity, so that the length of the line BG is the tangent of the angle θ.

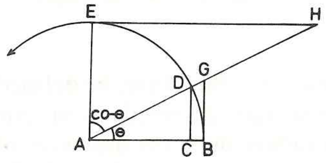


Fig. 1·6

If a straight line be drawn tangentially to the arc at E, which is 90° from B, to cut the radius AD produced at H, the length of the line EH is the cotangent of the angle θ. In other words EH is the tangent of the complement of the angle θ.

The triangles ACD and ABG are similar, so that the ratio between corresponding sides is constant. Thus:

$$CD/AC = BG/AB$$

or:

$$\sin\theta/\cos\theta = \tan\theta/1$$

The tangent of an angle may, therefore, be defined as the ratio between the sine and cosine of the angle.

Secant and Cosecant. The secant of an angle is the length of the line drawn from the centre of a circle through one extremity of an arc to the tangent drawn from the other extremity of the same arc in a circle of unit radius. In fig. 1·6 the secant of the angle θ is represented by the line AH. It should also be noted that the length of this line is equal to the cosecant of the complement of θ.

The triangles ACD and ABG are similar, so that:

$$AG/AB = AD/AC$$

and:

$$\sec\theta/1 = 1/\cos\theta$$

The secant of an angle may, therefore, be defined as the reciprocal of the cosine of the angle. Similarly, the cosecant of an angle may be defined as the reciprocal of the sine of the angle.

4. The Signs of the Trigonometrical Ratios of Angles between 90° and 180°

In the practice of navigation, when solving triangles, angles having values of up to 180° only are involved. For this reason angles over 180° need never be considered in the practical work of solving triangles.

Two angles are said to be supplementary when their sum is 180°. It will be noted from the graphs of the sines and cosines of angles illustrated in fig. 1·5, that the sine of an angle is equal to the sine of its supplement, and that the cosine of an angle is equal to minus the cosine of its supplement.

In the second quadrant, which refers to angles between 90° and 180°, the tangent and the cotangent are negative because the sine is positive and the cosine is negative. The cosecant of an angle in the second quadrant is positive because the sine, whose reciprocal it is, is also positive. The secant of an angle in the second quadrant is negative because the cosine, whose reciprocal it is, is also negative.

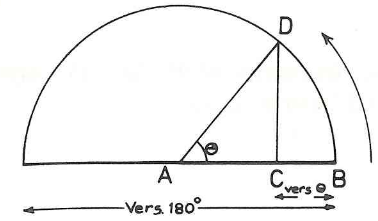


Fig. 1·7

Versine and Haversine. The versine of an angle is defined as "one minus the cosine of the angle", and the haversine of an angle is a half of the versine of the angle.

It is noticed from fig. 1·7 that when the angle θ increases from 0° to 180°, the versine of θ—which is equal to the length of the line BC—increases from 0 to +2.

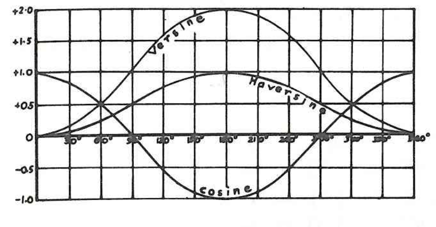


Fig. 1·8

Fig. 1·8 illustrates the graphs of the versines and haversines, and the cosines, of angles between 0° and 360°.

The principal feature of the functions versine and haversine is that they are always positive. In contrast to cosines, which in the second quadrant are negative, versines and haversines are easily handled in computations because they are always positive. It is for this reason that they are used in preference to cosines for solving certain navigational triangles.

5. The Standard Formulae

From the foregoing remarks it will be observed that sines and cosines are lengths of lines *within* a circle; tangents and cotangents are the length of lines which *touch* a circle; and that secants and cosecants are lengths of lines which *cut* a circle.

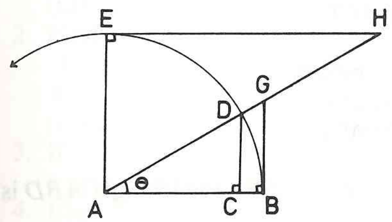


Fig. 1·9

Consider the triangles ACD, ABG, and AEH, in fig. 1·9. These are similar triangles, so that:

$$CD/AC = BG/AB = AE/EH$$

or: $\sin\theta/\cos\theta = \tan\theta = 1/\cot\theta$

Also: $CD/AD = BG/AG = AE/AH$

or: $\sin\theta/1 = \tan\theta/\sec\theta = 1/\operatorname{cosec}\theta$

Also: $AC/AD = AB/AG = EH/AH$

or: $\cos\theta/1 = 1/\sec\theta = \cot\theta/\operatorname{cosec}\theta$

By Pythagoras' Theorem:

$$CD^2+AC^2 = AD^2$$

so that: $\sin^2\theta+\cos^2\theta = 1$

Also: $BG^2+AB^2 = AG^2$

so that: $\tan^2\theta+1 = \sec^2\theta$

Also: $EH^2+AE^2 = AH^2$

so that: $1+\cot^2\theta = \operatorname{cosec}^2\theta$

The above formulae are known as the Standard Formulae. They are sometimes useful in navigational work, and are worth remembering.

6. Special Angles

It is a comparatively easy matter to derive the trigonometrical ratios of 0°, 30°, 45°, 60° and 90°. It is useful to memorise the ratios of these so-called "special angles".

It has already been established that:

$$\begin{aligned} \sin 0° &= 0 \\ \cos 0° &= 1 \\ \sin 90° &= 1 \\ \cos 90° &= 0 \end{aligned}$$

It follows that:

$$\begin{aligned} \tan 0° &= \sin 0°/\cos 0° = 0/1 = 0 \\ \tan 90° &= \sin 90°/\cos 90° = 1/0 = \infty \\ \sec 0° &= 1/\cos 0° = 1/1 = 1 \\ \sec 90° &= 1/\cos 90° = 1/0 = \infty \\ \operatorname{cosec} 0° &= 1/\sin 0° = 1/0 = \infty \\ \operatorname{cosec} 90° &= 1/\sin 90° = 1/1 = 1 \\ \cot 0° &= 1/\tan 0° = 1/0 = \infty \\ \cot 90° &= 1/\tan 90° = 1/\infty = 0 \end{aligned}$$

The trigonometrical ratios of 30° and 60° may be found as follows:

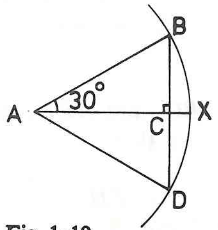


Fig. 1·10

Let the radius of the circular arc illustrated in fig. 1·10 be of any unit length. Let AX be rotated about A through an angle of 30° to AB. From B drop a perpendicular onto AX and produce to D on the arc. Join AD. In the triangles ABC and ACD:

$$\begin{aligned} AC &= AC \text{ (common)} \\ AB &= AD \text{ (radii)} \\ ACB &= ACD \text{ (90°)} \end{aligned}$$

The triangles ABC and ACD are, therefore, congruent triangles. Also the triangle ABD is equilateral.

Therefore:

$$BC = \tfrac{1}{2} BD = \tfrac{1}{2} AB$$

Also $\quad ABC = 60°$

Now $\quad AB = 1$

Thus, $\quad BC = \tfrac{1}{2}$

By Pythagoras' Theorem:

$$\begin{aligned} AC^2 &= AB^2 - BC^2 \\ &= 1^2 - \tfrac{1}{2}^2 \\ &= \tfrac{3}{4} \end{aligned}$$

Thus $\quad AC = \sqrt{3}/2$

It follows that:

$$\begin{aligned} \sin 30° &= & BC &= \tfrac{1}{2} = \cos 60° \\ \sin 60° &= & AC &= \sqrt{3}/2 = \cos 30° \\ \tan 30° &= \sin 30°/\cos 30° &&= 1/\sqrt{3} = \cot 60° \\ \tan 60° &= \sin 60°/\cos 60° &&= \sqrt{3} = \cot 30° \\ \operatorname{cosec} 30° &= 1/\sin 30° &&= 2 = \sec 60° \\ \operatorname{cosec} 60° &= 1/\sin 60° &&= 2/\sqrt{3} = \sec 30° \end{aligned}$$

The trigonometrical ratios of 45° may be found as follows.

Let the radius of the circular arc depicted in fig. 1·11 be of any unit length. Rotate OX about O to OA so that the angle AOB is 45°. The triangle OAB is, therefore, isosceles, and AB is equal to OB. By Pythagoras' Theorem:

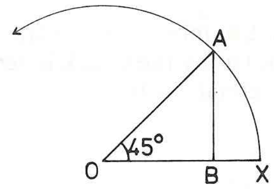


Fig. 1·11

$$\begin{aligned} OA^2 &= AB^2 + OB^2 \\ &= 2\,AB^2 \\ \text{Thus:} \quad AB^2 &= OA^2/2 \\ \text{and} \quad AB &= 1/\sqrt{2} = OB \\ \text{Therefore:} \quad \sin 45° &= \cos 45° = 1/\sqrt{2} \\ \tan. 45° &= \cot 45° = 1 \\ \sec 45° &= \operatorname{cosec} 45° = \sqrt{2} \end{aligned}$$

Exercises on Chapter 1

1. Find by scale drawing:
   (i) tan 26°
   (ii) cos 62°
   (iii) sec 48°
2. Find by scale drawing the acute angle whose:
   (i) cotangent is 0·7
   (ii) secant is 1·4
   (iii) sine is 0·6
3. If the sine of an acute angle is 0·4 find, without using tables, the remaining five trigonometrical ratios of the same angle.
4. Explain why cos 56° equals sin 34°.
5. Prove: (i) $\sin^2 A + \cos^2 A = 1$
   (ii) $\operatorname{cosec}^2 a - 1 = \cot^2 a$
6. Show that tan 100° is equal to negative tan 80°.
7. Prove: $\cos 30° = \sqrt{3}/2$.
8. Explain what is meant by the versine of an angle.
9. What is the principal advantage of a versine table compared with a cosine table?
10. If the haversine of an angle is 0·6 find the angle by scale drawing.
11. Derive the trigonometrical ratios of 0°, 90°, 180° and 270°.
12. Prove: sin 45° cosec 45° sec 30° sin 60° = 1.
13. Prove graphically that sin (180° − A) is equal to sin A given that A is an acute angle.
14. Demonstrate graphically that the secant of 38° is approximately equal to the tangent of 38°.
15. Find angle θ given that: vers θ = ¾ cos θ.

CHAPTER 2

CIRCULAR MEASURE

1. The Radian

The length of the circumference of any circle is a constant number of times the length of the diameter of the same circle. This constant number is denoted by the Greek letter Pi or π. It is an incommensurable quantity; but, to four places of decimals, it is:

$$\pi = 3{\cdot}1416$$
$$= 22/7 \text{ approximately}$$

It follows that the length of the circumference of a circle is 2π times the length of the radius of the same circle. Thus, if the radius is fitted around the circumference, as shown in fig. 2·1, it will be found that 2π, or 6·28..radii will equal the length of the circumference.

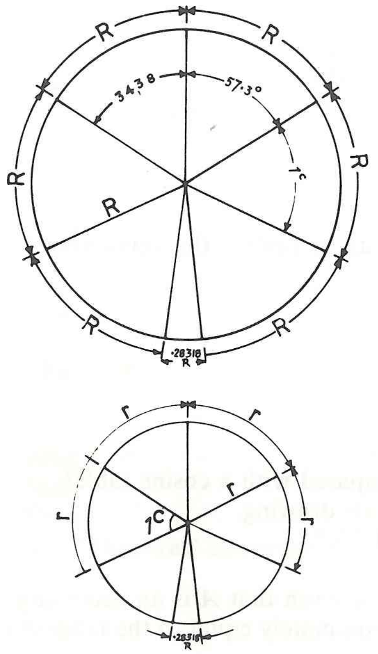


Fig. 2·1

Since the angle subtended at the centre of a circle by the circumference is 360°, it is evident that the angle subtended by an arc whose length is equal to the radius of the circle is $360/2\pi$, or 360/6·28 . . degrees. This angle is 57°·3 or 3438′ approximately. It is the unit of circular measure known as the radian, the symbol for which is a small letter *c*.

Circular measure provides a method of measuring or denoting angles which simplifies the solutions of certain navigational problems. Thus, instead of saying that the angle at the centre of a circle subtended by the circumference is 360°, we may say that it is 2π radians.

Thus:

360° is equivalent to 2π radians
180° is equivalent to π^c
and 57°·3 is equivalent to 1^c approximately

To find the number of radians corresponding to a given number of degrees, the latter is divided by 57·3. To find the number of radians in a given number of minutes of arc the latter is divided by 3438.

For example:

$$360° = 360/57{\cdot}3 = 6{\cdot}28\,..^c$$
$$90° = 90/57{\cdot}3 = 1{\cdot}57\,..$$
$$121° = 121/57{\cdot}3 = 2{\cdot}11\,..^c$$
$$5°43'{\cdot}8 = 343'{\cdot}8 = 343{\cdot}8/3438 = 0{\cdot}1^c$$

If the length of the arc of a circle of given radius is known it is an easy matter to find the number of radians contained at the centre of the circle subtended by that arc. A circle of radius 10 cms. has a circumference of 2π . 10 cms. or 62·83 cms. The circular measure of the angle at the centre subtended by the circumference is 6·28 . .^{c}. An arc which is one-quarter of the circumference in length, that is to say 15·707 cms., subtends an angle which is equal to one-quarter of 6·28 . . , which is 1·57 . .^{c}. This may be found by dividing the length of the arc by that of the radius. Similarly, an arc of 23 cms. in a circle of radius 10 cms. subtends an angle of 23/10, or 2·3^c.

Thus:

Circular Measure = Arc length/Radius
or Arc length = Circular Measure × Radius

Example 2·1—A circle has a radius of 20·0 miles. What length of its perimeter subtends an angle of 2·3^c at its centre?

Arc length = Circular Measure × Radius
= 2·3 × 20·0
= 46·0

Answer—Length = 46·0 miles.

Example 2·2—Find the radius of a circle if an arc of 3·4 cms. subtends an angle of 1·60 radians at its centre.

Radius = Arc length/Circular Measure
= 3·4/1·60
= 2·12 . .

Answer—Radius = 2·12 . . cms.

The following examples illustrate some of the practical applications of circular measure to navigation.

Example 2·3—A headland is kept abeam at a distance of 5·0 miles. Find the distance travelled by the ship in changing the bearing of the headland 95°.

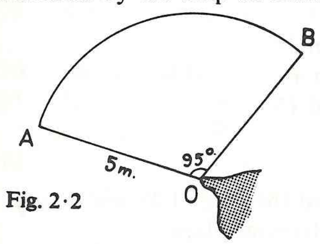


Fig. 2·2

Radius = 5·0 miles
AOB = 95°
= 95/57·3 radians
Arc AB = radius × AOB^c
= 5·0 (95/57·3)
= 8·3

Answer—Distance travelled = 8·3 miles.

Example 2·4—The horizontal angle between the extremities of a small circular island having a diameter of 1·1 miles is 5°30′. Find the distance off.

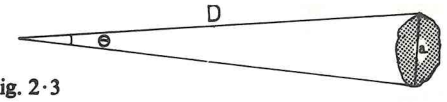


Fig. 2·3

$\theta = 5° \ 30' = 330/3438^c$
$a = 1{\cdot}1$ miles
$D = a/\theta = 1{\cdot}1\ (3438/330) = 11{\cdot}5$

Answer—Distance off = 11·5 miles.

2. Trigonometrical Ratios of Small Angles

Circular measure is particularly useful when dealing with very small angles because, as fig 2·4 shows, the lengths of the sine, tangent and arc, of a small angle, are very nearly equal to one another. The smaller the angle the more nearly so is this. In practice the degree of smallness of the angle involved determines the accuracy required in the final result. In general, for navigational purposes, angles of less than about 6° may be considered to be small.

Referring to fig. 2·4 it may readily be seen that the arc *BD* is very nearly equal to each of the lines *CD* and *BE*.

Now:

$BD/OB = BOD^c$
$BE/OB = \tan BOD$
$CD/OB = \sin BOD$

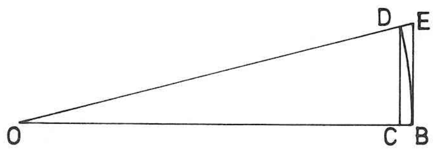


Fig. 2·4

If arc *BD* and lines *BE* and *CD* are considered to be equal in length; then, for practical purposes:

$BD/OB = BE/OB = CD/OB$

Therefore: $\theta^c = \tan\theta = \sin\theta$

This means that when the circular measure of a small angle is required, the sine or tangent of the angle may be used without introducing material error. Conversely, when the sine or the tangent of a small angle is required, the angle in radians may be used.

For small values of θ the graphs of sine θ and tangent θ, against angle θ, are almost coincident straight lines. It follows that the sine or tangent of a small angle is proportional to the angle itself. That is to say:

$$\sin\theta \propto \theta$$
$$\tan\theta \propto \theta$$

It follows that:

$$\sin 1°/1° = \sin\theta/\theta$$

and,

$$\tan 1°/1° = \tan\theta/\theta$$

In other words:

$\sin\theta \doteqdot \theta° \sin 1°$ and $\tan\theta \doteqdot \theta^c \tan 1°$

or, $\sin\theta \doteqdot \theta' \sin 1'$ and $\tan\theta \doteqdot \theta' \tan 1'$

or, $\sin\theta \doteqdot \theta'' \sin 1''$ and $\tan\theta \doteqdot \theta'' \tan 1''$

The following examples should be studied carefully.

Example 2·5—The sine of 1° to four decimal places is 0·0175. Find the sine of 4° and verify from trigonometrical tables that your answer is correct to three decimal places.

$\sin 1° = 0{\cdot}0175$ to 4 decimal places

Thus: $\sin 4° = 4 \times 0{\cdot}0175$

$= 0{\cdot}070$ to 3 decimal places

From tables:

$\sin 4° = 0{\cdot}0698$ to 4 decimal places

Example 2·6—Find without using tables:

(i) $\sin 0° \ 30'$

(ii) $\cot 0° \ 20'$

(i) $\sin\theta = \theta^c$ when θ is small

Therefore:

$\sin 30' \doteqdot 30/3438 = 0{\cdot}0088$ approximately

(ii) $\cot 20' = 1/\tan 20'$

$\tan 20' \doteqdot 20/3438$

Therefore:

$\cot 20' = 3438/20$

$= 171{\cdot}9$ approximately

Answers—$\sin 30' = 0{\cdot}0088$

$\cot 20' = 171{\cdot}9$.

Exercises on Chapter 2

1. Find the values of the following angles in radians: (i) 137°, (ii) 59° 51′, (iii) 37° 52″.
2. Given the following arc lengths and corresponding radii, find the angles in radians:
   (i) Arc length 32″, radius 15″.
   (ii) Arc length 7·2 miles, radius 3·73 miles.
   (iii) Arc length 120·0 feet, radius 0·073 nautical miles (*Note:*1 nautical mile = 6080 feet).
3. Find the length of an arc subtended by an angle of $2{\cdot}731^c$ in a circle of radius (i) 17·2 cm., (ii) 100·0 yards.
4. Prove that: $a = r\,\theta$ where *a* is the length of an arc of a circle of radius *r*, and θ is the angle subtended by the arc in circular measure.
5. Show that if θ is a small angle:
   $\sin\theta \doteqdot \tan\theta \doteqdot \theta^c$
6. A nautical mile on a spherical Earth is defined asd the length of an arc of a greater circle of the Earth, the extremities of the arc subtending an angle of 1′ at the Earth's centre. What is the Earth's diameter in nautical miles?
7. A ship is conned around a point of land at a constant distance of 3·5 miles. Find the distance between the instants when the point bore respectively N. 20° E. and S. 50° E.
8. One angle of a plane triangle is $\pi/4^c$, another is $3\pi/8^c$. Find the third angle in degrees and radians.
9. A tower 200·0 feet high subtends a vertical angle of 1° 40′. Find the distance off without using tables.
10. Show that for a small angle θ: $\sin\theta \doteqdot \theta' \sin 1'$.
11. Given $\sin 1° = 0{\cdot}0174524$, find approximate values of: (i) sin 5°, (ii) cosec 2°, (iii) tan 30′, (iv) cot 89°.
12. Given $\tan 1' = 0{\cdot}0002909$, find approximate values of: (i) tan 1°, (ii) cot 40′. comment upon the degree of accuracy of the answers given to questions 11 and 12.

CHAPTER 3

THE TRAVERSE TABLE AND THE SOLUTION OF PLANE RIGHT-ANGLED TRIANGLES

1. Introduction

Perhaps the most useful of all nautical tables is the Traverse Table. This table, simple in construction, is nothing more than an orderly collection of solutions of plane right-angled triangles. Although the traverse table may be used for solving any plane right-angled triangle, its principal uses in the hands of a navigator are to find:

(*a*) the direction of one given terrestrial position from another,
(*b*) the distance between two given positions on the Earth's surface,
(*c*) a ship's position after she has travelled from a given position in a given direction for a given distance.

These purely navigational problems are investigated in Part 2: in this chapter we shall examine the principles of the traverse table.

2. Plane Right-angled Triangles and their Traverse Table Solutions

The hypotenuse of any right-angled plane triangle lies opposite to the right angle, and the two non-90° angles are complementary angles.

Consider the plane right-angled triangle illustrated in fig. 3·1. The lengths of the sides AC and BC are functions of the angle A (and angle B), and of the length of the hypotenuse AB.

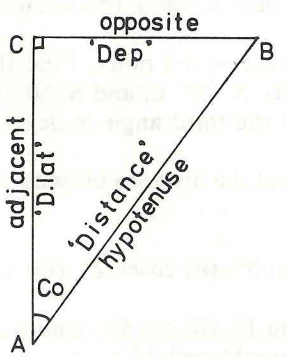


Fig. 3·1

Values of the sides AC and BC are tabulated in the traverse table against values of angle A (and B) and of the hypotenuse AB. The tabulated lengths of AB are usually given at intervals of one unit from 0 to 600 units.

The values of the three sides of every triangle that can be solved directly by means of the traverse table are tabulated in three vertical columns labelled Hypotenuse (Distance); Opposite (Departure); and Adjacent (D. Lat.). A complete page is given for each whole degree of angle A (and B), which is designated Course Angle.

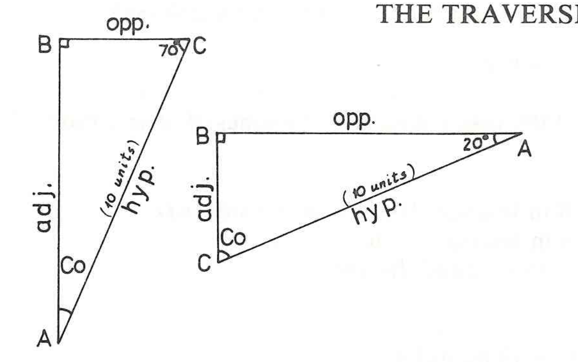


Fig. 3·2

To solve a 20° right-angled triangle having a hypotenuse of 10 units, the traverse table is entered at the page corresponding to a Course Angle of 20°. Abreast of 10 in the hypotenuse (Distance) column will be found the values of the sides adjacent and opposite to the angle 20° in the triangle to be solved. In this case, the side opposite is 3·4 units and the side adjacent to the 20° angle is 9·4 units.

To solve a 70° right-angled triangle having a hypotenuse of length 10 units, the table may be entered at angle 20°—which is the complement of 70°—and the value of the side adjacent to the angle 70° will be found in the column labelled Opposite; and that of the side opposite to the angle 70° will be found in the column labelled Adjacent. The reason for this will be clear from an examination of fig. 3·2.

It is not necessary to extend the traverse table beyond 45°. To solve any right-angled triangle the table may be entered with the smaller of the two non-90° angles, one of which must be smaller than or equal to 45°. The values of the lengths of the oppositae and adjacent sides may then be lifted from the table remembering, that the larger of the two sides forming the right angle is that which faces the larger of the two non-90° angles.

To facilitate the use of the traverse table angles between 45° and 90° are printed at the bottoms of the pages, such that the sum of the angles at the top and bottom of the page is 90°. Thus, at the top of the page which is labelled 25°, the angle 65°—the complement of 25°— will be found at the bottom. The columns labelled Opposite and Adjacent, respectively, at the top of the page are labelled Adjacent and Opposite, respectively, at the bottom of the page.

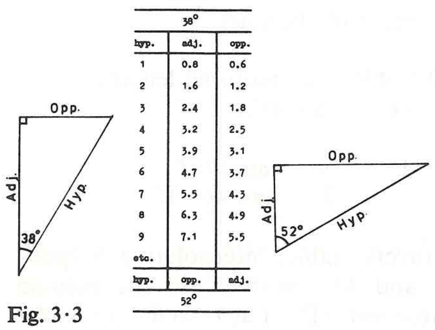


Fig. 3·3

The layout of the traverse table is illustrated in fig. 3·3.

The solution of the straightforward right-angled triangle problem, in which the given angle and side have integral values, is simple. If however, the given angle is not an integral degreee and/or the given side is a fractional quantity, interpolation may be necessary, and this is tedious and warrants considerable care. It is for this reason that awkward right-angled triangle problems are usually computed instead of being solved by inspection. It should be borne in mind, however, that the traverse table affords a ready check on even complex right-angled triangle solutions.

Example 3·1—From a vessel heading 110° a lighthouse bore 050° at the same time as a steeple bore 072°. After travelling for 8·0 miles the lighthouse and the steeple were in transit abeam. Find the distance between the lighthouse and the steeple.

Hint—Draw a diagram and then plan a solution.

In fig. 3·4 *A* and *B* denote the positions of the vessel at each of the times of observation. *C* denotes the steeple and *D* the lighthouse.

Plan—1. Using angle *DAB* and side *AB* in triangle *DAB*, find the side *BD*.
2. Using angle *CAB* and side *AB* in triangle *ACB*, find the side *BC*.
3. Subtract *BC* from *BD* to give the required distance.

From traverse table:

$$BD = 13{\cdot}86 \text{ miles}$$
$$BC = 6{\cdot}25 \text{ miles}$$
$$CD = BD - BC = 7{\cdot}61 \text{ miles}$$

Answer—Distance = 7·6 miles (to the nearest 0·1 of a mile).

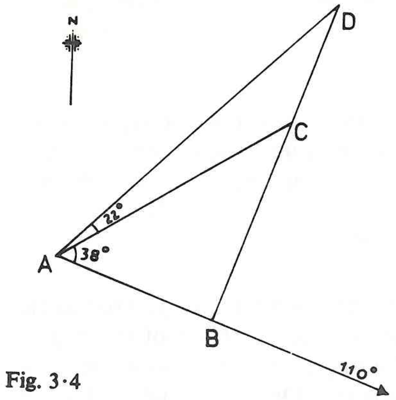


Fig. 3·4

Note—To find the sides *BD* and *BC* the traverse table was entered with 80, not 8, in the column labelled Adjacent. The decimal point was than shifted one place to the left to give the result to the nearest second place of decimals.

It should be appreciated that the accuracy of the result of any computation can never be greater than that of the data used in the computation. If, for example, the distance given in Example 3·1 had been 8 miles *to the nearest mile, BD* and *BC to the nearest mile* would have been 14 and 6 respectively, and the required distance would have been 8 miles *to the nearest mile.*

Example 3·2—The shadow of a vertical flag staff is 25 metres long at a time when the Sun's altitude is 46° 15′. Find the length of the staff.

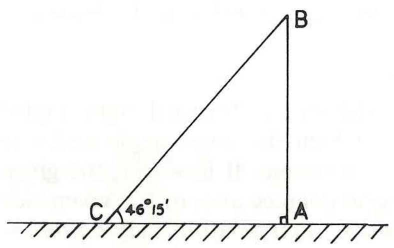


Fig. 3·5

In fig. 3·5 *AB* is the required length.

$$AB/AC = \tan BCA$$

Therefore:

$$AB = AC.\tan BCA$$
$$= 25 \times \tan 46°\ 15′$$

From traverse table, interpolating between angles 46° and 47°, with 25 in the column labelled Adjacent (D. Lat.), *AB*, from the column labelled Opposite (Departure), is 26 metres to the nearest unit.

Answer—Length = 26 metres.

Although the Hypotenuse (Distance) column in the traverse table extends only to 600 units, this in no way limits the use of the table. When the value of the hypotenuse of a triangle exceeds 600, the other two side may be solved by entering the table with any fraction of the hypotenuse, such as half or quarter, and then multiplying the tabulated values by the

reciprocal of the fraction. An alternative method is to find the values corresponding to the hypotenuse value of 600, and to add to these the values corresponding to a hypotenuse equal to the excess of the given hypotenuse over 600.

To add the lengths of the adjacent and opposite sides in a 40° right-angled triangle having a hypotenuse of 842·0, the two methods are used with reference to fig. 3·6 (*a*) and (*b*), respectively.

Method 1—Enter traverse table at angle 40°, and with hypotenuse equal to half the given value, that is to say, with 421·0, we have:
Tabulated adjacent = 322·5. This multiplied by 2 gives 645·0.
Tabulated opposite = 270·6. This multiplied by 2 gives 541·2.

Method 2—Enter traverse table with angle 40°, and with 600 and 242 in turn.

Adjacent for 600 = 459·6	Opposite for 600 = 385·7
Adjacent for 242 = 185·4	Opposite for 242 = 155·6
Required Adjacent = 645·0	Required Opposite = 541·3

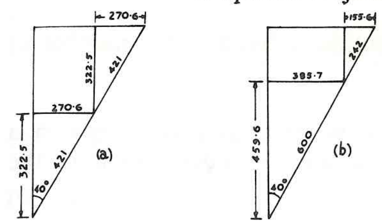


Fig. 3·6

The following examples illustrate that the traverse table may be employed for solving any problem in which a simple ratio is involved. This enlarges the scope of the table, and a wide variety of problems suggests traverse table solutions. In view of the versatility of the traverse table every endeavour should be made to become efficient in its use.

Example 3·3—Find, by means of the traverse table: (i) cot 50°, (ii) hav 32°.

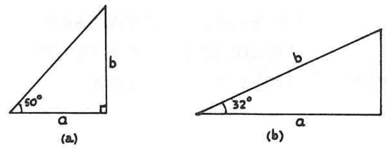


Fig. 3·7

(i) Referring to fig 3·7 (*a*)

$$\cot 50° = a/b$$
$$= a/100 \text{ (Note the convenient denominator.)}$$

Enter traverse table at angle 50°. The value of *a* is found in the column labelled Adjacent abreast of 100 in the column labelled Opposite.

$$\cot 50° = 83{\cdot}9/100$$
$$= 0{\cdot}839$$

(ii) Referring to fig. 3·7 (*b*)

$$\text{hav } 32° = \tfrac{1}{2}(1 - \cos 32°) = \tfrac{1}{2}(1 - a/b)$$
$$\text{Let } b = 100$$

Enter traverse table at angle 32°. The value of *a* will be found abreast of 100 in the column labelled Hypotenuse (Distance).

Thus: $\text{hav } 32° = \frac{1}{2}(1 - 84{\cdot}8/100)$
$= \frac{1}{2}(1 - 0{\cdot}848)$
$= \frac{1}{2}.0{\cdot}152$
$= 0{\cdot}076$

Answers—cot 50° = 0·842; hav 32° = 0·076.

Example 3·4—Find the area of a triangle *ABC* given that *AB* = 14·0 cms, *AC* = 20·0 cms. and angle *BAC* = 58°.

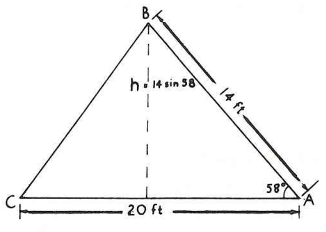


Fig. 3·8

Area of triangle = ½ base × height
= ½.20·0.14·0 . sin 58°

From traverse table:
sin 58° = 0·848

Therefore:
Area = 10·0 . 14·0 . 0·848
= 118·7 sq. cms.

Answer—Area = 118·7 sq. cms.

Example 3·5—Given that 38 statute miles is equal to 33 nautical miles, find the number of nautical miles in 187 statute miles.

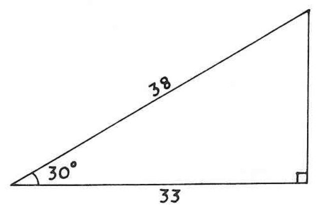


Fig. 3·9

The problem is equivalent to finding the angle in a triangle such that the ratio of any two sides containing the angle is 38:33.

From the traverse table it may readily be seen that the ratio between Hypotenuse and Adjacent is 38:33 when the angle is 30°. The page of the traverse table for 30° is, therefore, a conversion table for converting nautical into statute miles or *vice versa*. Refer to fig. 3·9.

Answer—187 statute miles = 162 nautical miles.

Exercises on Chapter 3

1. Explain clearly the construction of the traverse table. Why is it unnecessary to extend the traverse table beyond 45°?
2. Explain why the traverse table may be used to find the trigonometrical ratios of any angle. Find, by means of the traverse table, (i) sin 50°, (ii) sec 64°, (iii) tan 36°.
3. Explain how the traverse table may be used as a conversion table. Convert, using the traverse table, 18° Centigrade into degrees Fahrenheit.
4. What pages of the traverse table may be used for converting pounds weight into kilograms, given that 2·2 kg = 1 lb. wt.
5. Given the relationship: Convergency = D. Long. . sin Lat., find Convergency if latitude is 56° and D. Long. is 16°.
6. If 84 Lire are equivalent to 26 Francs, find (i) the number of Lire in 130 Francs, and (ii) the number of Francs in 524 Lire.
7. Given the relationships: Departure = D. Long. . cos Latitude and: Error in Latitude = Error in Departure. cot Azimuth, find the Error in D. Long. if the Error in the Latitude is 11′·0, the Azimuth 047°, and the Latitude 29°.

8. Find the area of an equilateral triangle given the length of a side as 14·8 miles.
9. Two adjacent sides of a plane triangle have lengths of 10·0 cms. and 8·5 cms. respectively, and the included angle is 36°. Find the area of the triangle.
10. Given the relationship: D. Long./D.M.P. = Tan course, find course if D. Long. is 457′ E. and D.M.P. is 642 N.
11. To find Latitude from an observation of the altitude of the Pole Star the formulae: Latitude = Altitude − *c*, and: *c* = *p* . cos *P* are used. Find Latitude given Altitude = 42°10′; *p* = 61′; and *P* = 37°30′.
12. Parallax-in-Altitude = Horizontal parallax . cos Altitude. Find parallax-in-Altitude if Altitude is 36° 00′ and Horizontal parallax is 60′·0.
13. Compass Deviation due to Force *P* is proportional to Sine Compass Course. If the maximum deviation due to *P* is 10°·0 W. on East by Compass, find the deviation due to *P* on (i) 030° Compass, (ii) 240° Compass.
14. Deviation due to Force *Q* is proportional to the Cosine of the Compass Course. If the maximum deviation due to *Q* is 6°·5 E. on North by Compass, find the deviation due to *Q* on (i) 320° Compass, (ii) 120° Compass.
15. If $x = B.\sin\theta$, and $x = 10°{\cdot}0$ when $\theta = 90°$, find *x when* $\theta = 40°$.
16. If $y = C.\cos\theta$, and $y = 6°{\cdot}5$ when $\theta = 0°$, find y when $\theta = 60°$.
17. A lighthouse bearing 010° subtended a vertical angle of 1° 30′. The ship travelled on a course of 340° for a distance of 10·0 miles, when the vertical angle of the lighthouse was again 1° 30′. Find the distance off the lighthouse at the time of the second observation.
18. The vertical angle of a cliff was 30°. At a point 1 metre nearer to the foot of the cliff the vertical angle was 60°. Find the height of the cliff.
19. The vertical angle of a cliff was 20°. At a point 10 metres nearer to the foot of the cliff it was 70°. Find the height of the cliff.
20. Find the course to steer to make good a course of due East in order to counteract the effect of a current setting due South at 4·0 knots given the speed of the ship through the water as 12·0 knots.
21. From a vessel heading due North two beacons were observed to bear 030° and 045° respectively. After having travelled for 10·0 miles the beacons were in transit abeam. Find the distance between the beacons.
22. A vertical rod 10·0 metres in length casts a shadow 4·0 metres long at noon when the Sun is at an equinox. Find the Latitude of the place.
23. Find the area of a regular pentagon the length of the side of which is 5·0 cms.
24. The length of a shadow of a vertical pole of length 10·0 metres is 20·0 metres. Find the Sun's altitude.

CHAPTER 4

COMPOUND ANGLES

A knowledge of the contents of this chapter is necessary if it is required to understand the derivation of certain navigational formulae.

1. Trigonometrical Ratios of the Sum of Two Angles

In fig. 4·1:

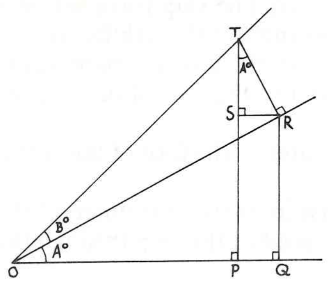


Fig. 4·1

$$\sin(A+B) = \frac{TP}{OT}$$
$$= \frac{RQ + ST}{OT}$$
$$= \frac{RQ}{OT} + \frac{ST}{OT}$$
$$= \frac{RQ\,\cdot}{\cdot\,OT} + \frac{ST\,\cdot}{\cdot\,OT}$$
$$= \frac{RQ}{OR}\cdot\frac{OR}{OT} + \frac{ST}{TR}\cdot\frac{TR}{OT}$$

That is: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\cos(A+B) = \frac{OP}{OT}$$
$$= \frac{OQ}{OT} - \frac{SR}{OT}$$
$$= \frac{OQ\,\cdot}{\cdot\,OT} - \frac{SR\,\cdot}{\cdot\,OT}$$
$$= \frac{OQ}{OR}\cdot\frac{OR}{OT} - \frac{SR}{TR}\cdot\frac{TR}{OT}$$

That is: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Note that if $A = B$, then:

$$\sin(A+B) = \sin 2A = 2\sin A\cos A$$
$$\cos(A+B) = \cos 2A = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$= 2\cos^2 A - 1$$

Also Note that:

$$\sin A = \sin(A/2 + A/2) = 2\sin A/2 \cos A/2$$
$$\cos A = \cos(A/2 + A/2) = \cos^2 A/2 - \sin^2 A/2$$
$$= 1 - 2\sin^2 A/2$$
$$= 2\cos^2 A/2 - 1$$

$\text{vers } A = 1 - \cos A$

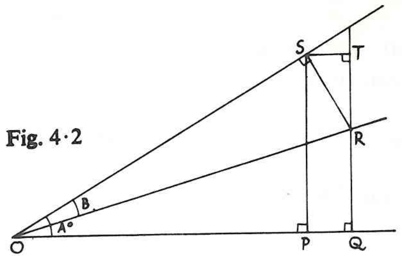
But: $1 = \sin^2 A + \cos^2 A$

Thus: $\text{vers } A = 2\sin^2 A/2$

And: $\text{Hav } A = \sin^2 A/2$

2. Trigonometrical Ratios of the Difference of Two Angles

In fig. 4·2:



$$\sin(A-B) = \frac{QR}{OR}$$
$$= \frac{SP}{OR} - \frac{TR}{OR}$$
$$= \frac{SP\,\cdot}{\cdot\,OR} + \frac{TR\,\cdot}{\cdot\,OR}$$
$$= \frac{SP}{OS}\cdot\frac{OS}{OR} - \frac{TR}{RS}\cdot\frac{RS}{OR}$$

That is: $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$\cos(A-B) = \frac{OQ}{OR}$$
$$= \frac{OP}{OR} + \frac{ST}{OR}$$
$$= \frac{OP}{OS}\cdot\frac{OS}{OR} - \frac{ST}{SR}\cdot\frac{SR}{OR}$$

That is: $\cos(A-B) = \cos A\cos B + \sin A\sin B$

3. Products as Sums and Differences

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

By addition: $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

By subtraction: $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

By addition: $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

By subtraction: $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

or: $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

4. Sums and Differences as Products

Let $X = \tfrac{1}{2}(X+Y) + \tfrac{1}{2}(X-Y)$

and $Y = \tfrac{1}{2}(X+Y) - \tfrac{1}{2}(X-Y)$

Then:

$$\begin{aligned}\sin X + \sin Y &= \sin \tfrac{1}{2}(X+Y)\cos \tfrac{1}{2}(X-Y) \\ &\quad + \cos \tfrac{1}{2}(X+Y)\sin \tfrac{1}{2}(X-Y) \\ &\quad + \sin \tfrac{1}{2}(X+Y)\cos \tfrac{1}{2}(X-Y) \\ &\quad - \cos \tfrac{1}{2}(X+Y)\cos \tfrac{1}{2}(X-Y) \\ &= 2\sin \tfrac{1}{2}(X+Y)\cos \tfrac{1}{2}(X-Y)\end{aligned}$$

Similarly:

$$\begin{aligned}\sin X - \sin Y &= 2\cos \tfrac{1}{2}(X+Y)\sin \tfrac{1}{2}(X-Y) \\ \cos X + \cos Y &= 2\cos \tfrac{1}{2}(X+Y)\cos \tfrac{1}{2}(X-Y) \\ \cos X - \cos Y &= -2\sin \tfrac{1}{2}(X+Y)\sin \tfrac{1}{2}(X-Y) \\ \cos Y - \cos X &= 2\sin \tfrac{1}{2}(X-Y)\sin \tfrac{1}{2}(X+Y)\end{aligned}$$

Exercises on Chapter 4

1. Using the expansions for $\sin (A+B)$ and $\cos (A+B)$ derive expressions for $\tan (A+B)$ and $\cot (A+B)$.
2. Using the expansion for $\sin (A+B)$ derive an expression for $\sin 3A$.
3. Using the expansion for $\cos (A+B)$ derive an expression for $\cos 3A$.
4. Prove that: $\sin 2A + 1 = (\sin A + \cos A)^2$.
5. Without using trigonometrical tables find $\cos 15°$ and $\sin 75°$.: (*Hint* $15 = (45 - 30)$, and $75 = (45+30)$).
6. Without using trigonometrical tables find $\tan 15°$ and $\tan 75°$.

CHAPTER 5

OBLIQUE-ANGLED TRIANGLES AND THEIR SOLUTIONS

1. Introduction

Any triangle which does not contain a right angle is known as an Oblique-Angled Triangle. Any oblique-angled triangle may be divided to form two right-angled triangles, simply by dropping a perpendicular from any corner of the triangle onto the opposite side or side produced. In this way an oblique-angled triangle may be solved by the methods described in Chapter 3. But such methods are indirect: in this chapter we shall be concerned with the DIRECT methods of solving oblique-angled triangles.

2. The Sine Formula

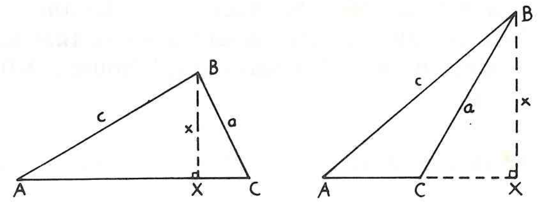


Fig. 5·1

Let the triangle ABC illustrated in fig. 5·1 denote any oblique-angled triangle. From any vertex, say B, drop a perpendicular onto the opposite side or side produced to X. Let the length of this perpendicular be x. In the two right-angled triangles so formed we have:

$$x = c \sin A$$

and $x = a \sin C$ (Remember that $\sin \theta = \sin (180 - \theta)$)

Therefore: $c \sin A = a \sin C$

By dropping a perpendicular from either of the other two vertices onto the opposite side or side produced it may be shown that, in general:

$$a/\sin A = b/\sin B = c/\sin C$$

This relationship is known as the Sine Formula. Stated in words, it is:

The ratio between any two sides of a plane triangle is equal to the ratio between the sines of the angles opposite to the respective sides.

When two sides of a plane triangle and an angle opposite to one of the sides are given, or when two angles and a side opposite to one of them are given, the unknown parts of the triangle may be solved by means of the Sine Formula.

Example 5·1—The horizontal angle at a point A between two other points B and C is 30°. The horizontal angle between A and C at point B is 100°. Find the distance between B and C if the distance between A and B is 8·20 miles.

Referring to fig. 5·2:

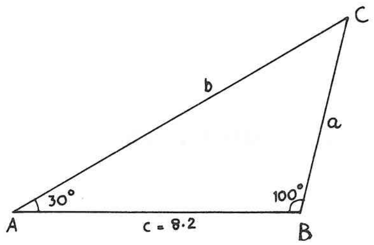


Fig. 5·2

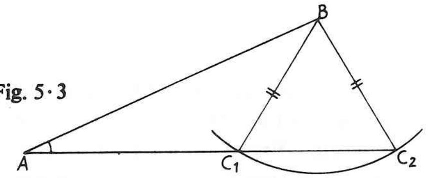
To find a:

$$a/c = \sin A/\sin C$$
$$a = c \sin A \operatorname{cosec} C$$
$$\log c = 0{\cdot}91381$$
$$\log \sin A = \bar{1}{\cdot}69897$$
$$\log \operatorname{cosec} C = 0{\cdot}11575$$
$$\log a = 0{\cdot}72853$$
$$a = 5{\cdot}35$$

Answer—BC = 5·35 miles.

If two sides and an angle of a plane triangle are given, and the shorter of the two given sides is opposite to the given angle, two values will satisfy each of the unknown parts of the triangle. This is known as the Ambiguous Case. This is illustrated in Example 5·3.

Fig. 5·3



Example 5·2—A vessel heading 340° bears 200° at a distance of 10·50 miles from a lighthouse. Find the distance the vessel travels so that the distance between her and the lighthouse is 8·00 miles.

It is evident from fig. 5·4 that, because the smaller of the given sides is opposite to the given angle, two values satisfy the solution.

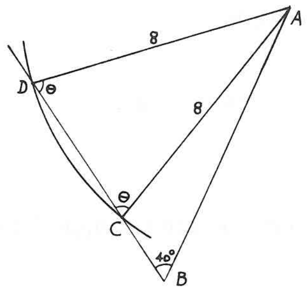


Fig. 5·4

(1) *To find* θ

$$\sin \theta/AB = \sin 40°/AD$$
$$\sin \theta = (10{\cdot}50 \,.\, \sin 40°)/8{\cdot}00$$
$$\log 10{\cdot}50 = 1{\cdot}02119$$
$$\log \sin 40° = \bar{1}{\cdot}80807$$
$$\log \text{product} = 0{\cdot}82926$$
$$\log 8{\cdot}00 = 0{\cdot}90309$$
$$\log \sin \theta = \bar{1}{\cdot}92617$$
$$\theta = 57° 31\tfrac{3}{4}'$$

(2) *To find BD*

$$BAD = 180° - (\theta + 40°)$$
$$= 82° 28\tfrac{3}{4}'$$
$$BD/\sin A = AD/\sin 40°$$
$$BD = 8{\cdot}00 \,.\, \sin 82° 28\tfrac{1}{4}' \,.\, \operatorname{cosec} 40°$$
$$\log 8{\cdot}00 = 0{\cdot}90309$$
$$\log \sin A = \bar{1}{\cdot}99624$$
$$\log \operatorname{cosec} 40° = 0{\cdot}19193$$
$$\log BD = 1{\cdot}09126$$
$$BD = 12{\cdot}3 \text{ miles}$$

(3) *To find CD*

$$CD/2 = 8{\cdot}00 \,.\, \cos \theta$$
$$CD = 16{\cdot}00 \,.\, \cos 57° 31\tfrac{3}{4}'$$
$$\log 16{\cdot}00 = 1{\cdot}20412$$
$$\log \cos \theta = \bar{1}{\cdot}72987$$
$$\log CD = 0{\cdot}93399$$
$$CD = 8{\cdot}60 \text{ miles}$$

(4) *To find BC*

$$BC = BC - CD$$
$$= 12{\cdot}3 - 8{\cdot}6$$
$$= 3{\cdot}7 \text{ miles}$$

Answer—Distance travelled = 3·7 miles or 12·3 miles.

3. The Cosine Formula

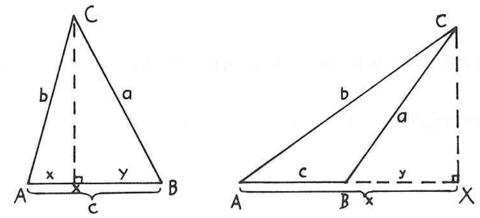


Fig. 5·5

Let ABC illustrated in fig. 5·5 be any plane triangle. Drop a perpendicular from any vertex, say C, onto the opposite side or side produced.

In the acute-angled triangle ABC: $c = x + y$
In the obtuse-angled triangle ABC: $c = x - y$
In both triangles: $c = b \cos A + c \cos B$. (I)

(Note that B is an obtuse angle and that its cosine is negative.)

By dropping perpendicular from A and B onto sides BC and AC respectively. It may be shown that:

$$b = c \cos A + a \cos C \;\ldots\ldots \quad \text{(II)}$$
$$a = b \cos C + c \cos B \;\ldots\ldots \quad \text{(III)}$$

From equations (I), (II) and (III), the plane Cosine Formula is deduced as follows: Multiply equation (I) by a; equation (II) by b; and equation (III) by c, thus:

$$a^2 = ab \cos C + ac \cos B$$
$$b^2 = bc \cos A + ab \cos C$$
$$c^2 = bc \cos A + ac \cos B$$

By subtracting each of these from the sum of the other two, we have:

$$b^2 + c^2 - a^2 = 2bc \cos A$$

or

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

or

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Similarly:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

or

$$b^2 = a^2 + c^2 - 2ac \cos B$$

And

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

or

$$c^2 = a^2 + b^2 - 2ab \cos C$$

These formulae are not suitable for use with logarithms, since addition and subtraction do not require logarithyms. Their use, therefore, should be confined to solving triangles whose sides are integral numbers which can easily be squared arithmetically, and the whole problem—except perhaps for the extraction of a square root—completed by simple arithmetic.

The plane Cosine Formula may be used for:

1. Finding an angle given the three sides of a triangle.
2. Finding a side of a triangle given the opposite angle and the other two sides.

Example 5·3—Find the largest angle in a plane triangle whose sides are 5·0 miles, 8·0 miles and 4·0 miles, respectively. Refer to fig. 5·6.
(Note that the largest angle of a triangle is opposite to the longest side.)

By the Cosine formula:

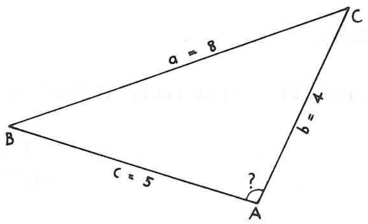


Fig. 5·6

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{16 + 25 - 64}{40}$$
$$= -\frac{23}{40}$$
$$= -0{\cdot}575$$
$$A = 180° - 54°54'$$
$$= 125°06'$$

Answer—Largest Angle = 125°06′.

4. The Haversine Formula

The Haversine Formula, derived from the Cosine Formula, is useful for solving an angle in a plane triangle given the three sides of the triangle. Being suitable for use with logarithms, it is a more useful formula than the Cosine Formula for solving angles.

In any plane triangle: *ABC*:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

By multiplying each side by − 1, adding + 1 to each side, and dividing each side by 2, we have:

$$\frac{1 - \cos A}{2} = \frac{1 - \dfrac{b^2 + c^2 - a^2}{2bc}}{2}$$

The left-hand side of this expression is equal to the haversine of *A*, thus:

$$\text{hav } A = \frac{2bc - b^2 - c^2 + a^2}{4bc}$$
$$= \frac{a^2 (- b^2 + c^2 - 2bc)}{4bc}$$
$$= \frac{a^2 - (b - c)^2}{4bc}$$
$$= \frac{(a - b + c)(a + b - c)}{4bc}$$
$$= \frac{(a + b + c - 2b)(a + b + c - 2c)}{4bc}$$
$$= \frac{(2s - 2b)(2s - 2c)}{4bc} \quad \text{where } s = (a + b + c)/2$$
$$= \frac{4(s - b)(s - c)}{4bc}$$

or:

$$\text{hav} A = \frac{(s - b)(s - c)}{bc}$$

Similarly:

$$\text{hav } B = \frac{(s - a)(s - c)}{ac}$$

$$\text{hav } C = \frac{(s - a)(s - b)}{ab}$$

Example 5·4—Point *Q* lies 17·540 miles from Point *P*. Point *R* lies 13·600 miles from Point *Q* and 18·770 miles from Point *P*. Find the angle between the directions of *Q* and *R* from *P*.

In fig 5·7:

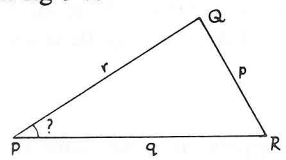


Fig. 5·7

$$p = 13{\cdot}600$$
$$q = 18{\cdot}770$$
$$r = 17{\cdot}540$$
$$2)49{\cdot}910$$
$$s = 24{\cdot}955$$

By the Haversine Formula:

$$\text{hav } P = \frac{(s-q)(s-r)}{qr}$$

$$= \frac{6{\cdot}185 \times 7{\cdot}415}{18{\cdot}770 \times 17{\cdot}540}$$

$$\begin{aligned}
\log 6{\cdot}185 &= 0{\cdot}79134 \\
\log 7{\cdot}415 &= 0{\cdot}87011 \\
\log \text{num}^r &= 1{\cdot}66145 \\
\log 18{\cdot}770 &= 1{\cdot}27346 \\
\log 17{\cdot}540 &= 1{\cdot}24403 \\
\log \text{den}^r &= 2{\cdot}51749 \\
\log \text{hav } P &= \bar{1}{\cdot}14396 \qquad P = 43^\circ\ 50'
\end{aligned}$$

Answer—Required Angle = 43° 50′.

5. The Tangent Formula

We have, by the Sine Formula, for triangle *ABC*;

$$a/\sin A = b/\sin B = c/\sin C = x$$

Therefore:

$$a = x \sin A$$

and

$$b = x \sin B$$

Therefore:

$$\frac{a+b}{a-b} = \frac{x \sin A + x \sin B}{x \sin A - x \sin B}$$

$$= \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)}{2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B)}$$

$$= \tan \tfrac{1}{2}(A+B) \cot \tfrac{1}{2}(A-B)$$

Or

$$\frac{a+b}{a-b} = \frac{\tan \tfrac{1}{2}(A+B)}{\tan \tfrac{1}{2}(A-B)}$$

This formula may be used instead of the Cosine or Haversine Formula for finding an angle of a plane triangle given any two sides and the included angle.

The quantity ½(*A* + *B*) is equal to the complement of *C*, because the sum of the angles of a plane triangle is 180°. Thus, if ½(*A* − *B*) is known, the angles *A* and *B* may be found by addition and subtraction.

Example 5·5—A yacht sails for a distance of 5·0 miles on a course of 060°, and for 4·0 miles on a course of 190°. Find the course to steer and the distance to sail to regain the starting position.

In fig. 5·8:

$$C + A = 130^\circ$$

$$\tfrac{1}{2}(C + A) = 65^\circ$$

Plan (i) Find *C* by the Tangent Formula.
(ii) Find *b* by the Sine Formula.

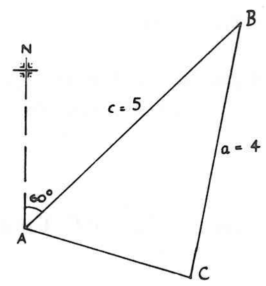


Fig. 5·8

$$\frac{\tan \tfrac{1}{2}(C-A)}{\tan \tfrac{1}{2}(C+A)} = \frac{c-a}{c+a}$$

$$\begin{aligned}
\tan \tfrac{1}{2}(C-A) &= \frac{(c-a)}{(c+a)} . \tan \tfrac{1}{2}(C+A) \\
&= 1/9 . \tan 65^\circ \\
&= 1/9 . 2{\cdot}14451 \\
&= 0{\cdot}23828 \\
\tfrac{1}{2}(C-A) &= 13^\circ 24' \\
\tfrac{1}{2}(C+A) &= 65^\circ\ 00' \\
C &= 78^\circ\ 24' \\
\text{Course} &= \text{N. } 68^\circ\ 24' \text{ W.}
\end{aligned}$$

$$\begin{aligned}
b &= 5 \sin 50^\circ \text{ cosec } 78^\circ\ 24' \\
\log 5 &= 0{\cdot}69897 \\
\log \sin 50^\circ &= \bar{1}{\cdot}88425 \\
\log \text{cosec} &= 0{\cdot}00896 \\
\log b &= 0{\cdot}59218 \\
b &= 3{\cdot}91
\end{aligned}$$

Answer—Course = 291½°, Distance = 3·9 miles.

Exercises on Chapter 5

1. From a vessel at anchor a Point *A* bore 030° and a lighthouse *B* bore 070°. *B* lies 190° at a distance of 5·40 miles from *A*. Find the distance between *B* and the vessel.

2. The horizontal angle between two vessels *A* and *B* is 62°. *A* is 4·0 miles from an observer and *B* is 5·0 miles from him. Find the distance between the vessels.

3. In a plane triangle *ABC*, *AB* is 3·0 miles; *AC* is 7·0 miles; and *BC* is 6·0 miles. Find the three angles of the triangle.

4. In a plane triangle *PQR*, *Q* is 97° 00′; *PQ* is 7·40 miles and *QR* is 6·70 miles. Find the distance *PR*.

5. Two vessels *A* and *B* are 6·30 miles apart. The horizontal angle between *B* and a rock *C* is 20° 32′, and that between *A* and *C* at *B* is 70° 15′. Find the distance *BC*.

6. Find the required course to steer to counteract the effect of a current which sets 100° at a rate of 4·0 knots, given the course to make good is 025° and the ship's speed through the water is 13 knots.

7. A point of land bears 305° at a distance of 46·0 miles. Find the course to steer and the time taken to reach the point in a current which sets 045° at 3·5 knots, and the speed of the vessel through the water is 15·5 knots.

8. A point of land bore 034°. After travelling for 5·60 miles on a course of 268°, it bore 050°. Find the distance off at the time of the second observation.

9. A point of land bore 066° from a vessel heading 030°. After travelling for 30 minutes the point bore 146° at a distance of 6·0 miles. Find the speed of the vessel.

10. *A, B* and *C*, are three buoys in a harbour. The distance between *A* and *B* is 125·6 metres; that between *B* and *C* is 130·4 metres and that between *C* and *A* is 112·0 metres. At a vessel *D, A* and *B*, and *A* and *C*, subtend, respectively, angles of 48° 58′ and 25° 52′. Find the distances between the vessel and each of the buoys.

11. In a plane triangle *ABC, AB* is 562 yards, *BC* is 320 yards, and angle *B* is 128° 04′. Find *AC*.

12. In a triangle *ABC*, the sides *AB* and *BC* are 345 cms. and 232 cms. respectively, and the angle *A* is 37° 20′. Find the angle *B*.

CHAPTER 6

SPHERICAL TRIGONOMETRY

1. The Geometry of the Sphere

A sphere is a three-dimensional shape every point on the surface of which is equidistant from a fixed point known as the Centre of the Sphere. It may be defined as the shape swept out by rotating a circle about any fixed diameter through an angle of 180°. Such a circle is the largest possible circle that may be drawn on the surface of the sphere produced.

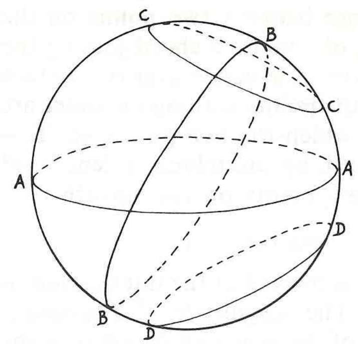


Fig. 6·1

It will be noticed in fig. 6·1 that the centre of the sphere illustrated lies on the planes of the circles *AA* and *BB*. Any circle on he surface of a sphere, on whose plane the centre of the sphere lies, is known as a Great Circle of that sphere. *AA* and *BB* in fig. 6·1 are examples of great circles. A great circle divides a sphere into two Hemispheres. Any circle on the surface of a sphere which is not a Great Circle is known as a Small Circle. In fig 6·1 *CC* and *DD* are examples of small circles. Notice that the word "small" in this context has no reference to the actual size of a circle: it is used merely to distinguish between great cirles on whose plane the centre of the sphere rests from those circles on whose planes the centre does not rest.

An arc of a great circle is measured in angular units: it is a measure of the angle at the centre of the sphere subtended by the two radii which terminate at the extremities of the arc. The measure of a complete circle is 360°; that of a semi-great circle is 180°; and that of a quadrant or quarter of a great circle is 90°. The measure of an arc of a great circle is known as a Spherical Distance.

Two points on a great circle which are diametrically opposed to one another, that is to say, two points which are separated by a spherical distance of 180°, are known as Antipodal Points, each being an Antipodes of the other.

A point on the surface of a sphere which is 90° from every point on a particular great circle is known as the Pole of that great circle. The diameter of a sphere which connects the two poles of a great circle is known as the Axis of the Great Circle.

Any semi-great circle which connects the poles of a given great circle is referred to as a Secondary to the given great circle which is known, in this case, as the Primary Great Circle. A secondary cuts its primary great circle at an angle of 90°. It follows that the axis of a primary great circle lies in the plane of every secondary.

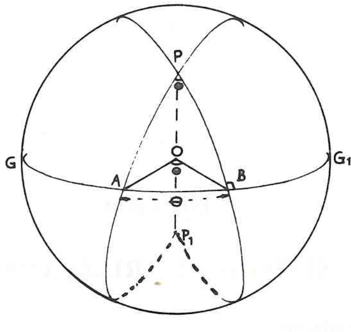


Fig. 6·2

In fig. 6·2, P and P_1 are the poles of a great circle GG_1. PA and PB are 90° arcs of the secondaries PAP_1 and PBP_1 respectively. The point G is the antipodes of the point G_1.Note that the arc AB has a spherical distance equal to the angle AOB at the centre of the sphere.

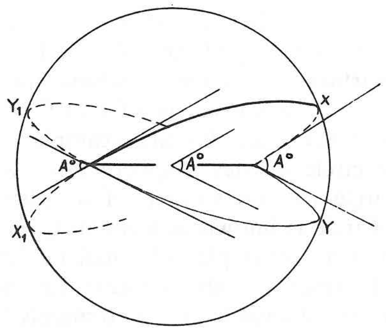


Fig. 6·3

The shortest distance between two points on the surface of a sphere is, of course, a chord joining the two points; but the shortest distance over the surface of the sphere between the points is along the lesser arc of the great circle on which the two points lie. This readily may be verified by stretching a length of chord between any two points on the surface of a model globe.

A Spherical Angle is formed at the intersection of two great circle arcs. The magnitude of a spherical angle is equal to that of the plane angle between the tangents to the great circles at the point of intersection.

The angle between the two great circles XX_1 and YY_1 illustrated in fig. 6·3, is $A°$. This is equal to the angle between the tangents as indicated in the figure.

A Spherical Triangle is formed on the surface of a sphere by the intersecton of three great circle arcs.

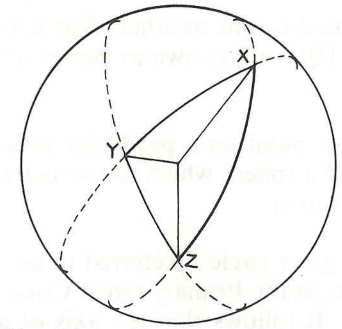


Fig. 6·4

Fig. 6·4 depicts a typical spherical triangle formed by great circle arcs XY, YZ and ZX.

The sum of the three angles of any plane triangle is always 180°. The sum of the three angles of any spherical triangle, however, is always greater than 180°, by an amount known as Spherical Excess. It is impossible to construct a plane triangle on the surface of a sphere, but the smaller is a spherical triangle on a given sphere the more nearly is the sum of its three angles equal to 180°. In some navigational problems small spherical triangles are treated as if they are plane, and no material error results.

The maximum value of any one angle of a spherical triangle is 180°, or 2 right angles. The sum of the three angles of a spherical triangle must, therefore, be less than 540° or 6 right angles. It follows that a spherical triangle cannot be larger than a hemisphere.

The spherical distance of any side of a spherical triangle never exceeds 180° or 2 right angles. Because a spherical triangle cannot be larger than a hemisphere, it follows that the sum of the three sides cannot be more than 360° or 4 right angles.

2. Propertise of a Spherical triangle.

i) Each side of a spherical triangle is an arc of a great circle. The length is measured in degree and minutes.
ii) Each angle is measured between great circles forming adjacent sides and is measured in degree and minutes.
iii) The sum of the length of the sides is less than 360°.
iv) The sum of the angles lies between 180°-540°.
v) No angle or side is greater than 180°.
vi) The greatest side is opposite the greatest angle and similarly, the smallest side and angle are opposites.
vii) Two or more sides and angles can be 90° in size.
viii) If two sides are equal in size, so are angles opposite to them.
ix) If three sides are equal, the angles are equal but not necessarily equal to 60°.

3. Spherical Trigonometry

The main use of spherical trigonometry is in the solving of spherical triangles. If any three parts of a spherical triangle are known, any of the remaining parts may be solved direct by one of the three so-called Fundamental Formulae of spherical trigonometry. These are the Sine Formula, the Cosine Formula and the Four Parts Formula. In addition to the fundamental formulae there are numerous *derived* formulae. The spherical trigonometrical formulae used in the practice of navigation and nautical astronomy will be considered in Chapter 7.

In Chapter 10 we shall see that the Earth's shape is not that of a perfect sphere. Nevertheless the spherical formulae used by navigators, despite the fact that they give accurate results only for perfectly spherical surfaces, yield results which are generally sufficiently accurate for all navigational problems.

Exercises on Chapter 6

1. Define: Sphere, Great circle, Poles of a Great Circle, Axis of a Great Circle.
2. Define: Secondary, Primary Great Circle.
3. What are antipodal points?
4. Show that the magnitude of a spherical angle is equal to that of the plane angle between the tangents at the intersection points of the two great circle arcs which form the spherical angle.
5. Prove that the sum of the three angles of a spherical triangle cannot exceed 540°, and that the sum of the three sides cannot exceed 360°.

CHAPTER 7

THE STEREOGRAPHIC PROJECTION AND THE GRAPHICAL SOLUTIONS OF SPHERICAL TRIANGLES

1. Introduction

The representation on a plane surface of points and lines on an object as they appear in the eye, is known as Perspective or Geometrical Projection. The plane surface is called the Plane of the Projection, and the position of the eye as the Point of Projection. A straight line extending from the Point of Projection to any point on the object is called a Line of Projection.

There are many methods of projecting a spherical surface onto a plane surface. All maps and charts, for example, are projections of the whole or part of the spherical Earth's surface, and some of these are perspective projections. Non-perspective projections, such as the Mercator Projection (see Chapter 12), are known as Conventional Projections.

2. The Stereographic Projection

A very useful method of projecting a spherical surface onto a plane surface is the stereographic projection, which is a perspective projection. In the stereographic projection a great circle of the sphere to be projected is assumed to lie in the plane of the projection. This great circle is known as the Primitive, and the point of the projection is one of the poles of the primitive great circle.

On a stereographic projection of a sphere all projected arcs of circles, great and small, are straight lines or arcs of circles. This property makes it possible to construct a stereographic projection geometrically by means of straightedge and drawing compasses. Other interesting properties of the stereographic projection are that more than a hemisphere can be projected, and that the projection is Orthomorphic, which means that angles at any point on the plane of projection are without distortion.

The stereographic projection is often used for Star Maps, and it has been used as the basis of a variety of instruments designed for solving spherical triangles, especially in nautical astronomy. Although we shall indicate how to construct a stereographic projection and how to measure angles and spherical distances, our main intention here is to assist students in visualizing the relative positions of points and arcs on a spherical surface, especially that of the celestial sphere, thereby leading them to an understanding and appreciation of certain navigational and nautical astronomical problems.

3. The Principles of the Stereographic Projection

Fig. 7·1 serves to illustrate the principle of the stereographic projection.

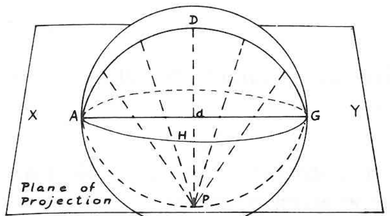


Fig. 7·1

The great circle AHG illustrated in fig. 7·1 lies in the plane of the projection. It is, therefore, the primitive, and it is upon the plane of the primitive that the sphere's surface is to be projected.

Consider a great circle $ADGP$ which stands perpendicular to the plane of the projection. If lines of projection are drawn from the point of projection to several points on the great circle, the points at which these lines cut the plane of the projection lie on the straight line AdG. This line, therefore, is the projection of the great circle arc ADG, and the point d, which is the projection of D, lies at the centre of the primitive.

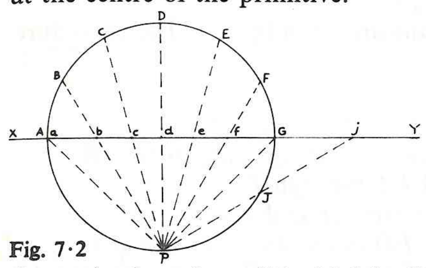


Fig. 7·2

In fig. 7·2, P is the point of projection and the straight line $AdGY$ lies in the plane of the projection. Points A, B, C, D, E, F, G and J, are points on a great circle which is a secondary to the primitive. These points are 30° apart and they are projected at points a to j respectively. It will be noticed that the projected lengths of these equal arcs are not uniform. The projected arc Ab is longer than that of CD, and the projection of arc GJ, which is Gj, is even longer than the projection ab of arc AB. At the edge of the primitive there is no distortion of arcs; within the primitive distortion is such that equal arcs of the sphere are projected as increasingly smaller lines as the centre of the projection is approached; and that outside the primitive the distortion is such that equal arcs of the sphere are projected as increasingly longer lines as distance from the primtive increases.

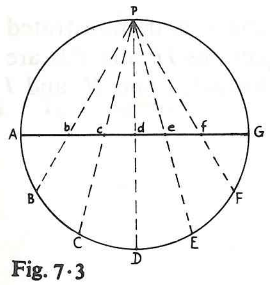


Fig. 7·3

All great circles which cut the primitive at right angles, that is to say, all the secondaries to the primitive, are projected as straight lines which pass through the centre of the projection.

Imagine the sphere illustrated in fig. 7·2 to be rotated about the diameter AdG through an angle of 90°. The points a, b, c, etc., remain stationary because they lie on the axis of rotation. The points B, C, D, etc., however, move to new positions on the plane of the projection. The projection viewed from directly above now appears as in fig. 7·3.

Fig. 7·3 serves to demonstrate that when constructing a stereographic projection all projecting is made from a point on the circumference of the primitive. It is to be realized, however, that the completed projection is a view of the sphere's surface from the pole of the primitive.

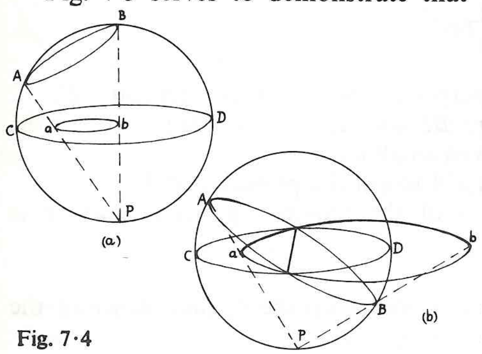


Fig. 7·4

Fig. 7·4 illustrates that all circles on the sphere are projected as circles or straight lines. Let the point P, in fig. 7·4, be the point of projection, and CD the primitive. In fig. 7·4(a) AB denotes any small circle on the sphere, and in fig. 7·4(b) AB denotes any great circle on the sphere. In both cases the circle is projected as circle ab.

4. To Project a Great Circle about a Given Point as Pole

Case 1—If the given point is at the centre of the primitive, the required projection is the primitive itself.

Case 2—If the given point is on the circumference of the primitive the required projection is a diameter lying perpendicular to the diameter on which the given point lies.

Referring to fig. 7·5, let the circle *ACBD* be the primitive and the point *P* the pole of the required projection.

Case 3—If the given point is within the primitive, as illustrated in fig. 7·5. The procedure in this case is as follows:

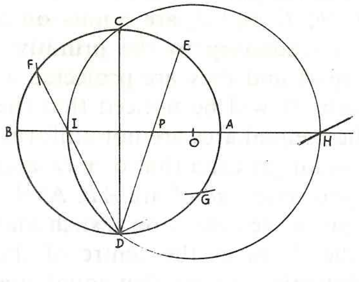


Fig. 7·5

(i) Draw diameter *AB* through *P*.
(ii) Draw diameter *CD* at right angles to *AB*.
(iii) Draw chord *DE* through *P*.
(iv) Describe 90° arcs *EF* and *EG*.
(v) Draw chord *FD* to cut diameter *AB* at *I*.
(vi) Draw chord *FD* (produced) to cut diameter *AB* (produced) at *H*.
(vii) Bisect *IH* at *O*, which is the centre of the required projection, radius *OI* or *OH*.

In practice it is customary to find the centre *O*, which is known to lie on the diameter *AB*, or *AB* produced, by trial and error.

The construction described above is demonstrated thus. Because arcs *EF* and *EG* of the primitive are each 90°, the projections *IP* and *PH* are each 90°. *P*, therefore, is the centre of the great circle which passes through *I* and *H*, and *I* and *H* are projections of antipodal points.

5. To Project a Small Circle about a Given Point as Pole

Case 1—If the given pole is at the centre of the projection.

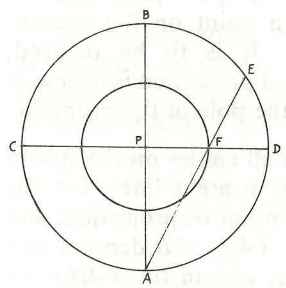


Fig. 7·6

Referring to fig. 7·6:

(i) Draw any two perpendicular diameters *AB* and *CD*.
(ii) Mark off an arc *BE* (or *AE*, *CE* or *DE*) equal to the radius of the given small circle.
(iii) Draw the chord *EA* to cut the primitive at *F*.
(iv) *PF* is the radius of the required projection which is centred at *P*.

Case 2—If the given pole is on the circumference of the primitive.

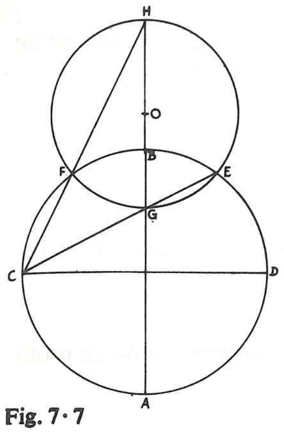


Fig. 7·7

Referring to fig. 7·7:

(i) Let *B* be the given pole.
(ii) Draw the perpendicular diameters *AB* and *CD*.
(iii) Mark off arcs *BE* and BF each equal to the radius of the given small circle.
(iv) Draw *EC* to cut *AB* at *G*.
(v) Draw *CF* and produce to cut *AB* produced at *H*.
(vi) Bisect *GH* at *O* which is the centre of the required projection of radius *OG* or *OH*.

Case 3—If the given pole is within the primitive but not at the centre.

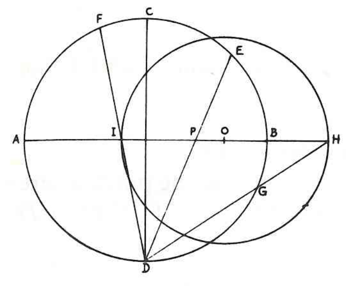


Fig. 7·8

Referring to fig. 7·8.

(i) Let *ACBD* be the primitive and *P* the given pole.
(ii) Draw diameter *AB* through *P* the given pole.
(iii) Draw diameter *CD* perpendicular to *AB*.
(iv) Draw the chord *DP* and produce to *E*.
(v) Centre at *E* and describe arcs *EF* and *EG*, each equal to the radius of the given small circle.
(vi) Draw *DF* to cut diameter AB at *I*, and *DG* produced to cut *AB* produced at *H*.
(vii) Bisect *IH* at *O*, which is the required centre, the radius of the required projection being *OI* or *OH*. (It is instructive to compare this construction with that given in fig. 7·5).

6. To Find the Locus of Centres of All Great Circles which Pass Through a Given Point

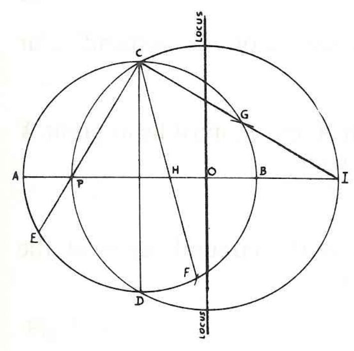


Fig. 7·9

Referring to fig. 7·9:

(i) Let the circle *ACBD* be the primitive and *P* the given point.
(ii) Draw diameter *AB* through *P*.
(iii) Draw diameter *CD* perpendicular to *AB*.
(iv) Draw the chord *CE* through *P*.
(v) Centre at *E* and radius 90°, describe arc *EF*.
(vi) Centre at *F* and radius 90°, describe arc *FG*.
(vii) Draw chord *CF* to cut *AB* at *H*.
(viii) Draw *CG* and produce to cut *AB* produced at *I*.
(ix) Bisect *PI* at *O*.
(x) Draw a perpendicular to *AB* through *O*, which is the required locus.

The proof of this construction is as follows:

Because *EF* and *FG* are each 90°, it follows that arcs *PH* and *HI* are each 90°. The point *I*, therefore, is 180° from *P*, so that *P* and *I* are antipodal points. Every circle passing through

P and *I* must be a great circle. The locus of centres of all circles passing through *P* and *I* is the perpendicular bisector of *PI*, so that the required locus must be this bisector.

7. To Project a Great Circle through Two Given Points

Case 1—If one of the points is at the centre of the primitive, the required projection is the diameter drawn through the two points.

Case 2—If one of the points lies on the circumference of the primitive and the other point is not at the centre or the circumference of the primitive.

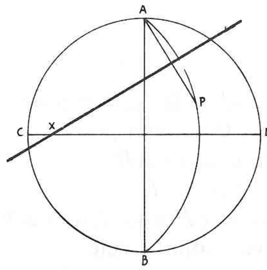


Fig. 7·10

Referring to fig. 7·10:

(i) Let the circle *ACBD* be the primitive and *A* and *P* the two given points.
(ii) Draw diameter *AB*.
(iii) Draw diameter *CD* perpendicular to *AB*.
(iv) The centre of the required projection is at the point where the perpendicular bisector of *AP* cuts *CD* or *CD* produced.

This is so because:

1. All great circles which pass through *A* must pass through the antipodes of *A*, that is to say, through *B*.

2. The locus of centres of all great circles which pass through *A* and *B* must lie on the diameter *CD* or *CD* produced.

3. The centres of all circles which pass through *A* and *P* must lie on the perpendicular bisector of the chord *AP*.

Therefore, the centre of the great circle which passes through *A* and *P*, must lie at point *X* which is the point at which perpendicular bisector of *AP* cuts *CD*.

Case 3—If neither of the given points lies at the centre or the circumference of the primitive.

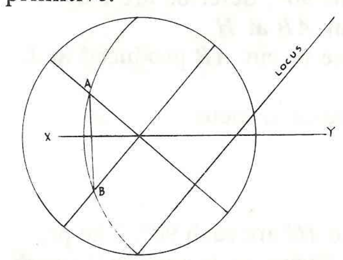


Fig. 7·11

Referring to fig. 7·11:

(i) Let *A* and *B* be the two given points.
(ii) Find the locus of centres of great circles through *A*.
(iii) Bisect the straight line joining *A* to *B* by the line *XY*.
(iv) The centre of the required projection is the point *O* at which the line *XY* cuts the locus of centres of great circles through *A*.

8. To Measure a Given Arc of a Projected Great Circle

Case 1—If the given arc is part of the primitive, the required measure is the angle at the centre of the primitive contained between the extremities of the given arc.

Case 2—If the arc is part of a great circle which is projected as a straight line.

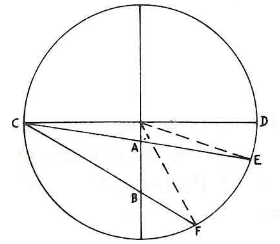


Fig. 7·12

Referring to fig. 7·12:

(i) Let *AB* be the projected arc.
(ii) Draw a diameter through *A* and *B*.
(iii) Draw a diameter perpendicular to *AB*.
(iv) The Draw chords *CE* and *CF* through *A* and *B* respectively.
(v) Arc *EF* is the required measure of the arc *AB*.

Case 3—If the given arc is part of a great circle which is inclined to the primitive.

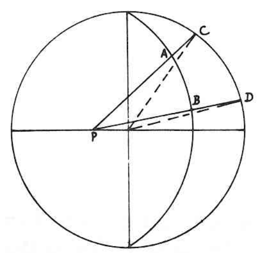


Fig. 7·13

Referring to fig. 7·13:

(i) Let *AB* be the given arc.
(ii) Find the pole *P* of the projected circle on which arc *AB* lies.
(iii) Draw *PA* and *PB* and produce each to cut the primitive at *C* and *D* respectively.
(iv) Arc *CD* is a measure of the arc *AB*.

9. To Measure a Projection of a Spherical Angle

The angle between two great circle arcs is equivalent to the plane angle between their radii or the tangents of the projected great circles at the point of intersection.

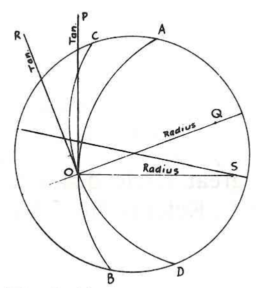


Fig. 7·14

In fig. 7·14 the angle between the projected great circles *AB* and *CD* is equal to the plane angle *POR* or the plane angle *SOQ*.

10. Examples

The following examples serve to show how the stereographic projection may be used to solve spherical triangles by construction. The student is advised to read and understand Chapter 4 before considering Examples 1 to 4 inclusive, and to read and understand Parts 4 and 5 before considering Examples 5 to 10 inclusive.

Example 7·1—Construct a stereographic projection of the Earth's northern hemisphere. Project meridians at intervals of 30° from the Greenwich meridian and the parallels of Latitude of 30° and 60° North. Project the positions of London (Lat. 51° N., Long. 0°); New York (Lat. 41° N., Long. 74° W.); Moscow (Lat. 56° N., Long. 38° E.); and Tokyo (Lat. 36° N., Long. 139° E.). Refer to fig. 7·15.

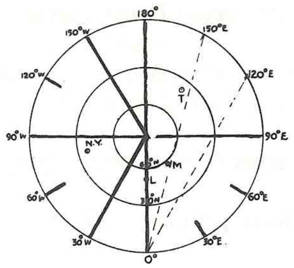


Fig. 7·15

Example 7·2—Construct a stereographic projection of the Earth on the plane of the Greenwich meridian. Project the parallels of 30° and 60° N. and S., and the meridians at intervals of 45° from the meridian of Greenwich. Project the positions of Panama (Lat. 9° N., Long. 80° W.) and Cape Horn (Lat. 55° S., Long. 66° W.). Refer to fig. 7·16.

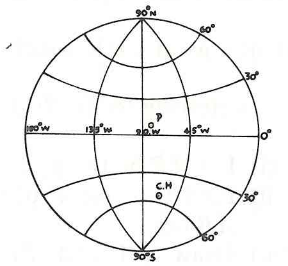


Fig. 7·16

Example 7·3—Find, by scale drawing, the initial course of the great circle route from *A* in Lat 20° N., Long. 100° W., to *B* in Lat. 60° N., Long. 40° W. (Note that because *B* lies to the eastwards of *A*, it is convenient to project on the plane of *A*'s meridian). Refer to fig. 7·17.

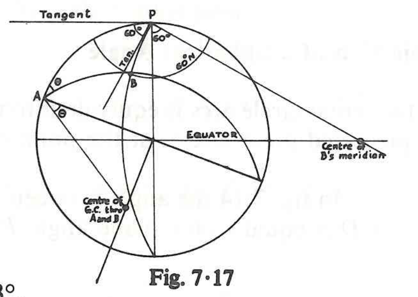


Fig. 7·17

Answer—Initial Course = 033°.

Example 7·4—Find by means of a stereographic projection, the great circle distance between *X* in Lat. 30° S., Long. 90° E., and *Y* in Lat. 40° N., Long. 40° E. Refer to fig. 7·18.

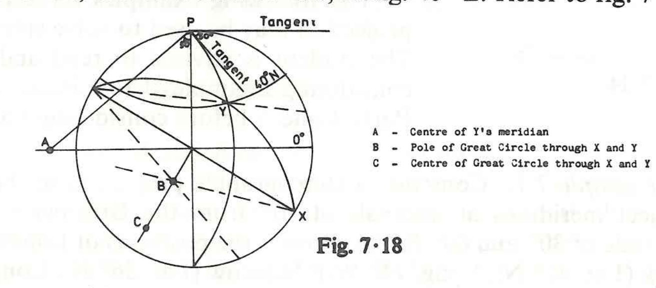


Fig. 7·18

Answer—Great Circle Distance = 85° or 5,100 miles.

Example 7·5—Construct a stereographic projection, of the celestial sphere on the plane of the celestial equator. Project celestial meridians at intervals of 45° from the meridian of the First Point of Aries. Project parallels of declination 30° and 60° N., and the positions of Capella (dec. 46° N., S.H.A. 283°), and Arcturus (dec. 19½° N., S.H.A. 147°). Refer to fig. 7·19.

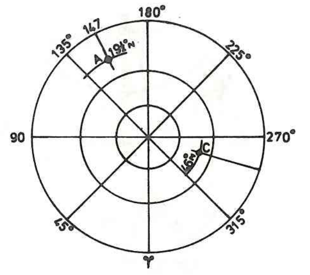


Fig. 7·19

Example 7·6—Construct a stereographic projection, of the celestial sphere on the plane of the celestial horizon of an observer in Lat. 50° N. Project the celestial equator and hour circles at intervals from 3 hr. from the Observer's lower celestial meridian. (*Note that*—Latitude of Observer = Altitude of Celestial Pole). Refer to fig. 7·20.

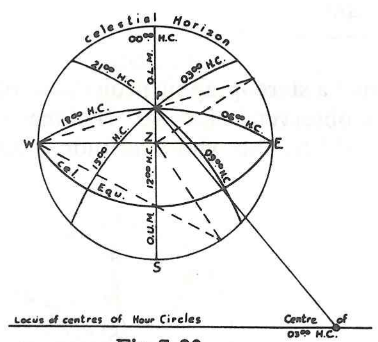


Fig. 7·20

Example 7·7—Construct a stereographic projection, of the celestial sphere on the plane of the celestial horizon of an observer in Lat. 30° S. Project the celestial equator and parallels of declination of 30° N., 30° S. and 60° S. Refer to fig. 7·21.

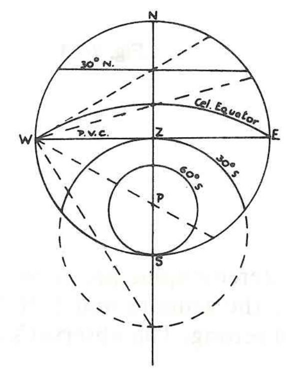


Fig. 7·21

Example 7·8—Construct a stereographic projection, of the celestial sphere on the plane of the celestial meridian of an observer in Lat. 30° N. Project the position of a star whose declination is 40° N., and whose altitude is 50° and decreasing. Measure the azimuth and the Local Hour Angle of the star. Refer to fig. 7·22.

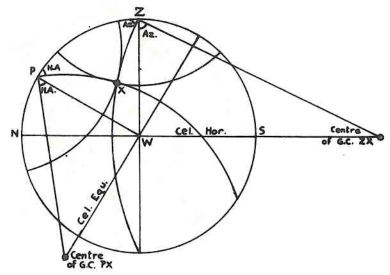


Fig. 7·22

Answer—Azimuth = 298°

L.H.A. = 46°.

Example 7·9—Construct a stereographic projection, of the celestial sphere on the plane of the celestial horizon of an observer in Lat. 40° N. Measure the L.H.A. and the azimuth of a star whose declination of 35° S. and whose altitude is 35° and rising. Refer to fig. 7·23.

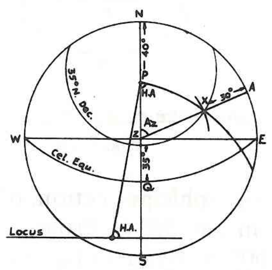


Fig. 7·23

Answer—Azimuth = 065°

L.H.A. = 80°.

Example 7·10—Construct a stereographic projection, of the celestial sphere on the plane of the celestial equator. Measure the azimuth and L.H.A. of a star whose declination is 20° N. and whose altitude is 30° and setting. The observer's Latitude is 50° N. Refer to fig. 7·24.

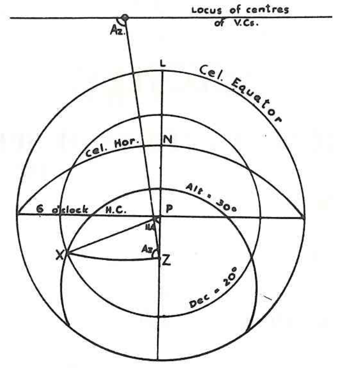


Fig. 7·24

Answer—Azimuth = 266°

L.H.A. = 67°.

11. Figure Drawing

Although in the practice of navigation and nautical astronomy solutions by scale drawing arenot,ingeneral,sufficientlyaccurate,freehandsketchesoftenhelptoclarifyproblems.Thestudent is recommended to illustrate his navigational and astronomical problems by freehand sketches, and to continue to do so until he is confident of solving his problems without the assistance of diagrams.

CHAPTER 8

THE TRIGONOMETRICAL SOLUTIONS OF SPHERICAL TRIANGLES

1. The Spherical Sine Formula

In any spherical triangle ABC:

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$$

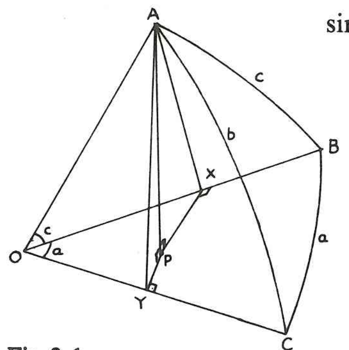


Fig. 8·1

Let ABC in fig. 8·1 be any spherical triangle on a sphere whose centre is at O. Drop a perpendicular from A onto the plane BOC at P. Drop a perpendicular from P onto the radii OC and OB at Y and X respectively. Join A to Y and A to X.

Because AY and AX are in the planes of the arcs AC and AB respectively, therefore:

Plane angle AYP = Spherical angle ACB
Plane angle AXP = Spherical angle ABC

Therefore:

$$\sin b/\sin B = (AY/AO)/(AP/AY) = (AY.AX)/(AO.AP) \quad \ldots\ldots \quad \text{(I)}$$
$$\sin c/\sin C = (AX/AO)/(AP/AY) = (AX.AY)/(AO.AP) \quad \ldots\ldots \quad \text{(II)}$$

From (I) and (II):

$$\sin b/\sin B = \sin c/\sin C$$

By dropping a perpendicular from B or C onto the opposite plane, it may similarly be shown that:

$$\sin b/\sin B = \sin a/\sin A$$

Therefore:

$$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C$$

Example 8·1—In the spherical triangle PZX, $P = 30°\ 00'$, $PX = 100°\ 00'$, and $ZX = 40°\ 00'$. Find Z.

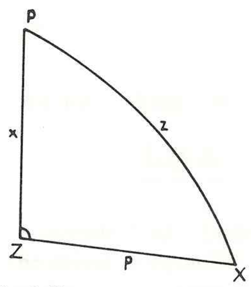


Fig. 8·2

In fig. 8·2:

$$\sin Z/\sin z = \sin P/\sin p$$
$$\sin Z = \sin P \sin z \operatorname{cosec} p$$
$$\log \sin P = \bar{1}{\cdot}69897$$
$$\log \sin z = \bar{1}{\cdot}99335$$
$$\log \operatorname{cosec} p = 0{\cdot}19193$$
$$\log \sin Z = \bar{1}{\cdot}88425 \qquad Z = 180° - 50°00' = 130°00'$$

Answer—Z = 130° 00′

It will be remembered that the sine of an angle between 90° and 180° has a positive value. Care, therefore, must be taken when using the Spherical Sine Formula to ensure that the correct quadrant is designated for the computed angle. In other words, it must not be thought that the computed angle is always less than 90°.

2. The Spherical Cosine Formula

In any spherical triangle ABC:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or:

$$\cos a = \cos A \sin b \sin c + \cos b \cos c$$

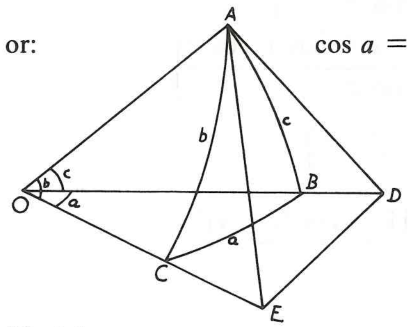


Fig. 8·3

Let ABC in fig. 8·3 be any spherical triangle on the sphere whose centre lies at O. At A draw tangents to the arcs AB and AC. These tangents lie in the planes of their respective arcs. Thus, the first must meet OB produced at D, and the second must meet OC produced at E. Join D to E.

Because AD and AE are tangents, the plane angle DAE is equal to the spherical angle BAC, and the angles OAE and OAD are right angles.

By the Plane Cosine Formula:

$$DE^2 = OD^2 + OE^2 - 2\,OD\,OE \cos a \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$
$$DE^2 = AD^2 + AE^2 - 2\,AD\,AE \cos A \quad \ldots\ldots\ldots\ldots \quad \text{(II)}$$

Subtract (II) from (I):

$$0 = (OD^2 + OE^2 - 2\,OD\,OE \cos a - AD^2 - AE^2 + 2\,AD\,AE \cos A$$
$$= (OD^2 - AD^2) + (OE^2 - AE^2) - 2\,OD\,OE \cos a + 2\,AD\,AE \cos A$$
$$= 2\,OA^2 - 2\,OD\,OE \cos a + 2\,AD\,AE \cos A$$

Therefore:

$$\cos A = \frac{OD\ OE \cos a - OA^2}{AD\ AE}$$

By dividing throughout by $OD\ OE$, we get:

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or:

$$\cos a = \cos A \sin b \sin c + \cos b \cos c$$

The Spherical Cosine Formula suffers from two disadvantages:

1. It is not convenient for logarithmic computations.
2. The cosines of angles in the second quadrant are negative so that great care must be taken in handling signs.

A formula similar to the Spherical Cosine Formula, but which does not suffer from the disadvantages of the Spherical Cosine Formula, is the Spherical Haversine Formula, which is easily derived from the former.

3. The Spherical Haversine Formula

In any spherical triangle ZYX:

$$\text{hav } X = \frac{\text{hav } x - \text{hav } (y \sim z)}{\sin y \sin z}$$

or:

$$\text{hav } x = \text{hav } X \sin y \sin z + \text{hav } (y \sim z)$$

Proof:

$$\text{hav } X = \tfrac{1}{2}(1 - \cos X)$$

$$= \tfrac{1}{2}\left[1 - \left(\frac{\cos x - \cos y \cos z}{\sin y \sin z}\right)\right]$$

$$= \tfrac{1}{2}\left[\frac{\sin y \sin z - \cos x + \cos y \cos z}{\sin y \sin z}\right]$$

$$= \tfrac{1}{2}\left[\frac{\cos x + \cos (z \sim y)}{\sin y \sin z}\right]$$

$$= \frac{\tfrac{1}{2}[1 - \cos x] - \tfrac{1}{2}[1 - \cos (z \sim y)]}{\sin y \sin z}$$

That is:

$$\text{hav } X = \frac{\text{hav } x - \text{hav } (z \sim y)}{\sin y \sin z}$$

or:

$$\text{hav } x = \text{hav } X \sin y \sin z + \text{hav } (z \sim y)$$

By doubling each side we have the Spherical Versine Formula, viz:

$$\text{vers } x = \text{vers } X \sin y \sin z + \text{vers } (z \sim y)$$

The principal advantage of the haversine or versine formula is that all the trigonometrical functions used for solving triangles by its means are positive.

Example 8·2—In the spherical triangle ABC, $a = 50°00'$, $b = 60°00'$, $c = 100°00'$. Find A.

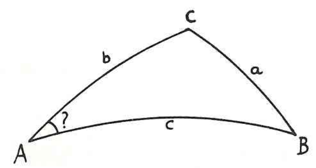


Fig. 8·4

Referring to fig. 8·4:

$$\begin{aligned}
a &= 50°00' \\
b &= 60°00' \\
c &= 100°00' \\
(c - b) &= 40°00' \\
\text{hav } A &= \text{hav } a - \text{hav } (c - b) \text{ cosec } b \text{ cosec } c \\
\text{nat hav } a &= 0{\cdot}17861 \\
\text{nat hav } c - b) &= 0{\cdot}11698 \\
\text{nat hav } \theta &= 0{\cdot}06163 \\
\text{log hav } \theta &= \bar{2}{\cdot}78978 \\
\text{log cosec } b &= 0{\cdot}06247 \\
\text{log cosec } c &= 0{\cdot}00665 \\
\text{log hav } A &= \bar{2}{\cdot}85890 \qquad A = 31°\ 11'
\end{aligned}$$

Answer—$A = 31°\ 11'$.

Example 8·3—In the spherical triangle XYZ, $X = 40°00'$, $z = 30°00'$, $y = 80°00'$. Find x.

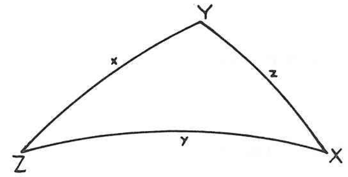


Fig. 8·5

Referring to fig. 8·5:

$$\begin{aligned}
X &= 40°00' \\
y &= 80°00' \\
z &= 30°00' \\
(y \sim z) &= 50°00' \\
\text{hav } x &= \text{hav } X \sin y \sin z + \text{hav } (y \sim z) \\
\text{log hav } X &= \bar{1}{\cdot}06810 \\
\text{log sin } y &= \bar{1}{\cdot}99335 \\
\text{log sin } z &= \bar{1}{\cdot}69897 \\
\text{log hav } \theta &= \bar{2}{\cdot}76042 \\
\text{nat hav } \theta &= 0{\cdot}05760 \\
\text{nat hav } (y \sim z) &= 0{\cdot}17861 \\
\text{nat hav } x &= 0{\cdot}23621 \qquad x = 58°09'
\end{aligned}$$

Answer—$x = 58°09'$.

4. The Four Parts Formula

In any spherical triangle if three of any four adjacent parts are known the unknown of the four parts may be found direct by means of the Four Parts Formula.

Referring to the spherical triangle ABC in fig. 8·6:

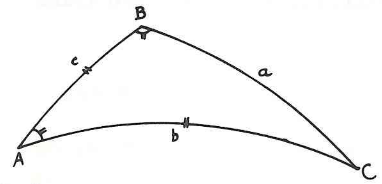


Fig. 8·6

B is the Outer Angle (O.A.)
c is the Inner Side (I.S.)
A is the Inner Angle (I.A.)
b is the Outer Side (O.S.)

The Four Parts Formula relating to these parts is:

$$\cos c \text{ (I.S.)} \cos A \text{ (I.A.)} = \sin c \text{ (I.S.)} \cot b \text{ (O.S.)} - \sin A \text{ (I.A.)} \cot B \text{ (O.A.)}$$

Proof:
By the Spherical Cosine Formula:

$$\cos b = \cos B \sin c \sin a + \cos c \cos a \quad \text{(I)}$$

$$\cos a = \cos A \sin b \sin c + \cos b \cos c \quad \text{(II)}$$

By the Spherical Sine Formula:

$$\sin a = (\sin A \sin b)/\sin B \quad \text{(III)}$$

Substitute (II) for cos a in (I), and (III) for sin a in (I).

Thus:

$$\cos b = \cos B \sin c\,[(\sin A \sin b)/\sin B] + \cos c\,(\cos A \sin b \sin c + \cos b \cos c)$$

That is:

$$\cos b = \cot B \sin c \sin A \sin b + \cos c \cos A \sin b \sin c + \cos b \cos^2 c$$

From which:

$$\cos b - \cos b \cos^2 c = \cot B \sin c \sin A \sin b + \cos c \cos A \sin b \sin c$$

That is:

$$\cos b\,(1 - \cos^2 c) = \sin b \sin c\,(\sin A \cot B + \cos c \cos A)$$

$$\frac{\cos b \sin^2 c}{\sin b \sin c} = \sin A \cot B + \cos c \cos A$$

That is:

$$\cot b \sin c = \sin A \cot B + \cos c \cos A$$

or:

$$\cos c \cos A = \sin c \cot b - \sin A \cot B$$

Exercises on Chapter 8

1. In the spherical triangle ABC: $B = 75°00'$, $C = 55°00'$, $Ac = 67°00'$. Find AB.
2. In the spherical triangle XYZ: $XZ = 105°00'$, $XY = 95°00'$, $XZ = 54°00'$. Find X.
3. In the spherical triangle PQR: $PQ = 65°10'$, $PR = 106°23'$, $Q = 43°10'$. Find RQ
4. In the spherical triangle DEF: $DE = 30°00'$, $DF = 60°00'$, $EF = 50°00'$. Find E.
5. In the spherical triangle ABC: $A = 35°00'$, $AB = 65°00'$, $B = 54°00'$. Find BC.
6. In the spherical triangle PQR: $PQ = 100°00'$, $RQ = 54°00'$, $Q = 67°00'$. Find P.
7. In the spherical triangle XYZ: $XY = 65°00'$, $ZX = 78°00'$, $Z = 34°00'$. Find X.
8. In the spherical triangle PZX: $P = 03h.24m.$, $PZ = 54°55'$, $PX = 87°10'$. Find Z and ZX.

CHAPTER 9

NAPIER'S RULES

1. Napier's Rules for Solving Right-angled Spherical Triangles

Any spherical triangle, right-angled or otherwise, may be solved using one or more of the formulae described in the preceding chapter. If, however, a spherical triangle contains a right angle, a shorter and simpler solution than that in which a formula for oblique-angled triangles is used, is made possible by Napier's Rules.

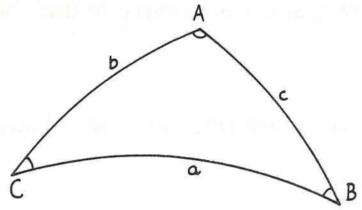


Fig. 9·1

In the spherical triangle illustrated in fig. 9·1 suppose that A, a and c are known, and that it is required to find C.

By the Spherical Sine Formula:

$$\sin C = \sin c \sin A \operatorname{cosec} A \quad \text{(I)}$$

Suppose that A, b and c in triangle ABC are known, and that it is required to find a:

By the Spherical Cosine Formula:

$$\cos a = \cos A \sin b \sin c + \cos b \cos c \quad \text{(II)}$$

Suppose that A, B and c are known and that it is required to find a:

By the Four Parts Formula:

$$\cos c \cos B = \sin c \cot a - \sin B \cot A$$

or

$$\cot a = (\cos c \cos B + \sin B \cot A)/\sin c \quad \text{(III)}$$

Now suppose that the angle A in the triangle ABC is 90°. Then, because $\cos 90° = 0$ and $\sin 90° = 1$, the three formulae (I), (II) and (III) reduce, respectively, to:

$$\sin C = \sin c \operatorname{cosec} a \quad \text{(IV)}$$

$$\cos a = \cos b \cos c \quad \text{(V)}$$

$$\cot a = \cot c \cos B \quad \text{(VI)}$$

It is possible to derive ten simple formulae which, collectively, provide the means for solving every possible case of a right-angled spherical triangle. Instead of deducing from these ten formulae so many distinct rules for the solution of the various cases, the whole, by the assistance of an ingenious contrivance, may be comprehended in two remarkably simple rules. These rules, named after their illustrious inventor, are known as Napier's Rules For Circular Parts.

The parts of a given right-angled spherical triangle, not including the right-angle, are written in order—either clockwise or anti-clockwise—in the five sectors of a cartwheel as illustrated in fig. 9·2.

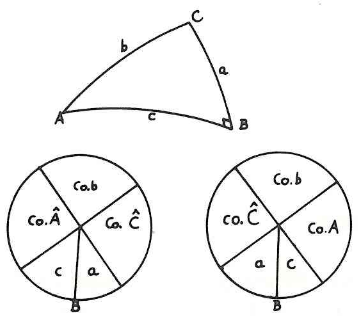


Fig. 9·2

Referring to fig 9·2 it will be noticed that the two angles and the side opposite to the right angle are prefixed with the letters "co" which denotes complement.

Of the three parts in the cartwheel one must be a Middle part, and the other two must be either Opposite or Adjacent parts.

Napiers Rules are:

sine middle part = product of the cosines of the opposites
sine middle part = product of the tangents of the adjacents

If, in the triangle ABC in fig. 9·2, the sides b and c are known, and it is required to find the remaining three unknown parts, the rules are:

(*a*) *To find C*

Of the three parts C, b and c, c is the Middle part, and C and b are Opposite parts. Thus:

$$\sin c = \sin \text{co-}C \cos \text{co } b$$

That is: $\sin c = \sin C \sin b$

and: $\sin C = \sin c \operatorname{cosec} b$ . . . . (I)

(*b*) *To find a*

Of the three parts a, b and c, b is the Middle part and a and c are Opposite parts. Thus:

$$\sin \text{co-}b = \cos a \cos c$$

That is: $\cos b = \cos a \cos c$

and: $\cos a = \cos b \sec c$ . . . . (II)

(*c*) *To find A*

Of the three parts A, b and c, A is the Middle part and b and c are Adjacent parts.

$$\sin \text{co-}A = \tan c \tan \text{co } b$$

That is: $\cos A = \tan c \cot b$ . . . . (III)

It is imprudent, in circumstances when it can be avoided, to solve a right-angled triangle using a part which has previously been calculated, and which may, therefore, be in error. Any error in such a part used to solve another part will cause unnecessary error in that part. When solving the three unknown parts of a right-angled spherical triangle it is advisable, therefore, to derive the three formulae before commencing the calculations. By so doing, not only is the time spent in entering tables reduced, but the possibility of blundering in the calculation is also reduced.

Before commencing to solve a right-angled spherical triangle it is advisable to ascertain whether or not the value of any unknown part is greater or less than 90°. This is easily done by means of the device now to be described.

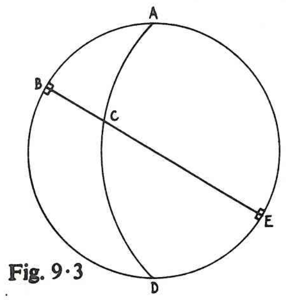


Fig. 9·3

Fig. 9·3 illustrates each of the four cases of right-angled spherical triangles.

In the triangle *ABC ALL* parts, except the right angle, are *LESS* than 90°.

In the triangle *BCD* only *D* and *BC* are less than 90°
In the triangle *CDE* only *C* and *DE* are less than 90°
In the triangle *ACE* only *Ac* is less than 90°.

By constructing such a simple figure the relative values of the unknown parts are readily seen.

Example 9·1—In the spherical triangle *PQR*: $P = 45°$, $r = 60°$, $Q = 90°$. Find the remaining parts.

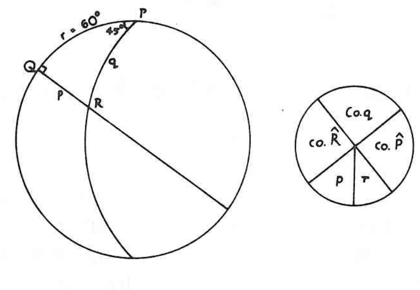


Fig. 9·4

Referring to fig. 9·4:

All parts are *LESS* than 90°

To find p:

$$\sin r = \tan p \tan \text{co-}P$$
$$\sin r = \tan p \cot P$$
$$\tan p = \sin r \tan P$$

$r = 60°$ log sin	$= \bar{1}{\cdot}93753$
$P = 45°$ log tan	$= 0{\cdot}00000$
log tan	$= \bar{1}{\cdot}93753$
p	$= 40° \ 54'$

To find R:

$$\sin \text{co-}R = \cos r \cos \text{co-}P$$
$$\cos R = \cos r \sin P$$
$$\cos R = \cos r \sin P$$

log cos	$= \bar{1}{\cdot}69897$
log sin	$= \bar{1}{\cdot}84949$
log cos	$= \bar{1}{\cdot}54846$
R	$= 69° \ 17'$

To find q:

$$\begin{aligned}
\sin \text{co-}P &= \tan r \tan \text{co-}q \\
\cos P &= \tan r \cot q \\
\cot q &= \cot r \cos P \\
\log \cot &= \bar{1}{\cdot}76144 \\
\log \cos &= \bar{1}{\cdot}84949 \\
\log \cot &= \bar{1}{\cdot}61093 \\
q &= 67^\circ\, 47'
\end{aligned}$$

Answer—$p = 40^\circ\, 54'$, $R = 69^\circ\, 17'$, $q = 67^\circ\, 47'$.

Example 9·2—In the spherical triangle PQR: $P = 90^\circ$, $R = 45^\circ$, $p = 110^\circ$. Solve the triangle.

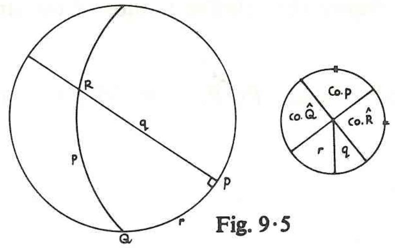


Fig. 9·5

Referring to fig. 9·5:

q and Q are more than 90° and r is less than 90°

To find q:

$$\begin{aligned}
\sin \text{co-}R &= \tan q \tan \text{co-}p \\
\cos R &= \tan q \cot p \\
\tan q &= \tan p \cos R \\
p = 110^\circ \quad \log \tan &= 0{\cdot}43893(-) \\
R = 45^\circ \quad \log \cos &= \bar{1}{\cdot}84949(+) \\
\log \tan q &= 0{\cdot}28842(-) \\
q &= 117^\circ\, 14'
\end{aligned}$$

To find r:

$$\begin{aligned}
\sin r &= \cos \text{co-}p \cos \text{co-}R \\
\sin r &= \sin p \sin R \\
\sin r &= \sin p \sin R \\
\log \sin &= \bar{1}{\cdot}97299(+) \\
\log \sin &= \bar{1}{\cdot}84949(+) \\
\log \sin r &= \bar{1}{\cdot}82248\ (+) \\
r &= 41^\circ\, 38'
\end{aligned}$$

To find Q:

$$\begin{aligned}
\sin \text{co-}P &= \tan \text{co-}Q \tan \text{co-}R \\
\cos P &= \cot Q \cot R \\
\cot Q &= \cos p \tan R \\
\log \cos &= \bar{1}{\cdot}53405(-) \\
\log \tan &= 0{\cdot}00000(+) \\
\log \cot &= \bar{1}{\cdot}53405(-) \\
Q &= 108^\circ\, 53'
\end{aligned}$$

Answer—$q = 117^\circ\, 14'$, $r = 41^\circ\, 38'$, $Q = 108^\circ\, 53'$.

2. Napier's Rules for Solving Quadrantal Spherical Triangles

A Quadrantal Spherical Triangle is one in which one of the sides has a value of 90°. Quadrantal triangles may be solved by a modification of Napier's Rules for Right-angled Triangles. The modifying rule is:

> "In a quadrantal triangle, if both Adjacents or both Opposites are both sides or both angles, change the final sign".

This modifying rule is derived from the fact that a quadrantal triangle may be solved by first solving a "related" right-angled spherical triangle. The following example serves to show this.

Example 9·3—In the quadrantal triangle ABC: $BC = 90^\circ 00'$, $BAC = 60^\circ 00'$, $ABC = 30^\circ 00'$. Find ACB.

Referring to fig. 9·6:

To find C use trhe right-angled spherical triangle ABD in which:

$$\begin{aligned}
D &= 90^\circ \\
BD &= 90^\circ \\
DAB &= \text{supplement of } BAC = 120^\circ \\
DBC &= \text{complement of } ABC = 60^\circ
\end{aligned}$$

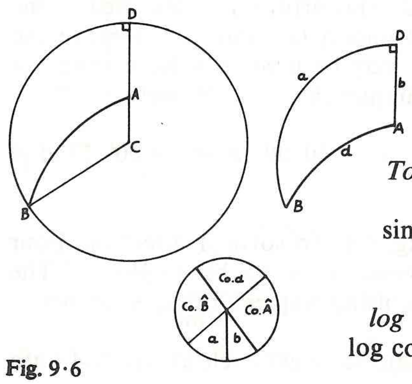


Fig. 9·6

To find a:

$$\begin{aligned}
\sin \text{co-}A &= \cos \text{co-}B \cos a \\
\cos A &= \sin B \cos a \\
\cos a &= \cos A \operatorname{cosec} B \\
\log \cos A &= \bar{1}{\cdot}69897(-) \\
\log \operatorname{cosec} B &= 0{\cdot}06247(+) \\
\log \cos a &= \bar{1}{\cdot}76144(-) \qquad a = 125^\circ\, 16'
\end{aligned}$$

Answer—$a = ACB = 125^\circ\, 16'$.

To solve a quadrantal triangle using the modifying rule, the procedure is as follows:

The parts of the quadrantal triangle are written in order in the sectors of the cartwheel as in fig. 9·7.

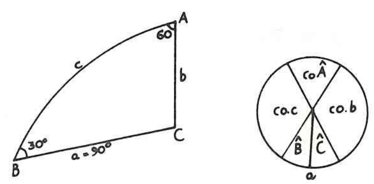


Fig. 9·7

Notice that in fig. 9·7 the angle opposite to the 90°-side and the other two sides are prefixed with the "co" to denote complement.

To find C:

$$\sin \text{co-}A = \cos B \cos C$$

Note that Both the Opposites (*B* and *C*) are angles in this case. Therefore the final sign will have to be changed as shown below.

$$\cos A = \cos B \cos C$$
$$\cos C = \cos A \sec B$$
$$\log \cos A = \bar{1}{\cdot}69897(+)$$
$$\log \sec B = 0{\cdot}06247(+)$$
$$\log \cos C = \bar{1}{\cdot}76144(+) \quad \text{This becomes } (-)$$

Therefore:

$$C = 180° - 54° 44'$$
$$= 125° 16'$$

Answer—C = 125° 16′.

3. The Solution of Oblique Spherical Triangles by Napier's Rules

Any oblique spherical triangle may be solved by Napier's Rules simply by dividing the triangle into two right-angled spherical triangles by dropping a perpendicular great circle arc from any vertex onto the opposite side or side produced. This artifice is often used in the construction of Short Method Tables used in nautical astronomy (see Part 5). Moreover, in many cases, particularly when the Four Parts Formula may be used to solve a spherical triangle, the solution by Napier's Rules is considerably simpler than the alternative.

Example 9·4—In the spherical triangle *ABC*, $B = 30°00'$; $c = 60°00'$, $b = 70°00'$. Find *A* using Napier's Rules.

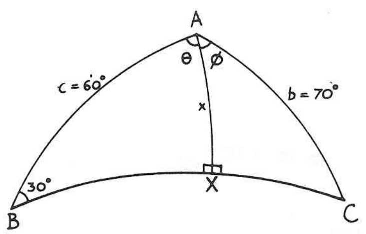


Fig. 9·8

Referring to fig. 9·8: To solve *A* direct the Four Parts Formula would have to be employed. The following solution, using Napier's rules, is simpler.

Drop a perpendicular great circle arc from *A* onto the side *BC* at *X*.

From the cartwheel illustrated in fig. 9·9 (*a*):

$$\sin \text{co-}c = \tan \text{co-}B \tan \text{co-}\theta$$
$$\cot \theta = \cos c \tan B$$

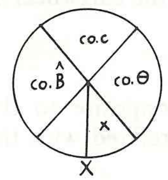


Fig. 9·9 (*a*)

$$\log \cos c = \bar{1}{\cdot}69897$$
$$\log \tan B = \bar{1}{\cdot}76144$$
$$\log \cot \theta = \bar{1}{\cdot}46041 \qquad \theta = 73° 54'$$

$$\sin x = \sin B \sin c$$
$$\log \sin B = \bar{1}{\cdot}69897$$
$$\log \sin c = \bar{1}{\cdot}93753$$
$$\log \sin x = \bar{1}{\cdot}63650 \qquad x = 25° 39'$$

From the cartwheel illustrated in fig. 9·9(*b*):

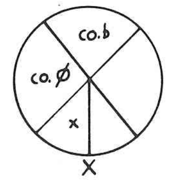


Fig. 9·9 (*b*)

$$\sin \text{co-}\varphi = \tan \text{co-}b \tan x$$
$$\cos \varphi = \cot b \tan x$$
$$\log \cot b = \bar{1}{\cdot}56107$$
$$\log \tan x = \bar{1}{\cdot}68158$$
$$\log \cos \varphi = \bar{1}{\cdot}24265 \qquad \varphi = 79° 56'$$

$$A = \theta + \varphi$$
$$= 73° 54' + 79° 56'$$
$$= 153° 50'$$

Answer—A = 153° 50′

Exercises on Chapter 9

1. In the spherical triangle *ABC*: $A = 40°00'$, $B = 90°00'$, $C = 65°00'$. Find the three sides.
2. In the spherical triangle *EFG*: $F = 90°00'$, $E = 28°45'$, $EF = 75°15'$. Find the unknown parts.
3. In the spherical triangle *XYZ*: $Z = 90°00'$, $ZY = 40°30'$, $Y = 102°30'$. Find the unknown parts.
4. In the spherical triangle *PQR*: $R = 90°00'$, $P = 115°35'$, $PQ = 98°40'$. Solve the triangle.
5. In the spherical triangle *ZYX*: $Z = 90°00'$, $X = 140°00'$, $ZX = 123°00'$. Solve the triangle.
6. In the spherical triangle *ABC*: $AB = 90°00'$, $C = 65°00'$, $B = 150°00'$. Solve the triangle.
7. In the spherical triangle *XYZ*: $XY = 90°00'$, $YZ = 55°00'$, $ZX = 70°00'$. Find the unknown parts.
8. In the spherical triangle *PZX*: $PX = 50°00'$, $ZX = 30°00'$, $P = 35°00'$. Explain why two values may be assigned to angle *Z*. Compute these values.
9. In the spherical triangle *ABC*: $AB = 75°00'$, $BC = 60°00'$, $A = 50°00'$. Find *B* and *C*.
10. In the spherical triangle *XYZ*: $XZ = 100°00'$, $X = 30°00'$, $Z = 40°00'$. Find *YZ*.

PART 2

THE SAILINGS

The "Sailings" embrace the several methods used to find the course to steer and the distance to travel in going from one position on the Earth's surface to another. The fact that vessels travel over a spherical surface made for difficulties in connection with sailing problems which were not overcome until the advent of the Mercator Chart in the sixteenth century. In this Part we shall first consider the shape and size of the Earth; the methods of defining position on the earth's surface; the nature of the tracks traced out by vessels moving over the sea; and the principles of the Mercator Chart. Following this a discussion on the several methods of computing courses and distances will be presented.

CHAPTER 10

THE SHAPE AND SIZE OF THE EARTH

1. The Earth

The Earth's shape is not quite spherical. In many navigational problems, however, the Earth is considered to be a perfect sphere—an assumption which leads to no appreciable error.

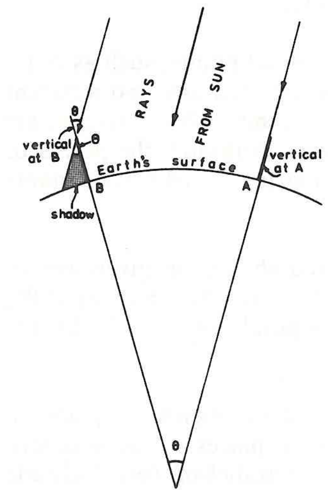


Fig. 10·1

It is believed that the notable Pythagoras, of right-angled triangle fame, taught that the Earth "is a ball suspended in space". It was not, however, until about three centuries after the time of Pythagoras, that the first recorded attempt at measuring the Earth was made by the Greek philosopher Eratosthenes. Eratosthenes noticed that at noon on the longest day of the year, at Syene in the upper Nile valley, the buidings cast no shadows. At Alexandria, situated to the north of Syene, the buildings did cast shadows at noon on the longest day of the year. Eratosthenes accounted for this by arguing that the Earth must be spherical, and that the parallel rays of the very remote Sun cast shadows of different lengths at the two places.

Fig. 10·1 illustrates the methods used by Eratosthenes for determining the circumference of the Earth. By finding the angle θ by measurement, and estimating the distance between *A* and *B*, the earth's circumference may be deduced from the relationship:

$$\text{Arc } AB : \theta^\circ :: \text{Circumference} : 360^\circ$$

From which:

$$\text{Circumference} = \frac{360 \, . \, AB}{\theta}$$

The Earth rotates about a fixed diameter known as the Earth's polar axis. The rate of the Earth's rotation is relatively slow: it spins once in twenty four hours.

The direction towards which points on the earth's surface are carried around the Earth's polar axis is known as East. The direction opposite to East is called West.

The extremities of the Earth's axis are known as the Earth's Poles. These two points are the poles of a great circle which lies in the plane of the Earth's rotation. This great circle is known as the Equator. i.e. The Equator is the great circle on the surface of the earth so that all points on it 90° away from each pole.

The Earth's pole at which the Earth's rotation is anticlockwise when viewed from above it is known as the North Pole. The other is called the South Pole.

An observer facing the direction of the North Pole from any point on the Earth's surface, would be looking in a direction which is 90° to the left of East. This direction is called North. The direction opposite to North is called South.

The four directions North, South, East and West, which are abbreviated to N., S., E. and W., respectively, are known as the Cardinal Points of the Compass. All other horizontal directions may be referred to the two adjacent cardinal points. Thus we may signify that a lighthouse bears N. 28° E., meaning that the horizontal angle between the direction of North and that of the lighthouse is 28°.

The complete "compass", that is to say, the horizontal circle through the cardinal points, is divided into 32 points. The term Point of the Compass sometimes refers to a direction and sometimes to the 32nd part of the compass, which is an arc of 11¼°.

The directions which lie midway between any two adjacent cardinal points, such as N.E., S.W., are called Half Cardinal Points. Those which lie midway between any two adjacent cardinal and half cardinal points, such as E.N.E., W.S.W., N.N.W., are called Intermediate or Three-Letter Points. The remaining points are called By-Points. Although the points of the compass are not used so extensively as in days gone by, every mariner worthy of the name should be able to Box the Compass.

The quadrantal system of denoting horizontal directions, noted above, has given way to the superior Three Figure Notation, in which North is referred to as 000°, East as 090°, South as 180°, West as 270°, and so on to 359° which corresponds to N.1°W. by the quadrantal notation.

The equator divides the Earth into the Northern and Southern Hemispheres. All places in the northern hemisphere are said to have North Latitude, and all places in the southern hemisphere, South Latitude. The equator may be defined as the Parallel of Zero Latitude because every point on the equator has a Latitude of 00° 00′ 00″.

Small circles on the surface of the Earth, which are parallel to the equator, are known as Parallels of Latitude. All points on a particular parallel of Latitude have the same latitude. The Latitude of a place on the Earth's surface, assuming the Earth to be perfectly spherical, is the angle at the Earth's centre, measured in the plane of a secondary to the equator, from the plane of the equator to the place.

Secondary great circles to the equator are called Meridians. Thus, the Latitude of a place is defined as the arc of a meridian intercepted between the equator and the place. Strictly speaking, meridians are semi-great circles which terminate at the Earth's Poles. In other words a secondary to the equator forms two antipodal meridians.

2. Describing a Terrestrial Position

Navigators employ one of two general methods of describing a position on the Earth's surface. The more common method is to state the parallel of Latitude and the meridian on

which the position to be described rests. The parallel of Latitude is denoted by stating the Latitude of the place, and the meridian is denoted by stating an angle called Longitude. Whereas the datum parallel from which Latitude is measured is the equator, the datum meridian from which Longitude is measured is the meridian of Greenwich. This meridian, the Prime Meridian, is generally called the Greenwich Meridian.

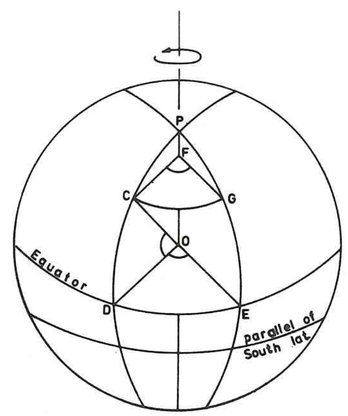


Fig. 10·2

In fig. 10·2, the angle *DOC* is the Latitude of the point *C* (and of every other point on *C*'s parallel of Latitude). If the meridian on which the point *G* lies is the Greenwich Meridian, then the West Longitude of *C* is given by the angle *EOD*.

But: Angle *EOD* = Angle *GFC*
= Angle *EPD*
= Arc *ED*

Notice in fig. 10·2 that the Latitude of the pole is 90°.

The Longitude of a place is the smaller angle at either pole, or the lesser arc of the equator, contained between the Greenwich Meridian and the meridian on which the place lies. Every point on the Greenwich Meridian has a Longitude of 00° 00′ 00″.

The Greenwich and the Antipodal, or 180th meridian, divides the Earth into the Eastern and Western Hemispheres. All places which lie to the east of the Greenwich Meridian and to the west of the 180th meridian, are said to have East Longitude. All places which lie to the west of the Greenwich Meridian and to the east of the 180th meridian have West Longitude.

By examining a world map it may be verified that Cardiff is in Latitude 51°30′N. Longitude 03°10′W., and that Capetown is in Latitude 34°00′S. Longitude 18°30′E.

The Difference of Latitude—abbreviated to D. Lat.—between two places is the arc of any meridian contained between the parallels of Latitude of the two places. If the two places have Latitudes of the same name, the D. Lat. is found by subtracting the smaller from the greater Latitude. If the two places have Latitudes of different names, the D. Lat. is found by adding the two Latitudes. D. Lat. is sometimes named North or South, according as the ship is moving northerly or southerly, respectively.

The Difference of Longitude—abbreviated to D. Long.—between two places is the smaller angle at either pole, or the lesser arc of the equator, contained between the meridians of the two places. If the two places have Longitudes of the same name the D. Long. is found by subtracting the smaller from the greater Longitude. When the two places have Longitudes of different names, and the Greenwich Meridian lies within the arc of D. Long., the D. Long. is found by adding the Longitudes. When, however, the 180th meridian lies within the arc of D. Long. between the two places, the D. Long. is found by adding the longitudes and subtracting the sum from 360°00′00″. D. Long. is named East or West, according as the ship moves easterly or westerly, respectively.

Example 10·1—Find the D. Lat. and D. Long. between the following pairs of positions:

(*a*)	From	Lat. 20° 38′ N.	Long. 96° 54′ W.
	To	Lat. 15° 22′ N.	Long. 35° 34′ W.
(*b*)	From	Lat. 15° 10′ S.	Long. 36° 06′ E.
	To	Lat. 06° 08′ N.	Long. 06° 55′ E.
(*c*)	From	Lat. 20° 33′ S.	Long. 04° 00′ W.
	To	Lat. 15° 36′ S.	Long. 05° 38′ E.
(*d*)	From	Lat. 54° 45′ N.	Long. 176° 25′ E.
	To	Lat. 00° 55′ S.	Long. 164° 52′ W.

(*a*)	From	Lat. 20° 38′ N.	Long. 96° 54′ W.
	To	Lat. 15° 22′ N.	Long. 35° 34′ W.
		D. Lat. 05° 16′ S.	D. Long. 61° 20′ E.
(*b*)	From	Lat. 15° 10′ S.	Long. 36° 06′ E.
	To	Lat. 06° 08′ N.	Long. 06° 55′ E.
		D. Lat. 21° 18′ N.	D. Long. 29° 11′ W.
(*c*)	From	Lat. 20° 33′ S.	Long. 04° 00′ W.
	To	Lat. 15° 36′ S.	Long. 05° 38′ E.
		D. Lat. 04° 57′ N.	D. Long. 09° 38′ E.
(*d*)	From	Lat. 54° 45′ N.	Long. 176° 25′ E.
	To	Lat. 00° 55′ S.	Long. 164° 52′ W.
		D. Lat. 55° 40′ S.	D. Long. 18° 43′ E.

In the alternative method of describing a terrestrial position the direction of the position and its distance from some known reference point are stated. The reference point is usually a prominent headland, a lighthouse, or an important landmark. The direction is given by stating a Bearing. The Bearing of an object indicates its compass direction. Thus we may say that a ship is in a position with Cape Hatteras bearing 265° at a distance of 16 miles. This means that the ship lies 085°—which is the opposite direction to the bearing of the Cape—16 miles from Cape Hatteras.

3. The True Shape of the Earth

Thus far the shape of the Earth has been considered to be a perfect sphere. For certain problems in navigation, notably in connection with the mariner's chart and the nautical unit of distance, it is necessary for us to consider the Earth's true shape. The actual shape of the Earth is that of an oblate Spheroid of Revolution. An oblate spheroid is the shape that would be swept out by rotating an ellipse about its minor diameter.

The Ellipticity of the terrestrial spheroid, that is to say, the ratio between the difference of the lengths of the equatorial and polar radii, and the length of the equatorial radius, is approximately 1/300. This very small fraction indicates that the Earth is almost a perfect sphere. The Earth's principal radii are:

$$\text{Equatorial radius} = 6{,}378{,}249 \text{ metres}$$
$$\text{Polar radius} = 6{,}356{,}515 \text{ metres}$$

The true shape of the Earth affects our earlier definition of Latitude, so that it is necessary to examine this closely.

The Vertical at any place is the direction perpendicular to the horizontal plane which touches the Earth's surface at the place. The angle contained between the vertical at a place and the plane of the equator is known as the Geographical Latitude of the place. It is the Geographical Latitude that is measured in astronomical observations for Latitude. For this reason it is often called True or Astronomical Latitude. When the term Latitude is used without qualification, it is understood to mean Geographical Latitude.

The angle at the Earth's centre contained between the equator and any place on the Earth's surface and measured in the plane of the meridian of a place, is called the Geocentric Latitude of the place. Except for places on the equator or at either pole, the Geocentric Latitude of a place is always smaller numerically than the Geographical Latitude of the place. For this reason Geocentric Latitude is sometimes called Reduced Latitude.

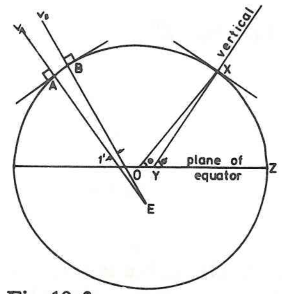


Fig. 10·3

In fig. 10·3:

$$\text{Geographical Latitude of } X = ZYX = \varphi$$
$$\text{Geocentric Latitude of } X = ZOX = \theta$$

The maximum difference between the Geographical Latitude of a place and its Geocentric Latitude occurs when the latitude of the place is 45°. The Geographical Latitude and Geocentric Latitude of any point on the equator is 00° 00′. The Geographical Latitude and the Geocentric Latitude of either pole is 90° 00′.

The difference between the lengths of the equatorial and polar radii is 21,734 metres. This is equivalent to 11·6 nautical miles.

4. The Nautical Mile

The important feature of the navigational unit of distance called the Nautical Mile is that it is related to a meridian spherical distance of one minute of arc. A nautical mile is the length of an arc of a meridian the Geographical Latitudes of the end points of which differ by 1′ of arc. In fig. 10·3, if the angle between *BE* and *AE*, which are the verticals at *B* and *A*, respectively, is exactly 1′, the arc-length *AB* is one nautical mile. It is for this reason that a nautical mile is sometimes defined as the length of an arc of a meridian between two points whose verticals are inclined to one another at an angle of 1′. Thus, if angle *AEB* is 1′, the arc *AB* is one nautical mile. The point *E* is at the centre of curvature of the small piece of the meridian contained between *A* and *B*. Because of the oblateness of the Earth, the radius of curvature of the meridian increases as the Latitude increases. As the radius of curvature increases, the arc-length corresponding to an angle of 1′ at the centre of curvature also increases. For this reason, the length of a nautical mile increases as the Latitude increases.

Length of nautical mile in Lat. 0° = 1842·787 metres or 6046 feet
Length of nautical mile in Lat. 90° = 1861·656 metres or 6108 feet

The average length of the nautical mile is 6077 feet, or 1852·221 metres. This corresponds to the length in Latuitude 45°. The figure 6077 is rounded off to 6080 and this latter figure is taken as the number of feet in the Standard Nautical Mile. The Standard Nautical Mile in metres is taken as 1852.

The length of the actual nautical mile in any Latitude φ is given by the formula:

$$\text{Length in Metres} = 1852 - 19 \cos 2\varphi$$
$$\text{or Length in Feet} = 6077 - 31 \cos 2\varphi$$

From this formula it may be verified that the standard nautical mile of 6080 feet or 1852 metres may be used without introducing error only in Latitude 49° approximately. In all other Latitudes, by using a distance-measuring instrument calibrated in standard nautical miles, an error proportional to the distance results. This error is, of course, greatest for any given distance when the latitude is zero or near 90°. The lengths of a minute of a meridian in latitudes 0° and 90° are, respectively, 0·995 and 1·005 nautical miles.

Example 10·2—Find the length of the nautical mile in metres in Lat. 60°00′.

$$\begin{aligned} \text{Length} &= 1852 - 19 \cos (2\times 60)^\circ \\ &= 1852 - 19 \cos 120^\circ \\ &= 1852 + 9{\cdot}5 \\ &= 1861{\cdot}5 \text{ metres} \end{aligned}$$

Answer—Length = 1861·5 metres.

A commonly used sub-multiple unit of the nautical mile is the cable. The cable is a tenth of a nautical mile. In practice it is usual to reckon a cable as being 600 feet or 200 yards. The nautical unit of speed is the Nautical Mile per Hour. This is called the Knot.

5. Reduction of the Geographical Latitude

We have noted above that the distance between the lengths of the equatorial and polar radii of the Earth is 11·6 nautical miles. The reduction R of the Geographical Latitude may, therefore, be found from the following formula:

$$R = 11{\cdot}6 \sin 2\varphi$$

where φ is Geographical Latitude.

Example 10·3—Find the Geocentric Latitude of a place whose Geographical Latitude is 30°00′.

$$\begin{aligned} \text{reduction} &= 11{\cdot}6 \,.\, \sin (2\times 30)^\circ \\ &= 11{\cdot}6 \,.\, \sin 60^\circ \\ &= 10{\cdot}0 \end{aligned}$$

$$\begin{aligned} \text{Geocentric Latitude} &= \text{Geographical Latitude} - \text{Reduction} \\ &= 30^\circ\, 00' - 10' \\ &= 29^\circ\, 50' \end{aligned}$$

Answer—Geocentric Latitude = 29° 50′.

6. The Geographical Mile

The length of a minute of arc of the equator is called a Geographical Mile. It is of interest to note that the equator is the only true great circle on the Earth. Meridians, because of the Earth's oblate shape, are semi-ellipses.

The geographical mile is 6087 feet or 1855·4 metres. This distance is used for computing the distance along a parallel of Latitude for purposes of surveying and large-scale mapping.

Exercises on Chapter 10

1. Define the Great Circle; Small Circle. Give examples of terrestrial great and small circles.
2. Describe the method used by Eratosthenes for measuring the circumference of the Earth.
3. State four proofs of the earth's rotundity.
4. Describe the true shape of the Earth. Explain why, in most navigational problems, the Earth may be assumed to be a perfect sphere.
5. What is a Meridian? In what direction would a ship be sailing were she steered along a meridian?
6. Define Bearing. Explain why the bearing of every point on Earth is 180° from the earth's North Pole.
7. Define Statute Mile; Geographical mile; Cable.
8. Explain clearly the meaning of Reduction of the Latitude.
9. Describe two systems of defining terrestrial positions.
10. Explain clearly the derivation of the Standard Nautical Mile.
11. Define Prime Meridian; Eastern Hemisphere; D. Lat.; D. Long.
12. What is the antipodal position of Lat. 20°S. Long. 15°W.?
13. A ship sailed due North for two days at 10 knots along the Prime Meridian. Find her final Latitude if her Departure position was in Lat. 54°00′S.
14. A ship sailed due West along the equator for 18 hours at 16 knots. Find her final position if her Departure position was in Long. 10°30′W.
15. What is the D. Lat. and D. Long. between the following pairs of positions:

(*a*)	From	Lat. 10° 43′ S.	Long. 05° 56′ W.
	To	Lat. 06° 34′ S.	Long. 18° 05′ E.
(*b*)	From	Lat. 34° 18′ N.	Long. 177° 08′ E.
	To	Lat. 22° 52′ N.	Long. 06° 18′ W.

16. What is meant by the term Ellipticity as it applies to the Earth?
17. Explain how the radius of curvature of the meridians changes with Latitude. How does this change affect the seaman's unit of distance?

18. Describe the error that results by using a patent log calibrated in Standard Nautical Miles when sailing in very low or very high Latitudes.
19. Define a Geocentric Latitude; Geographical Latitude. What is the Geocentric Latitude of a place whose True Latitude is 60°00′N.?
20. Calculate the length in metres of the nautical mile in Lat. 42°S.

CHAPTER 11

THE RHUMB LINE

1. Introduction

Ships are steered from place to place, when out of sight of land, by means of a magnetic or gyro compass which indicates a fixed horizontal direction irrespective of the movements of the ship. Compass Points are marked on the outer edge of the compass card; and radial lines, extending from the centre of the card to the several points, are known as Rhumbs. When a ship's head is steadied in a certain compass direction the fore-and-aft line of the ship lies in the vertical plane of the rhumb, and it is easy to visualise the path and track of the ship as extensions of the rhumb. For this reason a line of constant course is known as a Rhumb Line.

A rhumb line is usually defined as a line on the Earth's surface which cuts every meridian at the same constant angle. The most convenient path to travel along is a rhumb line path which connects the places of departure and destination. This is so because, in travelling along a rhumb line, the course of the vessel remains constant.

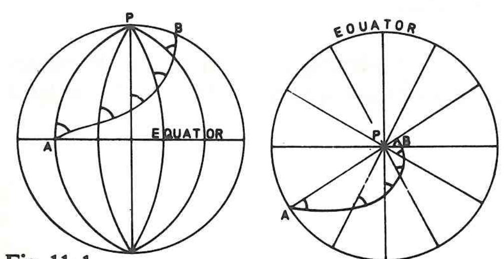


Fig. 11·1

In fig. 11·1, *AB* represents a typical rhumb line. Notice the constant angle which it makes with the meridians it crosses.

Special cases of rhumb lines are the equator, parallels of Latitude, and meridians. The equator and all other parallels of Latitude are rhumb lines because the course of a vessel travelling along a parallel is constantly 090° or 270°. Meridians are rhumb lines of constant course 000° or 180°

The art of sailing obliquely across the meridians is known as loxodromics, from the Greek words "loxos" and "dromos" meaning *oblique* and *running* respectively. For this reason all rhumb lines, other than parallels of latitude and meridians, are sometimes called Loxodromic Curves. When a vessel sails along a Loxodromic Curve her track is an equi-angular spiral which constantly approaches the Earth's Pole. This follows because the meridians close together, or converge, as the Latitude increases. Theoretically a Loxodromic Curve continually gets closer to but never reaches the Earth's Pole.

2. The Sailings

The Sailings comprise the various methods of finding the course and distance from one place on the Earth's surface to another. When the distance travelled by a vessel is relatively small it is usual, when practicable, to travel along the rhumb line connecting the points of departure and destination. For long distances, however, it is often advantageous to travel along the great circle arc connecting the points of departure and destination. In the latter case the distance to travel is less than the rhumb line distance.

There are four methods of Rhumb Line Sailing. Two of these will be dealt with in this chapter.

3. Parallel Sailing

When a vessel travels in any direction except due North or due South, she moves some distance towards due East or due West. This distance is known as Departure. Departure may be represented by an arc of a parallel of Latitude cut off between the meridians of the points between which the vessel travels.

If a vessel travels along the equator the D. Long. between the places left and arrived at is numerically equal to the Departure, the D. Long. being given in minutes of arc and the departure in nautical miles. This, of course, assumes the Earth to be a perfect sphere, in which case a nautical mile would the be length of any arc of the Earth's surface, the extremities of the arc subtending an angle of one minute at the Earth's centre.

When a vessel travels along any parallel of Latitude other than the equator; that is to say, when her course is due East or due West, the Departure measured in miles between the points left and arrived at is always numerically less than the D. Long. in minutes of arc between the two points. This is due to the Convergency of the meridians.

The relationship between D. Long., Departure, and Latitude, is given in the Parallel Sailing Formula, in which the Earth is assumed to be a perfect sphere.

4. The Parallel Sailing Formula

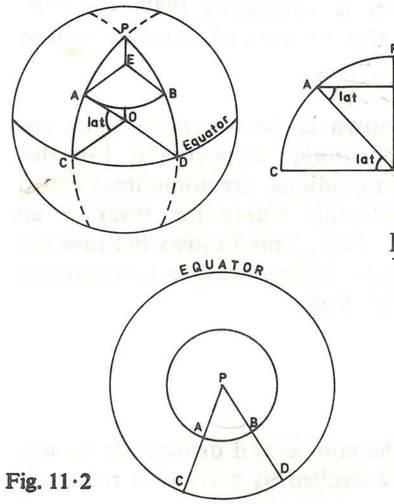


Fig. 11·2

In fig. 11·2:

AB = Departure in miles

CD = D. Long. in equatorial minutes of arc, these being equivalent to nautical miles on a spherical Earth.

Now:

$$\frac{\text{Departure}}{\text{D. Long.}} = \frac{AB}{CD}$$

$$= \frac{AE}{CO} \text{ (arcs of concentric circles subtended by the same angle are proportional to their radii)}$$

$$= \frac{AE}{AO} \text{ (} CO = AO \text{: radii of the same sphere)}$$

$$= \text{cos Lat. } A \text{ (or } B\text{)}$$

Therefore:

$$\frac{\text{Departure}}{\text{Departure}} = \text{Cos Lat.} \quad \ldots\ldots \quad (1)$$

$$\frac{\text{D. Long.}}{\text{Departure}} = \text{sec Lat.} \quad \ldots\ldots \quad (2)$$

$$\text{Departure} = \text{D. Long. . cos Lat.} \quad \ldots\ldots \quad (3)$$

$$\text{D. Long.} = \text{Departure . sec Lat.} \quad \ldots\ldots \quad (4)$$

These four relationships are variations of the Parallel Sailing Formula.

The traverse table is almost invariably used for solving problems in which the Parallel Sailing Formula is involved. The three columns of the table are labelled with supplementary headings: the top of the Distance column is labelled D. Long., and the top of the D. Lat. column is labelled Dep. In a traverse table which extends only to 45°, the bottoms of the Distance and Departure columns are labelled D. Long. and Dep. respectively.

Example 11·1—A vessel travelled 100 miles due East along the parallel of 50° 30′ N. If the Longitude of the point she left was 03° 50′ W., find her final Longitude.

D. Long = dep . sec Lat.
= 100 . sec 50° 30′

From traverse tables:

D. Long. = 158′
= 02° 38′ E.
Long. left = 03° 50′ W.
Final Long. = 01° 12′ W.

Answer—Final Longitude = 01° 12′ W.

Example 11·2—A vessel left a position in Lat. 39° 00′ S. Long. 30° 08′ W. and travelled due East until her Longitude was 25° 22′ W. How many miles did she travel?

Long. from = 30° 08′ W.
Long. to = 25° 22′ W.
D. Long. = 04° 46′ E.
= 286′ E.
Dep. = D. Long. . cos Lat.
= 286 . cos 39°

From Traverse Table:

Dep. = 222·3 miles

Answer—Distance = 222·3 miles.

Example 11·3—What is the Latitude where the D. Long. is numerically equal to three times the Departure?

$$\text{cos Lat.} = \frac{\text{Departure}}{\text{D. Long.}} = \frac{\text{Dep.}}{3 \times \text{Dep.}} = \frac{1}{3}$$

Lat. = 70° 32′ N. or S.

Answer—Latitude = 70° 32′ N. or S.

5. Plane Sailing

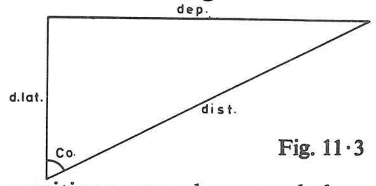


Fig. 11·3

When a vessel travels along a rhumb line, the acute angle which her fore-and-aft line makes with the meridians she crosses is known as the Course Angle. When a vessel travels along any rhumb line except a meridian or parallel of Latitude; the Distance steamed, the difference of Latitude, and the Departure between the first and final positions, may be regarded as forming the sides of a plane right-angled triangle with the Course Angle opposite to the side representing the Departure. This plane right-angled triangle is called the Plane Sailing Triangle. It must be borne in mind that this triangle does *NOT* represent a triangle on the Earth's surface: it is simply an artifice which shows the relationship between Rhumb Line Course, Distance, D. Lat. and Departure. If the Distance and Course Angle are known the D. Lat. and Departure may be found by solving a Plane Sailing Triangle. The formulae used in solving plane sailing triangles are called the Plane Sailing Formulae as indicated in fig. 11·3.

They are,

$$\text{Departure} = \text{Distance} . \sin \text{Course} \qquad (1)$$
$$\text{D. Lat.} = \text{Distance} . \cos \text{Course} \qquad (2)$$

By dividing (1) by (2) we get:

$$\frac{\text{Dep.}}{\text{D. Lat.}} = \tan \text{Course} \qquad (3)$$

6. Proof of Plane Sailing Formulae

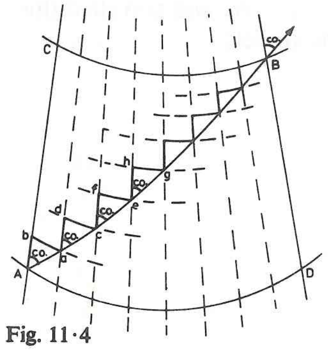


Fig. 11·4

Fig. 11·4 represents a portion of the Earth's surface showing parts of two meridians and two parallels of Latitude. The rhumb line between *A* and *B* is drawn. Let the rhumb line Course Angle be denoted by θ.

Imagine the distance *AB* to be divided into a sufficiently large number of small pieces, so that the triangles *Aab, acd, cef,* etc., may be considered to be plane. Strictly speaking the pieces *Aa, ac, ce,* etc., should be infinitely small. On this assumption the following proof holds good.

Ab, ad cf, etc., are pieces of D. Lat.
ab, cd, ef, etc., are pieces of Departure.

Then:

$$ab + cd + ef + \text{etc.} = Aa . \sin\theta + ac . \sin\theta + ce . \sin\theta + \text{etc.}$$
$$= \sin\theta\,(Aa + ac + ce + \text{etc.})$$

But, $Aa + ac + ce + \text{etc.} = \text{Distance } AB$

And, $ab + cd + ef + \text{etc.} = \text{Departure between } A \text{ and } B$

Therefore:

$$\text{Departure} = \text{Distance} . \cos\theta$$

Similarly it may be proved that:

$$\text{D. Lat.} = \text{Distance} . \sin\theta$$

Example 11·4—A vessel travelled for 245·0 miles on a Course of 062° S. Find the D. Lat. and Departure.

Course Angle = 62° Distance = 245·0 miles

From Traverse Tables:

Departure = 216·3′ E.
D. lat. = 115·0′ N.

Answer—Departure = 216·3 miles East.
D. Lat. = 115·0 miles North.

Example 11·5—A vessel left position Lat. 41° 44′ S. on a course of 158°. Find the distance she had travelled on reaching the parallel of 48° 16′ S.

Lat. from = 41° 44′ S. Course Angle = 22°
Lat. to = 48° 16′ S. D. Lat. = 392′ S.
D. Lat. = 06° 32′ S.

From Traverse Tables:

Distance = 423 miles

Answer—Distance = 423 miles.

Example 11·6—A vessel travelled for a distance of 346·0 miles and changed her Latitude by 4° 00′. If the course had been in the S.E. quadrant, find it.

D. Lat. = 240·0′ S.
Distance = 346·0 miles

From Traverse Tables:

Course Angle = 46°
Course S. 46° E. = 134°

Answer—Course = 134°

Example 11·7—A vessel travelled for a distance of 928·0 miles on a course of 306°. Find the D. Lat. and Departure.

Distance = 928·0 miles
= (600+328) miles
Course Angle = 54°

From Traverse Tables:

D. Lat. = 352·7+192·8
= 545·5′ N.
Departure = 485·4+265·4
= 750·8′ W.

Answer—D. Lat. = 545·5′ N.
Departure = 750·8′ W.

7. Traverse Sailing

When a vessel in travelling from one place to another has to make several courses, the irregular track she makes is known as a Traverse. Ths name is derived from the circumstance that a sailing vessel, in making a passage, would have to cross and recross the desired path several times because of the direction and/or change in direction of the wind during the passage. The problem of finding the course and distance the vessel would have made had it been possible for her to sail directly from the Departure position to the destination; that is to say, the Course and Distance Made Good, is known as Resolving a Traverse. This method of zig-zag sailing was, therefore, known as Traverse Sailing.

When making a traverse, the several legs of the track may be considered to be the hypotenuses of plane sailing triangles. The D. Lat. and Departure of each of these triangles may be solved by means of Plane Sailing. By summing the D. Lats. and Departures of the several plane sailing triangles, the D. Lat. and Departure between the points left and arrived at may be found.

The record of the courses and distances sailed on each leg of a traverse was known, in by-gone days, as the Ship's Reckoning. By means of the reckoning, and a knowledge of the initial Latitude, the Latitude of the vessel at any time could be found without recourse to observations. A Latitude so found was known as a Dead Reckoning, or D.R. Latitude.

In the modern practice of navigation, a D.R. position is one that has been worked up from the last Observed Position, making no allowance for current and/or leeway. The name Observed Position is given to any position obtained from observations of celestial or terrestrial objects, or from any electronic navigation instrument such as radar, Decca Navigator or Radio Direction Finder.

When observations are not possible, and a navigator wishes to know his vessel's position, he applies to the last Observed Position, courses and distances travelled through the water since the time of the Observed Position, and so finds his D.R. Position. To the D.R. he applies an estimated allowance for current, leeway, and any other disturbing factor which has influenced the vessel's movement. The position so found is referred to as an Estimated Position (E.P.). An Estimated Position is the most reliable position obtainable when direct observations are not available.

The traverse table lends itself admirably to the solution of traverse sailing. This is the reason, in fact, why the traverse table is so-named.

8. The Departure Position

When it is necessary to venture into the open sea it is essential, before the land is lost to sight, that the position of the vessel be found from terrestrial observations, in order to obtain a reliable Observed Position from which the course may be set. Such a position is known as a Departure Position. It is customary to describe a Departure Position as a bearing and distance from some conspicuous land- or sea-mark.

9. Current

The movement of the surface layers of the sea, due to meteorological causes, is known as current. The direction towards which the water in a current is moving is known as the Set, and the speed is known as the Rate of the current. The distance which a vessel is set in any given interval of time is called the Drift of the current.

If current is the only external factor influencing the movement of a vessel, the set and drift is equivalent to the course and distance from a D.R. Position to a corresponding Observed Position.

Example 11·8—A vessel takes her Departure from a position with Cape Sable in lat. 43° 25′ N. Long. 65° 30′ W., bearing 062° distance 12·0 miles. Course was set to 210°, log zero. When the log registered 14 the course was altered to 300° and when it registered 32 the course was altered to 223°. Find the vessel's D.R. Latitude when the log registered 49. Find also the course and distance the vessel made good.

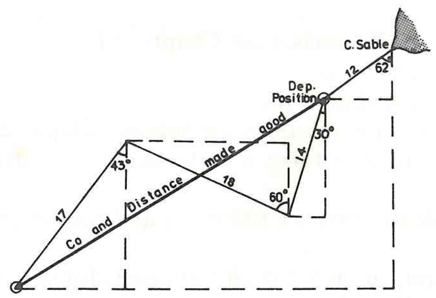


Fig. 11·5

In fig. 11·5

Departure Bearing is N. 62° E.
Departure Course is S. 62° W.

	D. Lat. N.	D. Lat. S.	Dep. E.	Dep. W.
Dep. course S. 62° W. distance 12 miles	—	5·6	—	10·6
1st course S. 30° W. distance 14 miles	—	12·1	—	7·0
2nd course N. 60° W. distance 18 miles	9·0	—	—	15·6
3rd course S. 43° W. distance 17 miles	—	12·4	—	11·6
		30·1′ S.		
		9·0′ N.		
D. lat. (for lat.)		21·1′ S.		34·2 = Dep.
		5·6		
D. Lat. (for Co.)		15·5′ S.		

From Traverse Tables:

Course and Distance Made Good = 246° ×38 miles
Lat. Cape Sable = 43° 25′ N.
D. Lat. = 21·1′ S.
D.R. Lat. ship = 43° 03·9′ N.

Answers—Course Made Good = 246°
Distance Made Good = 38 miles
D.R. Latitude = 43° 04′ N.

It is to be noted at this stage that it is not possible to find the final Longitude by using Plane or Traverse Sailing. When sailing obliquely across meridians, the Latitude changes constantly, and difficulty arises in finding the correct Latitude to use for converting Departure into D. Long. and *vice versa*. To find the position (Latitude and Longitude) of a vessel after she has travelled on a given course for a given distance such that her Latitude and

Longitude change is the problem of Mercator Sailing or Middle Latitude Sailing. These problems will be discussed in Chapter 13. The next chapter deals with the principle and the construction of the Mercator Chart, this leading to a discussion on the problems of Mercator and Middle Latitude Sailing.

Excercises on Chapter 11

1. Describe the properties of a rhumb line.
2. Define: Departure. What is the relationship between Departure and D. Long?
3. Explain clearly why the traverse table may be used for converting Departure into D. Long.
4. Construct a traverse table for distance 652 miles and course angles at 10° intervals from 000°.
5. Devise a graphical method, suitable for all Latitudes, for converting Departures into D. long. for distances up to 100 miles.
6. What is meant by Plane Sailing?
7. Define: D.R. Position; Estimated Position; Observed Position.
8. Prove that: D. Lat. = Distance . cos Course, for all distances on a spherical Earth.
9. A vessel travels 200 miles due East in Latitude 40° 30′ N. Find her change in Longitude.
10. How many miles must a vessel travel along the parallel of Latitude 56° 00′ S. in order to change her Longitude 10° 00′?
11. A vessel travels 250 miles due West and changes her Longitude by 8° 00′. Find the Latitude of the parallel along which she travelled.
12. Find the length of the parallel of Latitude 35°.
13. At what speed is a point in Latitude 60° 00′ carried around the Earth's axis?
14. A vessel travels for 12 hours at a speed of 10·0 knots due East along the parallel of Latitude 50° 10′S. Her Longitude changes 03° 24′. Find the set and rate of the current.
15. A vessel leaves a position in Latitude 40° 30′ N. Long. 16° 00′ W., and makes good the following courses and distances.

    (i) due East 300 miles — (iii) due West 300 miles
    (ii) due North 300 miles — (iv) due South 300 miles.

    Find her final position.
16. A vessel leaves a position in Lat. 30° 00′ S. Long. 178° 05′ E. and travels 200 miles due East. Find her final Longitude.
17. A vessel left a position in Lat. 20° 00′ S. Long. 18° 00′ E. and travelled due South for 120 miles, when her position was found to be in Lat. 22° 00′ S. Long. 18° 12′ E. Find the set and drift of the current.
18. Two vessels are 50 miles apart in latitude 35° 00′ N. They both travel due South until they are 55 miles apart. What is their present Latitude and how far has each vessel travelled?
19. A vessel on a course of 305° changed her Latitude by 4° 25′. Find the Departure and distance.
20. Find the departure and the change in latitude after having travelled on a course of 163° for a distance of 312 miles.
21. Find the distance and change in Latitude after having made a Departure of 218 miles on a course of 218°.
22. A vessel steamed between South and East, and in so doing made a Departure of 59 miles and changed her Latitude by 81 miles. Find the distance and course made good.
23. A vessel took her Departure off the South West coast of Ireland with the Fastnets

bearing 037° distant 12·0 miles. The log was set to zero and the following courses were steered:

    205° until the log registered 82
    208° until the log registered 196
    220° until the log registered 326

    Find the course and distance made good and the D.R. Latitude of the vessel when the log registered 326.
24. A vessel left a position off Callao in Lat. 14° 50′ S. Long. 76° 55′ W., and made the following courses and distances:

    195° for 45 miles
    165° for 160 miles
    170° for 82 miles

    The current was estimated to have set 290° for 27 miles. Find the course and distance made good and the vessel's present estimated Latitude.
25. A vessel left a position off the Cape of Good Hope with the Cape bearing 090° distance 20 miles. She travelled for 265 miles on a course of 330° and for 345 miles on a course of 324°. During the interval the current was estimated to have set 020° for 25 miles. Find the estimated course and distance made good by the vessel, and her estimated Latitude at the end of the interval.
26. How many miles are there in one degree of D. Long in Lat. 42°?
27. A vessel took her Departure from a position in the mouth of the River Plate in Lat. 34° 49′ S. Long. 54° 50′ W., and sailed along the parallel to a position off the Cape of Good Hope in Lat. 34° 49′ S. Long. 20° 00′ E. Find the distance travelled.

CHAPTER 12

THE MERCATOR CHART

1. The Navigator's Chart

The Earth is a three dimensional solid figure while the maps and charts used for navigational purposes, are only two dimensional. This causes problems of representation and whichever system is used some distortion is inevitable. The amount of type of distortion depends upon the method of projection used. The Mercator projection is almost universal for marine navigation purposes.

A chart or map used for navigation purposes should have four basic properties which can be represented acurately.

1. Area
2. Direction
3. Shape
4. Scale

This means that lines of latitude and longitude are required to be drawn in a continuous way, with distortion reduced to a minimum. We also need to be able to represent a course, a bearing and a distance with accuracy.

The Mercator Chart fulfils these requirements and, for this reason, nearly all navigational charts for use at sea are of this type.

The principle on which the Mercator Chart is constructed was first used in the sixteenth century by a German cartographer named Gerhard Kaufman, the latinised version of whose name (which in English means merchant) is Mercator. There seems to be doubt that Kaufman understood the exact mathematical principle of the chart which bears his name, and credit is given to the famous Elizabethan scholar Edward Wright for discovering the mathematical principle of the Mercator Chart. Wright published a description, and also a table for facilitating the construction of Mercator Charts, in an important book first published in the closing decade of the sixteenth century.

When the surface of a sphere is projected onto a plane surface there is bound to be distortion. The amount and type of distortion depends upon the method of projecting the spherical surface onto the plane surface. The Mercator Chart is based on a projection which is not a perspective projection. The Mercator projection is a mathematical projection described by cartographers as a Conventional or Non-Perspective Projection. In the Mercator projection, as in all other conventionals, distances representing the spacing of parallels and meridians must be calculated, in order to project. It is not possible to construct an accurate Mercator Chart geometrically, like, for example, a stereographic projection.

In order that angles on the projection are not distorted the exaggeration of the representation of any small area of the sphere's surface must be equal in the North/South to that in the East/West direction. A projection in which angles are not distorted is known as an Orthmorphic projection.

2. Features of a Mercator Chart

The characteristic features of a Mercator Chart are:

(1) All meridians are projected as equidistantly spaced parallel straight lines.
(2) All parallels of Latitude are projected as parallel straight lines perpendicular to the projected meridians.
(3) All rhumb lines are projected as straight lines.
(4) All arcs of great circles, with the exceptions of arcs of the equator or any meridian, are projected as curves which are concave to the projected equator.
(5) Angles, such as course and bearing angles are easily and accurately determined.

Because meridians are projected as parallel straight lines, whereas on the globe they converge towards the poles, it follows that the exaggeration of arcs of parallels of latitude increases polewards. In order for the map to be orthomorphic the distances between successive parallels of latitude must also increase polewards in the same ratio.

3. The Defects of a Mercator Chart

Although the Mercator Chart satisfies the principal needs of the navigator it does have defects. The principal defects of the Mercator Chart are:

(1) Every Latitude has a different scale of distance.
(2) Great circle arcs, except those of the equator or meridians, are projected as curves. This makes for difficulty in the practice of Great Circle Sailing.

The variation in the Latitude scale causes areas to be exaggerated proportional to their Latitudes. It will be noticed that on a Mercator map of the world, Greenland appears larger than the continent of South America, and yet the range of Latitude of Greenland is no more than about a quarter of that of South America.

4. Distortion of the Mercator Projection

The degree of exaggeration of lengths along parallels and meridians will now be examined. In doing so it will be convenient to think of the Earth reduced in size to a model globe from which the chart is to be projected. Let the radius of the globe be R.

Exaggeration of the Parallels of Latitude

On the Globe:

$$\text{Length of the Equator} = 2\pi R$$
$$\text{Length of the Pole} = 0 \text{ (Pole is a point)}$$

Therefore

$$\text{Length of any Parallel of Latitude } \theta = 2\pi R \cos \theta$$

On the chart:

$$\text{Length of the Equator} = 2\pi R$$
$$\text{Length of any Parallel} = 2\pi R$$

Now:

$$\text{Exaggeration} = \frac{\text{Length on Chart}}{\text{Length on Globe}}$$

Therefore:

$$\text{Exaggeration} = \frac{2\pi R}{2\pi R \cos\ \theta}$$

$$= \frac{1}{\cos\ \text{Lat.}}$$
$$= \sec\ \text{Lat.}$$

A parallel of Latitude, therefore, is projected with exaggeration which is proportional to the secant of the Latitude of the parallel.

Now the trigonometrical ratio of the secant, changes from unity when the angle is 0°, to infinity when the angle is 90°. It is impossible to represent a line that has been exaggerated to an infinite extent. Therefore, the poles of the Earth, whose Latitudes are 90°, cannot be represented on a Mercator Chart. Not only is it impossible to project the poles, but it is impracticable to project areas of very high Latitude. But this does not concern surface mariners, the vessels of whom trade in more temperate climes than those of very high Latitudes. The parallel of Latitude 60° is exaggerated two-fold because the secant of 60° is 2: the parallel of Latitude 70½° is exaggerated three-fold, because the secant of 70½° is 3, and so on.

Exaggeration of the Meridians

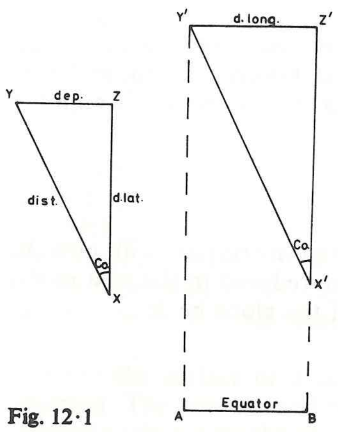


Fig. 12·1

Consider the rhumb line, illustrated in fig. 12·1, which cuts the meridians at an angle θ. A part XY of this rhumb line, if sufficiently small, may be regarded as forming the hypotenuse of a right-angled plane triangle containing the Course Angle θ. Again, if the length of this hypotenuse is sufficiently small the side opposite the course angle in the right-angled triangle may be considered to be equal to the Departure between the end points of the rhumb line XY. The side coinciding with the meridian through X then represents the D. Lat. between the end points.

If the point x is projected at point X^1 on a Mercator Chart; then, because the chart is orthomorphic, the angle θ is represented without distortion. This applies equally to the angles at Y and Z, so that these points are projected at Y^1 and Z^1 respectively, such that the triangle $X^1Y^1Z^1$ on the chart is similar to the triangle XYZ on the globe. This is strictly true only when the triangle XYZ is infinitely small. For this reason the term orthomorphism, when applied to a map projection, has a special meaning: *shape is preserved only for infinitely small areas.*

Because meridians are projected on a Mercator chart as parallel straight lines, the Departure between X and Y is represented by Y^1Z^1. This length also represents the D. Long. between X and Y.

$$\text{Exaggeration of Arc } XZ \text{ of Meridian} = \frac{\text{Length on Chart}}{\text{Length on Globe}}$$
$$= \frac{X^1Z^1}{XZ}$$

The triangles XYZ and $X^1Y^1Z^1$, are similar, so that the ratio between corresponding sides is constant.

Therefore:

$$\frac{X^1Z^1}{XZ} = \frac{Y^1Z^1}{YZ}$$

That is:

$$\text{Exaggeration} = \frac{\text{D. Long.}}{\text{Dep.}}$$

By the Parallel Sailing Formula:

$$\frac{\text{D. Long.}}{\text{Dep.}} = \sec\ \text{Lat.}$$

Therefore:

$$\text{Exaggeration of Arc } XZ \text{ of Meridian} = \sec\ \text{Lat.}$$

The exaggeration of the projection of an arc of a meridian is proportional to the secant of the latitude. This is to the same extent as the exaggeration of the projection of an arc of a parallel of Latitude. The Mercator projection is, therefore, orthomorphic.

5. Meridional Parts

The Longitude scale on a Mercator Chart is constant. On the other hand the Latitude scale is variable: it increases proportionally to the secant of the Latitude. Thus the unit of the Longitude scale is a convenient unit for certain purposes which we shall now discuss.

The number of minute-of-arc units of the Longitude scale contained in a projected piece of a meridian on a Mercator Chart between the projected equator and the projection of any given parallel of Latitude θ, is called the Meridional Parts for Latitude θ. One Meridional Part (m.pt.), therefore, is equivalent to a minute of arc of the Longitude scale.

The length of any piece of a projected meridian on a Mercator Chart between two given projected parallels of Latitude, expressed in m.pts., is called the Difference of Meridional Parts (D.M.P.) between the latitudes of the two parallels.

Meridional Parts are useful in two applications:
(1) in constructing Mercator Charts
(2) in Mercator Sailing.

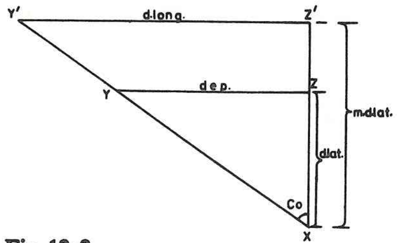


Fig. 12·2

Fig. 12·2 illustrates a part of a Mercator Chart with the rhumb line connecting projected positions X^1 and Y^1.

The number of units of the constant Longitude scale contained in arcs X^1Z^1 and Z^1Y^1 are the D.M.P. and D. Long. respectively between X^1 and Y^1.

Provided that the correct part of the Latitude scale is used, the number of units of the variable Latitude scale contained in arcs X^1Z^1 and Z^1Y^1 are D. Lat. and Departure, respectively, between X^1 and Y^1.

It follows that:

$$\frac{\text{D.M.P.}}{\text{D. Lat.}} = \frac{\text{D. Long.}}{\text{Dep.}}$$

By the Parallel Sailing Formula:

$$\frac{\text{D. Long.}}{\text{Dep.}} = \cos \lambda$$

where λ is an angle known as the Middle Latitude (see Chapter 13).

Therefore:

$$\frac{\text{D.M.P.}}{\text{D. Lat.}} = \sec \sec \lambda$$

This relationship is known as the Mercator Principle.

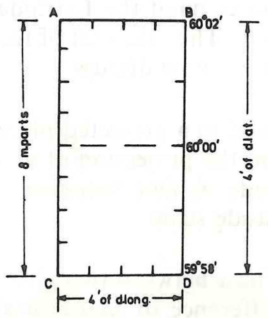


Fig. 12·3

Fig. 12·3 illustrates a part of a Mercator chart. The D. Lat. between A and C is 4'. If the Middle latitude is taken as 60°00', the D.M.P. between A and C, by the Mercator principle, is:

$$\begin{aligned} \text{D.M.P.} &= \text{D. Lat. sec } 60°00' \\ &= 4 \sec 60°00' \\ &= 4 \times 2 \\ &= 8 \end{aligned}$$

Therefore, the piece of the meridian cointained between A and C has been doubly magnified.

The D.M.P. between two Latitudes may be measured direct from a Mercator Chart simply by finding the number of minutes of the Longitude scale in the piece of any meridian contained between the parallels of the two Latitudes.

Before a Mercator Chart can be constructed, a table of meridional parts must be available. We shall now discuss how Edward Wright devised his table of m.pts. from which the first mathematically correct chart was constructed.

After explaining the Mercator principle of orthomorphism and expressing it in mathematical terms, viz. D.M.P./D. Lat = sec Middle Latitude, Wright divided the part of a meridian between the equator and a given parallel of Latitude into a number of equal pieces. He then multiplied the number of Latitude minutes in each piece by the secant of the *MEAN* Latitude of each piece. This gave a value for the Meridional parts in each piece. By adding the m.pts. of the pieces together, a value for the m.pts. for the given Latitude was obtained. When computing in m.pts. in this way the degree of accuracy of the results depends upon the number of pieces into which the part of the meridian between the equator and the given parallel is divided. The greater the number of pieces the more accurate is the result. The inaccuracy of a result arising through not taking a sufficiently large number of pieces, is due to the Mean Latitude not being equivalent to the Middle Latitude of the piece. In order to compute the exact number of m.pts. for any given Latitude it is required to take an infinite number of pieces. The precise computation, therefore, requires the use of the integral calculus. In the calculus notation the number of m.pts. (M) in any Latitude φ is given by:

$$M = \int_{\theta = 0}^{\theta = \varphi} \sec \theta \,.\, d\theta$$

Edward Wright computed his table of m.pts. before the integral calculus had been invented. Wright's table, which was based on the assumption that the Earth is a perfect sphere, was computed by dividing the meridian into 1' arc lengths.

The following example illustrates the principle of Wright's method of computing meridional parts.

Example 12·1—Compute the approximate Meridional Parts for Latitude 30°00' assuming the Earth to be a perfect sphere.

Method—Divide the arc of the meridian contained between the equator and the parallel of Latitude of 30° into (say) six equal pieces. Each piece is, therefore, 5° or 300' in length.

The piece between Lat. 0° and Lat. 5° is represented, approximately, on a Mercator Chart, by a length proportional to 300 sec 2½°.

The piece between Lat. 5° and Lat. 10° is represented by a length proportional to 300 sec 7½°.

The piece between Lat. 10° and Lat. 15° is represented by a length proportional to 300 sec 12½° and so on.

Thus:

$$\begin{aligned}\text{M.pts. for lat. } 30^\circ &= 300 \sec 2\tfrac{1}{2}^\circ + 300 \sec 7\tfrac{1}{2}^\circ + \\ &\quad 300 \sec 12\tfrac{1}{2}^\circ + 300 \sec 17\tfrac{1}{2}^\circ + \\ &\quad 300 \sec 22\tfrac{1}{2}^\circ + 300 \sec 27\tfrac{1}{2}^\circ \\ &= 300 (\sec 2\tfrac{1}{2}^\circ + \sec 7\tfrac{1}{2}^\circ + \sec 12\tfrac{1}{2}^\circ + \\ &\quad \sec 17\tfrac{1}{2}^\circ + \sec 22\tfrac{1}{2}^\circ + \sec 27\tfrac{1}{2}^\circ)\end{aligned}$$

$$\begin{aligned}\sec 2\tfrac{1}{2}^\circ &= 1{\cdot}00095 \\ \sec 7\tfrac{1}{2}^\circ &= 1{\cdot}00863 \\ \sec 12\tfrac{1}{2}^\circ &= 1{\cdot}02428 \\ \sec 17\tfrac{1}{2}^\circ &= 1{\cdot}04853 \\ \sec 22\tfrac{1}{2}^\circ &= 1{\cdot}08239 \\ \sec 27\tfrac{1}{2}^\circ &= 1{\cdot}12738 \\ \text{sum} &= 6{\cdot}29216 \\ &\quad \times 300 \\ &\quad 1887{\cdot}64800\end{aligned}$$

Answer—M.pts. Lat. 30° = 1887·65 approximately.

Note—Had the arc of the meridian been divided into a number of parts greater than six, the result would have been more accurate than that obtained above.

6. Meridional Parts for the Terrestrial Spheroid

The m.pts. table given in nautical tables such as Norie's and Burton's, are computed for a terrestrial spheroid having an ellipticity of about 1/300. Admiralty charts on the Mercator projection are constructed using these tables.

7. Constructing Mercator Charts

The Graticule, or network of projected parallels and meridians, of a Mercator Chart is drawn to a convenient scale using a straightedge. The first thing to do is to choose a suitable scale of Longitude. Given the range of Longitude of the proposed chart, the East/West extent of the chart may then be found. A straight line of this length is then drawn across the lower part of the sheet on which the graticule is to be constructed. This line is the projection of one of the limiting parallels of Latitude of the area to be portrayed. Straight lines, to represent the projected meridians, are then erected perpendicularly from this projected parallel. The range of Latitude is divided into a number of equal arcs. The D.M.P. between the limiting Latitudes of each of these arcs is found using the m.pts. table, and the spacing of the projected parallels is then computed. The following example illustrates this method.

Example 12·2—Construct a Mercator Chart between the limits of Latitudes 10° and 50° N., and between the meridians of 90° and 150° W. Project parallels and meridians every 10°.

Range of Longitude = 60°

Let the scale of Longitude be 1 unit to represent 10° or 600′ of Longitude.

$$\text{Width of chart} = \frac{60}{10} = 6 \text{ units}$$

$$\begin{aligned}\text{m.pts Lat. } 10^\circ &= 599{\cdot}01 \\ \text{m.pts Lat. } 20^\circ &= 1217{\cdot}14 \\ \text{D.M.P.} &= 618{\cdot}13 \text{ represented by } \frac{618{\cdot}13}{600} = 1{\cdot}030 \text{ units}\end{aligned}$$

$$\begin{aligned}\text{m.pts Lat. } 20^\circ &= 1217{\cdot}14 \\ \text{m.pts Lat. } 30^\circ &= 1876{\cdot}67 \\ \text{D.M.P.} &= 569{\cdot}53 \text{ represented by } \frac{659{\cdot}53}{600} = 1{\cdot}099 \text{ units}\end{aligned}$$

$$\begin{aligned}\text{m.pts Lat. } 30^\circ &= 1876{\cdot}67 \\ \text{m.pts Lat. } 40^\circ &= 2607{\cdot}64 \\ \text{D.M.P.} &= 730{\cdot}97 \text{ represented by } \frac{730{\cdot}97}{600} = 1{\cdot}218 \text{ units}\end{aligned}$$

$$\begin{aligned}\text{m.pts Lat. } 40^\circ &= 2607{\cdot}64 \\ \text{m.pts Lat. } 50^\circ &= 3456{\cdot}53 \\ \text{D.M.P.} &= 848{\cdot}89 \text{ represented by } \frac{848{\cdot}89}{600} = 1{\cdot}415 \text{ units}\end{aligned}$$

Range of Latitude 40° will be represented by:

$$1{\cdot}030 + 1{\cdot}099 + 1{\cdot}218 + 1{\cdot}415 = 4{\cdot}762 \text{ units}$$

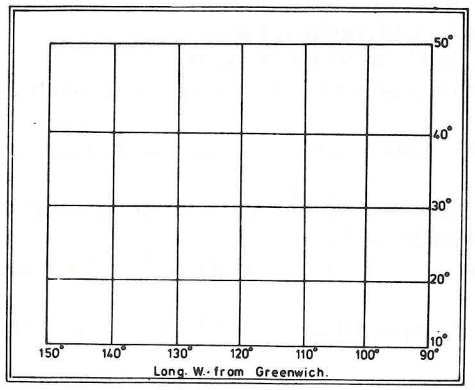


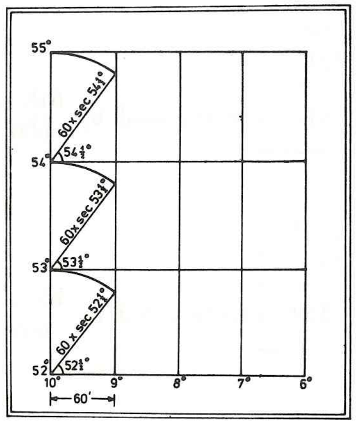
Fig. 12·4

Fig. 12·4 illustrates the required graticule.

An alternative method of constructing a Mercator Chart, which is suitable only when the range of Latitude is small, is shown in fig. 12·5 which illustrates Example 12·3. In this method, after having chosen a longitude scale and projected one of the limiting parallels from which the projected meridians are erected, an angle equal to the Middle Latitude of the limiting latitudes is constructed as shown in fig. 12·5. The hypotenuse of the triangle, by the Mercator Principle, is proportional in length to the D.M.P. between the two limiting parallels.

Example 12·3—Construct a Mercator Chart for the area contained between parallels of 52° and 55° N. and the meridians 06° and 10° W. Insert parallels and meridians at one-degree intervals.

Fig. 12·5



Range of Longitude = 4°

Let the Scale of Longitude be 1 unit to 1°.

Width of chart = 4 units

The construction is illustrated in fig. 12·5 which is the required projection.

Exercises on Chapter 12

1. What are the main requirements of a navigational chart?
2. Describe the features of a Mercator Chart. State the advantages and disadvantages of a Mercator Chart to a navigator.
3. Explain carefully the mathematical principle of the Mercator projection.
4. What is the meaning of the term Orthomorphism as it applies to a map projection?
5. Define: Meridian Parts for Lat. θ; Difference of Meridional Parts.
6. Explain how Edward Wright constructed his table of meridion parts.
7. Explain carefully how the graticule of a Mercator Chart of a small area, such as a harbour or estuary, may be constructed.
8. Describe how a small circle of diameter 600 miles lying on the equator appears on a Mercator Chart.
9. How many units of the Longitude scale of a Mercator Chart are contained in a part of a meridian between the parallels of 53° 20′N. and 53° 40′N.?
10. If the scale of Longitude on a Mercator Chart is 1 cms. to 1°, find the scale of Latitude in Latitude 65° 15′N.
11. If 1′ of Longitude on a Mercator Chart is represented by 2·52 cms., what length represents 1′ of Latitude in Lat. 42° 10′S.?
12. If 1′ of Latitude on a Mercator Chart is represented by 2·30 cms. in Lat. 32° 20′N., find the scale of Longitude.

13. If 1′ of Latitude on a Mercator Chart is represented by 3·25 cms. in Latitude 47° 30′, find the scale of Latitude in Lat. 40° 00′.
14. Construct a Mercator Chart for the area between the limits of Latitude 50° 00′N. and 58° N., and between the meridians of 4° 00′W. and 20° W., using a Longitude scale of 2·5 cms. to 1°. Insert parallels and meridians at one-degree intervals.

CHAPTER 13

MERCATOR SAILING AND MIDDLE LATITUDE SAILING

1. Introduction

When a vessel has travelled a given distance along a meridian her Departure is zero and her change in Latitude, in minutes of arc, is numerically equal to the number of miles she has sailed.

When a vessel has travelled a given distance along a parallel of Latitude her change in Latitude is zero and her Departure in miles is equal to the distance she has sailed. If the D. Long. corresponding to an unknown distance sailed along a parallel is known the distance may be found by means of the parallel Sailing Formula.

In both cases, of travelling along a meridian and along a parallel of Latitude, the sailing problems of finding position or course and distance present no difficulty.

It now remains to examine the general sailing problems in which a vessel changes both Latitude *AND* Longitude by travelling on an oblique rhumb line path.

There are two methods of solving the general sailing problem:
(1) by Mercator Sailing
(2) by Middle Latitude Sailing

2. Mercator Sailing

If a rhumb line path is drawn between two places A_1 and B_1 on a Mercator Chart, the plane right-angled triangle having the rhumb line path as its hypotenuse and containing the Course Angle, may conveniently be called the Chart Triangle. The two sides which meet to form the right angle in the Chart Triangle, when measured on the constant scale of Longitude, give the D.Long. and D.M.P. respectively between A_1 and B_1.

The Chart Triangle is geometrically similar to the Plane Sailing Triangle corresponding to the rhumb line path AB which is projected onto the Mercator Chart as A_1B_1.

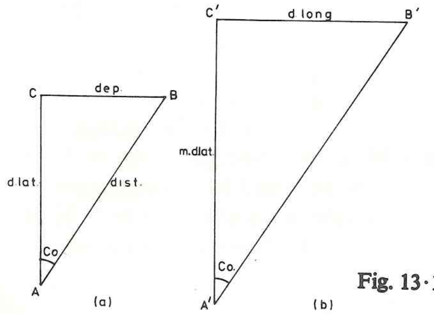


Fig. 13·1

Fig 13·1 (*a*) and (*b*) illustrate corresponding Plane Sailing and Chart Triangles.

The two general rhumb line sailing problems are:
(1) Finding the rhumb line course and distance from one given position to another.
(2) Finding the position of arrival after having travelled on a given rhumb line course for a given distance from a given position.

To solve the first of these problems, using Mercator Sailing, the procedure is as follows:
(1) Find D. Lat. and D. Long.
(2) Using m.pts. table find D.M.P.
(3) In the Chart Triangle, using D. Long. and D.M.P. find rhumb line Course Angle.
(4) In the Plane Sailing Triangle, using Course Angle and D. Lat., find the rhumb line Distance.

An example will make this clear.

Example 13·1—Find the rhumb line Course and Distance using Mercator Sailing, from A in Lat. 49° 50′ N. Long. 05° 30′ W. to B in Lat. 37° 50′ N. Long. 25° 40′ W.

Lat. A = 49° 50′ N.	m.pts. = 3441·05	Long. A = 05° 30′ W.
Lat. B = 37° 50′ N.	m.pts. = 2441·23	Long. B = 25° 40′ W.
D. Lat. = 12° 00′ S.	D.M.P. = 999·82	D. Long. = 20° 10′ W.
= 720′ S.		= 1,210′ W.

Referring to fig. 13·2:
In the Chart Triangle:

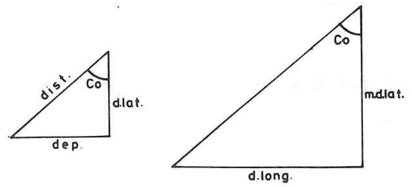


Fig. 13·2

$$\tan \text{Co} = \frac{\text{D. Long.}}{\text{D.M.P.}}$$

log D. Long. = 3·08279
log D.M.P. = 2·99991
log tan Co = 0·08288
Co = S., 50° 26′ W.

In the Plane Sailing Triangle:

Distance = D. Lat. sec Co
log D. Lat. = 2·85733
log sec Co = 0·19588
log Distance = 3·05321
Distance = 1130 miles

Answers—Course = 230½°
Distance = 1130 miles.

It should be noted that the answers are found by solving two right-angled plane triangles. The traverse table, therefore, may be used to check the calculated answers.

The second general rhumb line sailing problem is solved thus:
(1) In the Plane Sailing Triangle, using Course Angle and Distance find D. Lat.
(2) Find the final Latitude and thence the D.M.P.
(3) In the Chart Triangle, using D.M.P. and Course Angle, find the D. Long.
(4) Find the final Longitude.

The following example illustrates the method of solution.

Example 13·2—A vessel leaves a position in lat. 32° 00′ S. Long. 116° 05′ E. and sails for a distance of 1243 miles on a course of 322°. Find her D.R. position after making this run.

Referring to fig. 13·3:

In the Plane Sailing Triangle:

D. Lat. = distance cos Co
log Distance = 3·09447
log cos Co = $\bar{1}$·89050
log D. Lat. = 2·98497
D. Lat. = 966·0′ N.
Lat. from = 16° 06′ N.
Lat to = 32° 00′ S.
= 15° 54′ S.
m.pts. Lat. from = 2015·98
m.pts. Lat. to = 960·08
D.M.P. = 1055·90

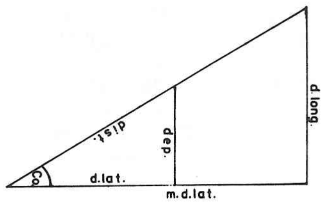


Fig. 13·3

In the Chart Triangle:

D. Long. = D.M.P. tan Co
log D.M.P. = 3·02362
log tan Co = $\bar{1}$·89281
log D. Long. = 2·91643
D. Long. = 825′ W.
= 13° 45′ W.
Long. from = 116° 05′ E.
Long. to = 102° 20′ E.

Answer—Lat. to = 15° 54′ S.
Long. to = 102° 20′ E.

3. Rhumb Line Sailing when Course Angle is Large

When the Course Angle is large—more than about 60°—the change in the secant of the course angle is large for a small change in the angle. Examination of the secant table will reveal this.

It will be noticed that when finding the rhumb line Distance in example 13·1, the Course was found using the tangent table, and then, for finding the Distance, the secant table was used. Now sec θ is equivalent to tan θ . cosec θ. Therefore, to facilitate finding the rhumb

line Distance in problems in which the Course Angle is large, instead of working as we have done in Example 13·1, it is better to use the formula:

Distance = D. Lat. . tan Co . cosec Co

The tangent of the course angle is found from the formula:

$$\tan Co = \frac{D.\ Long.}{D.M.P.}$$

and the cosecant of the course angle may be lifted from the tables without difficulty. When an angle is large, the change in its cosecant as the angle increases, is small.

The following example illustrates a case in which the course angle is large.

Example 13·3—A vessel sails from *A* in Lat. 40° 00′ S. Long. 149° 00′ E. to *B* in Lat. 37° 00′ S. Long. 173° 00′ E. Find the Course and Distance using Mercator Sailing.

Lat. *A* = 40° 00′ S.	m.pts. Lat. *A* = 2607·6	Long. *A* = 149° 00′ E.
Lat. *B* = 37° 00′ S.	m.pts. Lat. *B* = 2378·5	Long. *B* = 173° 00′ E.
D.Lat. = 03° 00′ N.	D.M.P. = 229·1	D. Long = 24° 00′ E.
= 180′ N.		= 1440′ E.

Referring to fig. 13·4:

In the Chart Triangle:

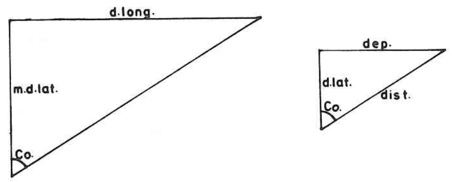


Fig. 13·4

$$\tan Co = \frac{D.\ Long.}{D.M.P.}$$

log D. Long. = 3·15836
log D.M.P. = 2·36003
log tan co = 0·79833
Co = N. 81° E.

In the Plane Sailing Triangle:

distance = D. Lat. . tan Co . cosec Co
log D. Lat. = 2·25527
log tan Co = 0·79833
log cosec Co = 0·00542
log Distance = 3·05902
Distance = 1146 miles

Answers—Course 081°
Distance 1146 miles.

4. Middle Latitude Sailing

Fig. 13·5 illustrates a portion of the Earth's surface with the rhumb line connecting the points *A* and *B*.

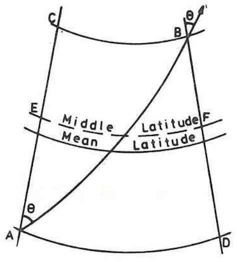


Fig. 13·5

The D. Lat. between *A* and *B*, in fig. 13·5, is equal to the arc of the meridian *AC* or *BD*. The Departure between *A* and *B*, however, is greater than arc *BC* and less than arc *AD*. The actual Departure between *A* and *B* may be represented by the arc *EF*. The Latitude of the parallel on which this arc lies is referred to as the Middle Latitude.

Considering the Earth to be a perfect sphere the Middle Latitude is always greater than the average or Mean Latitude between any two points in the same hemisphere on the Earth. This is not always the case, however, on a spheroidal Earth.

Middle Latitude may be defined as the angle the cosine of which is equal to the ratio between Departure and D. Long. Thus:

$$\text{Cos Middle Latitude} = \frac{\text{Departure}}{\text{D. Long.}}$$

This relationship is known as the Middle Latitude Sailing Formula.

The magnitude of the difference of the Middle and Mean Latitudes between two places depends upon the D. Lat. and the Mean Latitude of the two places. A table giving differences between Middle and Mean Latitudes for all convenient values of D. Lat. and Mean Latitude is given in the earlier edition of nautical tables such as Norie's and Burton's. Middle latitude is not longer in common use, hence the tables of corrections contained in Norie's and Burton's have now been omitted.

5. Crossing the Equator

When it is necessary to find the rhumb line Course and Distance between two places which lie on different sides of the equator, it is more convenient to use Mercator Sailing. In this case the D.M.P. is found by adding together the m.pts. for the two latitudes.

Example 13·4—Find, by Mercator Sailing, the course and distance from *A* in Lat. 10° 00′ S. Long. 90° 00′ W. to *B* in Lat. 08°30′ N. Long. 60° 00′ W.

Lat. *A* =	10° 00′ S.	m.pts. Lat. *A* =	599·0	Long. *A* =	90° 00′ W.
Lat. *B* =	08° 30′ N.	m.pts. Lat. *B* =	508·4	Long. *B* =	60° 00′ W.
D. Lat. =	18° 30′ N.	D.M.P. =	1107·4	D. Long. =	30° 00′ E.
=	1110′ N.			=	1800′ E.

Referring to fig. 13·6:
In the Chart Triangle:

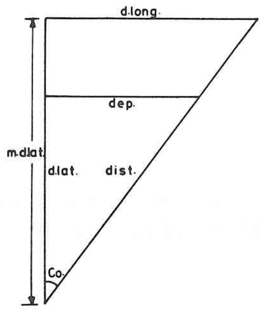


Fig. 13·6

$$\tan \text{Co} = \frac{\text{D. Long.}}{\text{D.M.P.}}$$

log D. Long. = 3·25527
log D.M.P. = 3·04430

log tan Co = 0·21097

Co = N. 58° 24′ E.

Distance = D. Lat. . sec Co
log D. Lat. = 3·04532
log sec Co = 0·28068

log Distance = 3·32600
Distance = 2118 miles

Answers—Course = 058½°
Distance = 2118 miles.

6. The Day's Run

In merchant vessels it is customary to determine the ship's position as accurately as possible at each noon. The distance travelled over the ground between successive noons is known as the Day's Run. This distance divided by the Steaming Time gives the average speed of the vessel for the day. When coasting, the Day's Run is measured direct from the chart, and the difference between the measured distance and the distance recorded by the patent log is usually regarded as Favourable or Adverse Current.

When navigating out of sight of land, the distance made good is found by calculation. Measuring the distance accurately from a chart is not possible in this case, because the ocean chart used has too small a scale of distance.

If, during the day, the vessel has made one course only, the distance made good is calculated by Mercator sailing. If, however, the vessel has made more than one course, the distance on each leg is estimated as accurately as possible, and the several distances are then summed to give the total run for the day.

In days of sail, when a vessel was forced to make a zig-zag course towards her destination, it was usual to ascertain the direct course and distance made good for the day. This has no value to a navigator on a power-driven vessel: he is interested only in the actual distance his vessel has steamed over the ground.

The vessel's D.R. Noon Position, reckoned from the observed position of the previous noon, is compared with the Observed Noon Position, to give the set and drift of the current experienced during the day.

An estimation of the distance travelled by a vessel may be found by means of a knowledge of the propeller revolutions. The engineer officer, at each noon, records the reading of the Counter which registers the number of revolutions of the propeller. From successive noon recordings, the revolutions made by the propeller during the day may be found. If the Pitch of the propeller is known the distance which should have been covered may be found.

$$\text{Engine Distance} = \frac{\text{Revs.} \times \text{Pitch}}{6080} \text{ miles}$$

Because of several factors including:
(*a*) current
(*b*) hull resistance
(*c*) faulty propeller

the engine distance is not generally the same as the Vessel's Distance as found from observations. The Engine Distance is usually greater than the Vessel's Distance by an amount called Slip.

Slip is usually calculated as a percentage of the Engine Distance from the formula:

$$\text{Slip (\%)} = \frac{\text{Engine Distance} - \text{Vessel's Distance}}{\text{Engine Distance}} \times 100$$

If the slip is estimated and the Engine Distance is known, the Vessel's Distance may readily be computed.

Example 13·5—If the Engine Distance is found to be 240·0 miles, and the slip is estimated to be 4%, find the Vessel's Distance.

Engine Distance = 240·0 miles
Slip = 4%

Now, $$\text{Slip} = \frac{\text{Engine Dist.} - \text{Vessel's Dist.}}{\text{Engine Dist.}} \times 100$$

Therefore, $$4 = \frac{240{\cdot}0 - \text{Vessel's Dist.}}{240{\cdot}0} \times 100$$

and $$\text{Vessel's Distance} = 240{\cdot}0 - \frac{96{\cdot}0}{10}$$
$$= 230{\cdot}4 \text{ miles}$$

Answer—Vessel's Distance = 230·4 miles.

7. The Day's Work

The process of finding a vessel's course and distance made good between successive noon positions, finding the average speed, and finding the set and rate of the current experienced during the day, is known as the Day's Work. An example of a typical Day's Work is as follows.

Example 13·6—At noon on 2nd January, Malin Head (Lat. 55° 22′ N. Long. 07° 24′ W.) was observed to bear 170° Distance 10·0 miles. The log was set to zero and the course was set to 330°. At the following noon the log registered 302·0 and the observed position was Lat. 59°

50·0′ N. Long. 11° 54·0′ W. The steaming time was 24 hrs. 00 mins. Find the Day's Run, the average speed and the set and drift of the current experienced during the day.

Departure Brg. = 170°
Departure Co and Distance = 350° by 10·0 miles

D. Lat. = 10′ N	Dep. = 1·7′ W.	D. Long. = 3·0′ W.	
Lat. point = 55° 22′ N.		Long. Point = 07° 24′ W.	
D. Lat. = 10′ N.		D. Long. = 3′ W.	
lat. Ship = 55° 32′ N.		Long. Ship = = 07° 27′ W.	
Co = 330°	Distance = 302 miles		

From Traverse Table:

D. Lat. = 261·5′ N.
= 4° 21·5′ N.
Lat. ship 2nd = 55° 32·0′ N.
D.R. Lat. 3rd = 59° 53·5′ N.

m. pts. Lat. from = 4004·8
m.pts. lat. to = 4494·1
D.M.P. = 489·3

Referring to fig. 13·7:

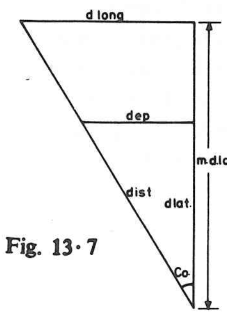


Fig. 13·7

D. Long. = D.M.P. tan Co
log D.M.P. = 2·68958
log tan Co = $\bar{1}$·76144
log D. Long. = 2·45102
D. Long. = 282·5′ W.
= 04° 42·5′ W.
Long. ship 2nd = 07° 27·0′ W.
D.R. Long. 3rd = 12° 09·5′ W.

D.R. Lat. 3rd = 59° 53·5′N.	D.R. Long. = 12° 09·5′ W.
Obs. Lat. 3rd = 59° 50·0′ N.	Obs. Long. = 11° 54·0′ W.
D. Lat. = 3·5′ S.	D. Long. = 15·5′ E.
	Dep. = 7·8′ E.

From Traverse Table:

Set = S. 66° E. Drift = 8·6 miles

Obs. Lat.2nd = 55° 32′ N.	m.pts. = 4004·8	Obs. Long. = 07° 27′ W.
Obs. Lat. 3rd = 59° 50′ N.	m.pts. = 4487·2	obs. Long. = 11° 54′ W.
D. Lat. = 4° 18′ N.	D.M.P. = 482·4	D. Long. = 4° 27′ W.
= 258′N.		= 267′ W

$$\tan Co = \frac{\text{D. Long.}}{\text{D.M.P.}}$$

log D. Long. = 2·42651
log D.M.P. = 2·68341

log tan Co = $\bar{1}$·74310

Co = N. 28° 58′ W.

Distance = D. Lat. sec Co.
log D. Lat. = 2·41162
log sec Co = 0·05804

log distance = 2·46966

Distance = 295 miles

$$\text{Average Speed} = \frac{\text{Distance}}{\text{Stmg. Time}}$$

log Distance = 2·46966
log Stmg. Time = 1·38021

log Speed = 1·08945

Average Speed = 12·29 knots

Answers—Course Made Good = 331°
Speed Made Good = 12·29 knots
Set = 114°
Drift = 8·6 miles.

Exercises on Chapter 13

1. Explain how the Engine Distance as found from Counter readings may be used to estimate the ship's position.
2. If the slip is estimated to be 2·5% and the Engine Distance is 346 miles, find the Vessel's Distance.
3. Find the Course and Distance from a position off the North Andaman, in Lat. 13° 05′ N. Long. 92° 10′ E. to a position off Trincomalee in lat. 08° 20′ N. Long. 82° 00′ E.

4. A vessel travelled 820 miles on a course of 060°, from Lat. 40° 00′ N. Long. 40° 00′ W. Find her present position.
5. A vessel left a position off the Lizard in Lat. 49° 50′ N. Long. 05° 10′ W. and travelled for a distance of 800 miles on a Course of 230°. What was the bearing and distance of Fayal in Lat. 38° 32′ N. Long. 28° 40′ W. after making this run?
6. A vessel sailed from a position in Lat. 40° 00′ N. Long. 60° 00′ W. and arrived in a position Lat. 42° 00′ N. Long. 20° 00′ W. Find the Course and Distance.
7. A vessel left a position in Lat. 00° 50′ S. Long. 91° 00′ W. (off Galapagos Islands) and travelled on a course of 136° until her Latitude was 12° 30′ S. What distance did she travel and what is her present Longitude assuming that the current set 316° throughout.
8. Find the present Latitude and the distance travelled since leaving a position in Lat. 26° 00′ S. Long. 109° 30′ W., if the course had been 040° and the present Longitude is 90° 30′ W.
9. Find the course and distance from a position in Lat. 05° 30′ S. Long. 32° 06′ W. to a position in lat. 08° 14′ N. Long. 25° 00′ W.
10. A vessel left a position in Lat. 385 40′ N. Long. 09° 00′ E. and travelled a distance of 120 miles on course of 130°. What was the bearing and distance of Cape Bon in Lat. 37° 06′ N. Long. 11° 05′ E. at the end of the run?
11. A vessel left a position in Lat. 42° 40′ N. Long. 170° 20′ E. and travelled 705 miles on a course of 112°. Find her position at the end of this run.
12. At noon, Tuskar Rock in Lat. 52° 12′ N. Long. 06° 12′ W. bore 042° at a distance of 12·0 miles. The log was set to zero and the course was set to 200°. At 1430 hours, log 25, the course was altered to 242°. At 2000 hours, log 79, the course was again altered to 260°. Find the Estimated Position of the vessel for midnight when the log registered 117, allowing for a current the average set and rate of which was estimated to be 100°, 1·2 knots.
13. A vessel found her position to be Lat. 31° 10′ N. Long. 72° 22′ W. She travelled 120 miles on a course of 340°, and then for 32 miles on a course of 320°, when her position was found to be Lat. 33° 15′ N. Long. 73° 25′ W. Find the set and rate of the current experienced.
14. A vessel took Departure with Cape St. Mary in Lat. 25° 39′ S. Long. 45° 06′ E., bearing 328° distance 12·0 miles. Course was set to 142°. What was the ship's E.P. when the log registered 498, assuming that the current had set 090° for 28 miles during the interval.
15. A vessel found that she was in Lat. 42° 06′ S. Long. 50° 15′ W. at noon. The log was set to zero and the course was set to 042°. At the following noon, when the log registered 345, the position was found to be in Lat. 37° 54′ S. Long. 45° 12′ W. Find the set and rate of the current.

CHAPTER 14

GREAT CIRCLE SAILING

1. Introduction

The advantage of rhumb line sailing is that the Course Angle is constant. To travel along a rhumb line path, the navigator simply joins the points of departure and destination with a straight line on the Mercator Chart. He then measures the Course Angle: this being the inclination of the parallel rulers with the projected meridians. The course is set and, if it is maintained during the voyage, the vessel will fetch up at her desired destination. The principal disadvantage of employing rhumb line sailing is that the path does not coincide with the shortest route between the points of departure and destination.

The shortest route over the Earth's surface between two terrestrial points is along the shorter arc of the great circle on which the two points lie. Now the angle which is a great circle track makes with the meridians constantly changes, except in those special cases when the great circle track is also a rhumb line track. Thus, if it is desired to travel along the shortest route from one place to another, the great circle path must be followed. In this event the course must be constantly changed as the voyage proceeds. For long ocean voyages the great circle path should be followed, when it is safe and practicable to do so, in the interests of economy.

The differences between the distances along the rhumb line and great circle arc between any two places depend upon the:

(1) distance between the places
(2) D. Long. between the places
(3) Latitudes of the places.

A considerable difference results when the D. Long. between the two places is great and the Latitudes of the places are high. If the D. Long. is small the rhumb line path, which lies almost due North or South, almost coincides with the great circle path. If the Latitudes of the two places are low the rhumb line path lies almost along the equator in which case it approximates to the great circle path.

In order to measure courses along a great circle route it is necessary to plot the route on a Mercator Chart. Because of the distortion of the Mercator Chart, the great circle track between any two places not on the equator or on the same meridian, is projected as a curved line concave to the equator. The rhumb line track appears as a straight line, and the two tracks together give the false impression that the rhumb line distance is shorter than that of the great circle. On a terrrestrial globe, however, it is readily seen that great circle tracks are shorter than corresponding rhumb line tracks.

Notice in figs. 14·1 and 14·2 that the rhumb line makes a constant angle θ with every meridian it crosses, whereas the direction of the great circle changes constantly.

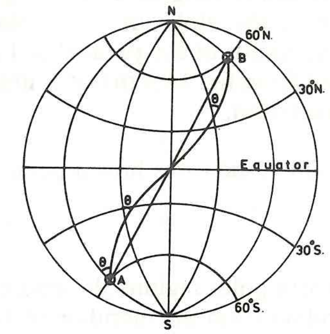


Fig. 14·1

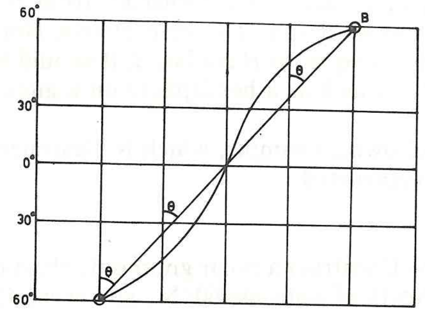


Fig. 14·2

On the globe, as depicted in fig. 14·1, the rhumb line appears as a curved line. The great circle arc, in contrast, appears as a straight line when viewed in its plane.

It will be noticed in fig. 14·1 that the rhumb line AB and the great circle arc AB intersect at the equator. This is the case only when the Latitudes of A and B have the same magnitude but opposite signs.

On a Mercator Chart, as depicted in fig. 14·2, the rhumb line AB appears as a straight line, whereas the great circle arc between A and B appears as a curved line.

Great sailing routes may be used to great advantage for many ocean passages, and for some coastal passages too. A good example of a suitable great circle coastal track is the route along the eastern seaboard of North America between Long Island and Florida. To ascertain which is the better route to choose when a choice presents itself, a Gnomonic Chart will be found to be of great use. A gnomonic chart is constructed on the Gnomonic Projection. Its most important feature is that great circle arcs are projected as straight lines.

2. The Gnomonic Chart

In the gnomonic projection, the sphere's surface is projected outwards from the sphere's centre onto a plane which is tangential to the sphere. The tangential point may be at the North or South Pole of the globe, in which case the resulting projection is called a Polar Gnomonic; or it may be at a point on the equator of the globe, in which case the projection is called a Transverse or Equatorial Gnomonic. If the tangential point is at a position other than the pole or a point on the equator (and this is generally the case with gnomonic charts used on board ship), the projection is called an Oblique Gnomonic.

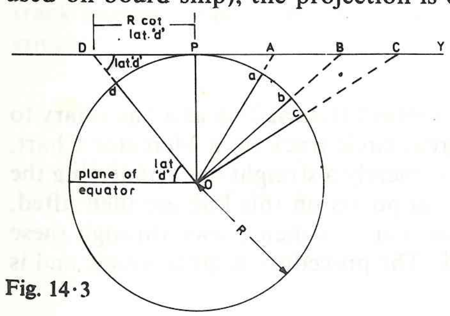


Fig. 14·3

On a polar gnomonic projection all meridians appear as straight lines which intersect, like the spokes of a bicycle wheel, at the projection of the Earth's Pole. Parallels of Latitude apppear as concentric circles centred at the projection of the pole. The radius of any parallel is proportional to the cotangent of the Latitude of the parallel. A polar gnomonic projection is easily constructed as illustrated in figs. 14·3 and 14·4.

In fig. 14·3, the plane of a polar gnomonic projection is tangential to the globe at *P*. Points *a, b, c* and *d*, are projected from the centre of the globe onto the plane of the projection at *A, B, C* and *D*, respectively. Note that the radius of the parallel of Latitude on which *d* lies is equal to *R* cot lat. *d*. It should be evident from fig. 14·3 that it is impossible to represent as much as a hemisphere on a gnomonic projection.

The following example, which is illustrated by fig. 14·4, indicates how a polar gnomonic chart is constructed.

Example—Construct a polar gnomonic chart of the North polar regions showing every tenth parallel North of Latitude 60° N., and every 45th meridian from the meridian of Greenwich.

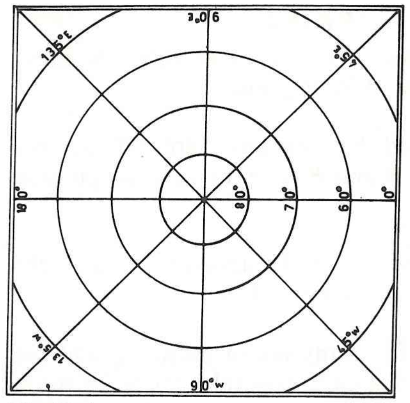


Fig. 14·4

Let the radius of the model globe from which the projection is made = 1 unit

Then:

Radius of projected parallel of Lat. 60° = 1 . cot 60° = 0·577 units

Radius of projected parallel of Lat. 70° = 1 . cot 70° = 0·346 units

Radius of projected parallel of Lat. 80° = 1 . cot 80° = 0·176 units

Fig. 14·4 illustrates the required projection.

A gnomonic chart having the tangential point at any position other than the Earth's Pole, is more difficult to construct than the polar gnomonic. In such a case the parallels of latitude, except the equator, are projected as hyperbolae.

A gnomonic chart is not suitable for measuring distances and courses. Position, however, may be lifted with ease.

3. Practical Great Circle Sailing

A gnomonic chart has great value in the practice of Great Circle Sailing as an auxiliary to a Mercator Chart. When it is desired to lay down a great circle track on a Mercator Chart, the track is first laid down on a gnomonic chart which is merely a straight line connecting the points of departure and destination. Positions of several points on this line are then lifted, and these are transferred to the Mercator Chart. A fair curve is then drawn through these plotted points: this being the required great circle track. The procedure is quite simple and is illustrated in figs. 14·5 and 14·6.

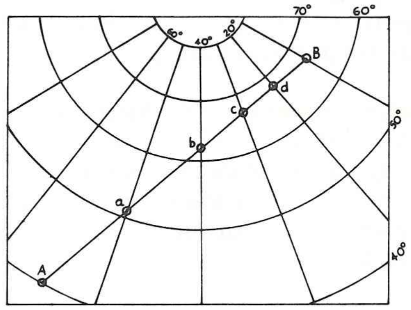


Fig. 14·5

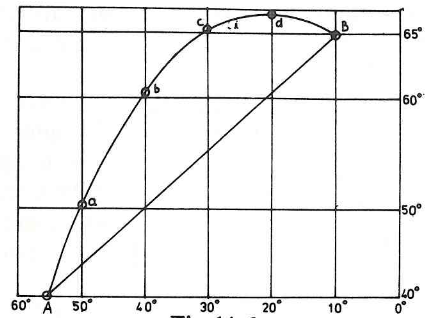


Fig. 14·6

It will be noticed in figs, 14·5 and 14·6 that the nearest approach to the pole of the great circle track *AB* is at position *d*. At this point the track cuts the meridian at an angle of 90°. To the eastwards of the point *d*, the course at any point on the track is South-Easterly, and to the westwards of point *d* the course is North-Easterly. At the point of highest Latitude on a great circle track the course is due East or due West.

The point on a great circle track at which the course changes from Northerly to Southerly, or from Southerly to Northerly, is known as a Vertex of the great circle. The course at the vertex is due East or due West. Every great circle has two vertices, one in the northern hemisphere and the other in the southern hemisphere. The vertices of a great circle are antipodal points.

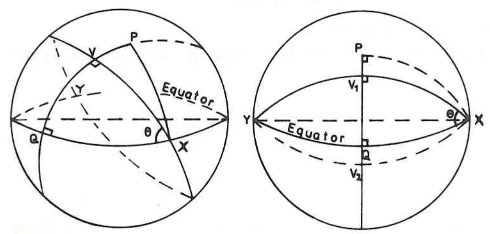


Fig. 14·7

Fig. 14·7 illustrates two views of a great circle which crosses the equator at the points *X* and *Y*. The vertices of this great circle are at V_1 and V_2.

Because the meridian of the vertex crosses the great circle track and the equator at an angle of 90°, the triangle XV_1Q is isosceles, and the arc *XV* is, therefore, equal to the arc *XQ*.

But arc $XV_1 =$ arc $V_1Y = 90°$

Therefore: arc $XQ = 90°$

Also, arc $VQ =$ angle VXQ

It follows that the Latitude of either vertex is equal to the angle at which the great circle track crosses the equator. The angle is equal to the complement of the Course Angle at the equator.

Latitude of Vertex = Complement of Course Angle at Equator

It is also evident from fig. 14·7 that the Longitude of the vertex differs by 90° from the Longitude of either of the two points where the greater circle track crosses the equator.

To steer along a great circle would render it necessary for the course to be continually altered. This is impracticable, if not impossible. In practice, when employing Great Circle

Sailing, the course is altered frequently, and the ship, therefore, is steered along a series of short rhumb line tracks which, collectively, approximate to the great circle track. This method of sailing is sometimes called Approximate Great Circle Sailing.

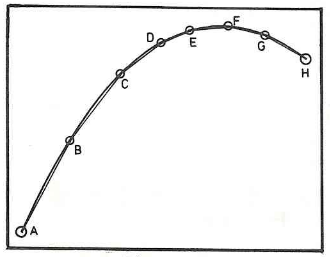


Fig. 14·8

In fig. 14·8, which illustrates part of a Mercator Chart, the straight lines *AB, BC, CD,* etc., represent rhumb line tracks which approximate to the great circle track from *A* to *H*. Where the curve of the great circle track is most pronounced, as at arc *CG*, the course is altered more frequently than at other parts of the track.

The general practice in great circle sailing is to find, from a gnomonic chart or otherwise, the new initial course every time the ship's position is found, and to alter heading if necessary, to this course.

In the event of a gnomonic chart not being available the positions of points on the great circle track, which are needed for plotting the track on the Mercator Chart, may be found by means of Azimuth or *A B C* tables. Short Method navigation tables may also be employed for this purpose; and, in the last resort, the positions may be computed using spherical trigonometry. The trigonmetrical method of solution, although providing useful practice at computation, is seldom used at sea. The method is:

(1) Find the Great Circle Distance using the Haversine Formula for a side.
(2) Find the initial course using the Haversine Formula for an angle.
(3) Find the position of the vertex by means of Napier's Rules.
(4) Calculate the Latitudes of points whose Longitudes differ by regular amounts from the Longitude of the vertex.

The following example illustrates the method.

Example 14·2—Find the Great Circle Distance and the Initial Course from a position in Lat. 51° 10′N. Long. 10° 00′W. (off S.W. Ireland) to a position off Belle Isle in Lat. 52° 00′N. Long. 55° 00′W. Calculate the Latitudes of points on the path whose Longitudes differ by multiples of 10° from the Longitude of the initial position.

In Fig. 14·9:

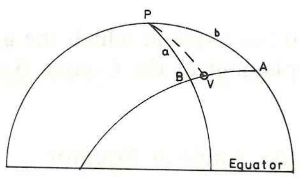


Fig. 14·9

A represents the Initial Position
B represents the Destination
P represents the Earth's North Pole
V represents the Northern Vertex

Given:

a = co. Lat. B = 38° 00′
b = co. Lat. A = 38° 50′
$\hat{P}$ = D. Long. AB = 45° 00′

To find p, the Great Circle Distance:

$$\text{hav } p = \text{hav } P \sin a \sin b + \text{hav } (a \sim b)$$

$$\begin{aligned} \log \text{hav } P &= \bar{1}{\cdot}16568 \\ \log \sin a &= \bar{1}{\cdot}78934 \\ \log \sin b &= \bar{1}{\cdot}79731 \\ \log \text{hav } \theta &= \bar{2}{\cdot}75233 \\ \text{nat hav } \theta &= 0{\cdot}05654 \\ \text{nat hav } (a \sim b) &= 0{\cdot}00005 \\ \text{nat hav } p &= 0{\cdot}05659 \qquad p = 27° \, 30' \end{aligned}$$

Distance = 1650 miles

To find A, the Initial Course:

$$\text{hav } A = \{\text{hav } a - \text{hav } (b \sim p)\} \text{ cosec } b \text{ cosec } p$$

$$\begin{aligned} \text{nat hav } a &= 0{\cdot}10599 \\ \text{nat hav } (b \quad p) &= 0{\cdot}00975 \\ \text{nat hav } \theta &= 0{\cdot}09624 \\ \log \text{hav } \theta &= \bar{2}{\cdot}98336 \\ \log \text{cosec } b &= 0{\cdot}20269 \\ \log \text{cosec } p &= 0{\cdot}33559 \\ \log \text{hav } A &= \bar{1}{\cdot}52164 \qquad A = 70° \, 24{\cdot}8' \end{aligned}$$

Initial Course = 289½°

To find x, the co-Lat of the vertex:

(refer to fig. 14·10)

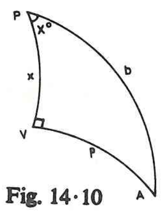


Fig. 14·10

$$\begin{aligned} \sin x &= \cos \text{co } b \cos \text{co } A \\ \sin x &= \sin b \sin A \\ \log \sin b &= \bar{1}{\cdot}79731 \\ \log \sin A &= \bar{1}{\cdot}97412 \\ \log \sin x &= \bar{1}{\cdot}77143 \qquad x = 36° \, 13' \end{aligned}$$

Lat V = 53° 47′