Springer Undergraduate Mathematics Series

S
U
M
S

Seán Dineen

Multivariate Calculus and Geometry

Third Edition

Springer

Springer Undergraduate Mathematics Series

For further volumes:
http://www.springer.com/series/3423

Seán Dineen

Multivariate Calculus and Geometry

Third Edition

Springer

Seán Dineen
School of Mathematical Sciences
University College Dublin
Dublin
Ireland

ISSN 1615-2085 ISSN 2197-4144 (electronic)
ISBN 978-1-4471-6418-0 ISBN 978-1-4471-6419-7 (eBook)
DOI 10.1007/978-1-4471-6419-7
Springer London Heidelberg New York Dordrecht

Library of Congress Control Number: 2014936739

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To my four godchildren
Anne-Marie Dineen, Donal Coffey,
Kevin Timoney, Eoghan Wallace

Preface to First Edition

The importance assigned to accuracy in basic mathematics courses has, initially, a useful disciplinary purpose but can, unintentionally, hinder progress if it fosters the belief that exactness is all that makes mathematics what it is. *Multivariate calculus* occupies a pivotal position in undergraduate mathematics programmes in providing students with the opportunity to outgrow this narrow viewpoint and to develop a flexible, intuitive and independent vision of mathematics. This possibility arises from the extensive nature of the subject.

Multivariate calculus links together in a non-trivial way, perhaps for the first time in a student's experience, four important subject areas: *analysis*, *linear algebra*, *geometry* and *differential calculus*. Important features of the subject are reflected in the variety of alternative titles we could have chosen, e.g. "Advanced Calculus", "Vector Calculus", "Multivariate Calculus", "Vector Geometry", "Curves and Surfaces" and "Introduction to Differential Geometry". Each of these titles partially reflects our interest but it is more illuminating to say that here we study differentiable functions, i.e.

functions which enjoy a good local approximation by linear functions.

The main emphasis of our presentation is on understanding the underlying fundamental principles. These are discussed at length, carefully examined in simple familiar situations and tested in technically demanding examples. This leads to a structured and systematic approach of manageable proportions which gives shape and coherence to the subject and results in a comprehensive and unified exposition.

We now discuss the four underlying topics and the background we expect—bearing in mind that the subject can be approached with different levels of mathematical maturity. Results from *analysis* are required to justify much of this book, yet many students have little or no background in analysis when they approach multivariate calculus. This is not surprising as differential calculus preceded and indeed motivated the development of analysis. We do not list analysis as a prerequisite, but hope that our presentation shows its importance and motivates the reader to study it further.

Since linear approximations appear in the definition of differentiable functions, it is not surprising that *linear algebra* plays a part in this book. Several-variable calculus and linear algebra developed, to a certain extent, side by side to their mutual benefit. The primary role of linear algebra, in our study, is to provide a suitable notation and framework in which we can clearly and compactly introduce concepts and present and prove results. This is more important than it appears since to quote T. C. Chaundy, *"notation biases analysis as language biases thought"*. An elementary knowledge of matrices and determinants is assumed and particular results from linear algebra are introduced as required.

We discuss the role of *geometry* in multivariate calculus throughout the text and confine ourselves here to a brief comment. The natural setting for *functions* which enjoy a good local approximation by *linear functions* are *sets* which enjoy a good local approximation to *linear spaces*. In one and two dimensions this leads to *curves* and *surfaces*, respectively, and in higher dimensions to *differentiable manifolds*.

We assume the reader has acquired a reasonable knowledge of *one-variable differential and integral calculus* before approaching this book. Although not assumed, some experience with partial derivatives allows the reader to proceed rapidly through routine calculations and to concentrate on important concepts. A reader with no such experience should definitely read Chapter 1 a few times before proceeding and may even wish to consult the author's *Functions of Two Variables* (Chapman and Hall 1995).

We now turn to the contents of this book. Our general approach is holistic and we hope that the reader will be equally interested in all parts of this book. Nevertheless, it is possible to group certain chapters thematically.

Differential Calculus on Open Sets and Surfaces (Chapters 1–4).

We discuss extremal values of real-valued functions on surfaces and open sets. The important principle here is the *Implicit Function Theorem*, which links linear approximations with systems of linear equations and sets up a relationship between graphs and surfaces.

Integration Theory (Chapters 6, 9, 11–15).

The key concepts are parameterizations (Chapters 5, 10 and 14) and oriented surfaces (Chapter 12). We build up our understanding and technical skill step by step, by discussing in turn line integrals (Chapter 6), integration over open subsets of $\mathbb{R}^2$ (Chapter 9), integration over simple surfaces without orientation (Chapter 11), integration over simple oriented surfaces (Chapter 12) and triple integrals over open subsets of $\mathbb{R}^3$(Chapter 14). At appropriate times we discuss generalizations of the fundamental theorem of calculus, i.e. *Green's Theorem* (Chapter 9), *Stokes' Theorem* (Chapter 13) and *the Divergence Theorem* (Chapter 15). Special attention is given to the parameterization of classical surfaces, the evaluation of surface integrals using projections, the change of variables formula and to the detailed examination of involved geometric examples.

Geometry of Curves and Surfaces (Chapters 5, 7–8, 10, 16–18).

We discuss signed curvature in $\mathbb{R}^2$ and use vector-valued differentiation to obtain the Frenet–Serret equations for curves in $\mathbb{R}^3$. The abstract geometric study of surfaces using Gaussian curvature is, regrettably, usually not covered in multivariate calculus courses. The fundamental concepts, parameterizations and plane curvature, are already in place (Chapters 5, 7 and 10) and examples from integration theory (Chapters 11–15) provide a concrete background and the required geometric insight. Using only curves in $\mathbb{R}^2$ and critical points of functions of two variables we develop the concept of Gaussian curvature. In addition, we discuss normal, geodesic and intrinsic curvature and establish a relationship between all three. In the final chapter we survey informally a number of interesting results from differential geometry.

This text is based on a course given by the author at University College, Dublin. The additions that emerged in providing details and arranging self-sufficiency suggest that it is suitable for a course of 30 lectures. Although the different topics come together to form a unified subject, with different chapters mutually supporting one another, we have structured this book so that each chapter is self-contained and devoted to a single theme.

This book can be used as a main text, as a supplementary text or for self-study. The groupings summarised above allow a selection of short courses at a slower pace. The exercises are extremely important as it is through them that a student can assess progress and understanding.

Our aim was to write a short book focusing on basic principles while acquiring technical skills. This precluded comments on the important applications of multivariate calculus which arise in physics, statistics, engineering, economics and, indeed, in most subjects with scientific aspirations.

It is a pleasure to acknowledge the help I received in bringing this project to fruition. Dana Nicolau displayed skill in preparing the text and great patience in accepting with a cheerful “OK, OK,” the continuous stream of revisions, corrections and changes that flowed her way. Michael Mackey’s diagrams speak for themselves. Brendan Quigley’s geometric insight and Pauline Mellon’s suggestions helped shape our outlook and the text. I would like to thank the Third Arts students at University College, Dublin, and especially Tim Cronin and Martin Brundin for their comments, reactions and corrections. Susan Hezlet of Springer provided instantaneous support, ample encouragement and helpful suggestions at all times. To all these and the community of mathematicians whose results and writings have influenced me, I say—thank you!

Department of Mathematics,
University College Dublin,
Belfield, Dublin 4,
Ireland.

Preface to Third Edition

Fifteen years have elapsed since the first edition was published and 5 years have gone by since I last taught a course on the topics in this book. It is nice to know that Springer still believes that new generations of teachers and students may still be interested in my approach and I am grateful to them for allowing me the opportunity to correct some errors, to revise some material, and to pass on to new readers comments of previous readers. I have, I hope, maintained the style, format, general approach and the results of previous editions. I have made changes in practically all chapters but the main changes occur in the final three chapters, which is an introduction to the differential geometry of surfaces in three-dimensional space. And now some important information which was not sufficiently stressed in earlier prefaces: as preparation to using this book readers should have completed a course in linear algebra and a first course on partial differentiation. Chapter 1 in this book is a summary of material that is presumed known and an introduction to notation that we use throughout the book.

It is a pleasure to thank Michael Mackey for his continued support and practical and mathematical help in preparing this edition. Joerg Sixt and Catherine Waite from Springer have been supportive and efficient throughout the period of preparation of this edition.

Dublin, Ireland

Seán Dineen
e-mail: sean.dineen@ucd.ie

Contents

Chapter 1
Introduction to Differentiable Functions

Summary *We introduce differentiable functions, directional and partial derivatives, graphs and level sets of functions of several variables.*

In this concise chapter we introduce continuous and differentiable functions between arbitrary finite dimensional spaces. We pay particular attention to notation, as appropriate notation is often the difference between simple and complicated presentations of several-variable calculus. Once this is in place many of our calculations follow the same lines as in the one dimensional calculus. We do not include proofs but, for readers familiar with analysis, we provide suggestions that lead to proofs along the lines that apply in the one variable calculus.

The following extremely simple example illustrates the type of calculation we will be executing frequently and the reader should practice similar examples until they become routine and the intermediate step is unnecessary.

Example 1.1 Let

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}, \ \ f(x, y, z) = xe^y + y^2z^3.$$

The *partial derivative* of f with respect to x, $\dfrac{\partial f}{\partial x}$ or f_x, is obtained by treating y and z as constants and differentiating with respect x in the usual one variable way. Thus if $A = e^y$ and $B = y^2z^3$ then $f(x, y, z) = Ax + B$ and

$$\frac{\partial f}{\partial x} = \frac{d}{dx}(Ax + B) = A = e^y.$$

Similarly if $C = x$ and $D = z^3$ then $f(x, y, z) = Ce^y + Dy^2$ and

$$\frac{\partial f}{\partial y} = \frac{d}{dy}(Ce^y + Dy^2) = Ce^y + 2Dy = xe^y + 2yz^3,$$

and, if $E = xe^y$ and $F = y^2$, then $f(x, y, z) = E + Fz^3$ and

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_1,

$$\frac{\partial f}{\partial z} = \frac{d}{dz}(E + Fz^3) = 3Fz^2 = 3y^2z^2.$$

We now recall concepts and notation from linear algebra. First we define the distance between vectors in $\mathbb{R}^n$. This will enable us to define convergent sequences, open and closed sets, continuous and differentiable functions, and state the fundamental existence theorem for maxima and minima.

If $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\|X\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and call $\|X\|$ the *length* (or *norm*) of X. If X and $Y = (y_1, \ldots, y_n)$ are vectors in $\mathbb{R}^n$ then $\|X - Y\|$ is the distance between X and Y. The *inner product* (or *dot product* or *scalar product*) of X and Y, $X \cdot Y$ or $\langle X, Y \rangle$, is defined as

$$\langle X, Y \rangle = X \cdot Y = \sum_{i=1}^{n} x_i y_i.$$

We have $\|X\|^2 = \langle X, X \rangle$ and two vectors X and Y are *perpendicular* if and only if their inner product is zero.

For $1 \leq j \leq n$, let $\mathbf{e}_j = (0, \ldots, 1, 0, \ldots)$, where 1 lies in the jth position. The set $(\mathbf{e}_j)_{j=1}^n$ is a *basis*, the standard *unit vector basis*,[1] for $\mathbb{R}^n$. If $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ then

$$X = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \sum_{i=1}^{n} x_i\mathbf{e}_i = \sum_{j=1}^{n} \langle X, \mathbf{e}_j \rangle \mathbf{e}_j.$$

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear if

$$T(aX + bY) = aT(X) + bT(Y)$$

for all $X, Y \in \mathbb{R}^n$ and all $a, b \in \mathbb{R}$. For $1 \leq i \leq m$ and $1 \leq j \leq n$ let $a_{i,j} = \langle T(\mathbf{e}_j), \mathbf{e}_i \rangle$. If $X = (x_1, \ldots, x_n)$ then, interchanging the order of summation, we obtain

$$\begin{aligned} T(X) = T\Big(\sum_{j=1}^{n} x_j\mathbf{e}_j\Big) &= \sum_{j=1}^{n} x_j T(\mathbf{e}_j) \\ &= \sum_{j=1}^{n} x_j \Big(\sum_{i=1}^{m} \langle T(\mathbf{e}_j), \mathbf{e}_i \rangle \mathbf{e}_i\Big) \\ &= \sum_{i=1}^{m} \Big\{ \sum_{j=1}^{n} x_j \langle T(\mathbf{e}_j), \mathbf{e}_i \rangle \Big\} \mathbf{e}_i \end{aligned}$$

[1] We use the same notation for the standard basis in $\mathbb{R}^n$ and $\mathbb{R}^m$. The context tells the dimension of the space involved. Otherwise we would be using more and more unwieldy notation.

$$= \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} a_{i,j} x_j \right\} \mathbf{e}_i .$$

This shows that $T(X) = A(X)$ where the $m \times n$ matrix $A = (a_{i,j})_{1 \le i \le m, 1 \le j \le n}$ operates on the *column* vector X by matrix multiplication. We may now identify the space of linear mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ with the space of $m \times n$ matrices, $\mathbb{M}_{m,n}$. To present in a reasonable form the product rule and chain rule (see below) we identify $\mathbb{R}^n$ with $\mathbb{M}_{n,1}$, that is the points in $\mathbb{R}^n$ are considered to be *column* vectors. The reader should however note that, for typographical convenience, this convention is often ignored and points in $\mathbb{R}^n$ are written as *row* vectors. However, in taking derivatives the correct convention should be followed to avoid meaningless expressions.

A subset U of $\mathbb{R}^n$ is *open* if for each $X_0 \in U$ there exists $\varepsilon > 0$ such that[2]

$$\{X \in \mathbb{R}^n : \|X - X_0\| < \varepsilon\} \subset U.$$

A subset A of $\mathbb{R}^n$ is *closed* if its complement is open and a set B is *bounded* if there exists $M \in \mathbb{R}$ such that $\|x\| \le M$ for all $x \in B$. Thus all points in an open set A are surrounded by points from A while any point that can be reached from a closed set B belongs to B. A crucial role in all aspects of calculus, analysis and geometry is played by sets which are both closed and bounded; such sets are said to be *compact*.

Example 1.2 The closed solid sphere $\{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$ and its boundary $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ are both compact subsets of $\mathbb{R}^3$ while the solid sphere without boundary $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ is an open bounded subset of $\mathbb{R}^3$. A line is a closed unbounded set while every open subset of $\mathbb{R}^3$ is a union of open spheres.

If $(X_k)_{k=1}^{\infty}$ is a sequence of vectors in $\mathbb{R}^n$ and $Y \in \mathbb{R}^n$ then we say that X_k converges to Y as k tends to infinity and write $X_k \longrightarrow Y$ as $k \longrightarrow \infty$ and $\lim_{k\to\infty} X_k = Y$ if

$$\|X_k - Y\| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

Convergence in $\mathbb{R}^n$ is thus reduced to convergence in $\mathbb{R}$. Moreover, if $X_k = (x_1^k, \ldots, x_n^k)$ and $Y = (y_1, \ldots, y_n)$ then

$$\lim_{k\to\infty} X_k = Y \Longleftrightarrow \lim_{k\to\infty} x_i^k = y_i, \quad 1 \le i \le n.$$

A function $F{:}U \subset \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $X_0 \in U$ if for each sequence $(X_k)_{k=1}^{\infty}$ in U

[2] The ε chosen will depend on X_0 and to be rigorous we should indicate this dependence in some way, e.g. by writing ε_{X_0}. This would lead to unnecessarily complicated notation. We hope that this simplification will not be the source of any confusion.

$$\lim_{k\to\infty} X_k = X_0 \Longrightarrow \lim_{k\to\infty} F(X_k) = F(X_0).$$

When this is the case we write $\lim_{X\to X_0} F(X) = F(X_0)$. If F is continuous at all points in U we say F is continuous on U.

We could also define, in different ways, a length function on $\mathbb{M}_{m,n}$, for instance by identifying $\mathbb{M}_{m,n}$ with $\mathbb{R}^{mn}$. Any such standard definition will be equivalent to the following: if $A^k \in \mathbb{M}_{m,n}$ for all k, $A^k = (a^k_{i,j})_{i,j}$ and $A = (a_{i,j})_{i,j}$ then

$$\lim_{k\to\infty} A^k = A \Longleftrightarrow \lim_{k\to\infty} a^k_{i,j} = a_{i,j}$$

for all (i, j), $1 \le i \le m$, $1 \le j \le n$. We may now define for positive integers l, m and n continuous mappings from $U \subset \mathbb{R}^l \longrightarrow \mathbb{M}_{m,n}$.

A function $f{:}A \subset \mathbb{R}^n \to \mathbb{R}$ has a *maximum* on A if there exists $X_1 \in A$ such that $f(X) \le f(X_1)$ for all $X \in A$. We call $f(X_1)$ the maximum of f on A and say that f achieves its maximum on A at X_1. The maximum, if it exists, is unique but may be achieved at more than one point. A point X_1 in A is called a *local maximum* of f on A if there exists $\delta > 0$ such that f achieves its maximum on $A \cap \{X : \|X - X_1\| < \delta\}$ at X_1. If, in addition, $f(X) < f(X_1)$ whenever $X \ne X_1$ we call X_1 a *strict local maximum*. Isolated local maxima are strict, i.e. if for some $\delta > 0$, X_1 is the only local maximum of f in $A \cap \{X : \|X - X_1\| < \delta\}$ then X_1 is a strict local maximum. In particular, if the set of local maxima of f is *finite* then all local maxima are strict local maxima. The analogous definitions of minimum, local minimum and strict local minimum are obtained by reversing the above inequalities.

Compact sets and continuity feature in the following *fundamental existence theorem for maxima and minima.*

Theorem 1.3 *A continuous real-valued function defined on a compact subset of* $\mathbb{R}^n$ *has a maximum and a minimum.*

The practical problem of finding maxima and minima often requires a degree of smoothness finer than continuity called differentiability. Continuity and differentiability of most functions we encounter can be verified by using functions from $\mathbb{R}$ into $\mathbb{R}$, addition, multiplication, composition of functions and linear mappings.

Definition 1.4 Let $F{:}U \subset \mathbb{R}^n \to \mathbb{R}^m$, U open, and let $X_0 \in U$. We say that F is differentiable at $X_0 \in U$ if there exists a function $A : U \longrightarrow \mathbb{M}_{m,n}$ which is continuous at X_0 such that

$$F(X) = F(X_0) + A(X)(X - X_0) \tag{1.1}$$

for all $X \in U$.

If $f{:}(a, b) \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ in the classical sense, that is if

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is finite, then letting

$$A(x) = \begin{cases} (f(x) - f(x_0))/(x - x_0), & \text{if } x \neq x_0, \\ f'(x_0), & \text{if } x = x_0, \end{cases}.$$

we see that f is also differentiable in the sense of Definition 1.4. By using (1.1) we see that the converse is also true.

The function A in Definition 1.4 is not necessary unique, However, if the matrix-valued functions A and B satisfy $F(X) = F(X_0) + A(X)(X - X_0)$ and $F(X) = F(X_0) + B(X)(X - X_0)$ and both are continuous at X_0 then

$$\lim_{X \to X_0} A(X) = \lim_{X \to X_0} B(X)$$

and we call the common value the derivative of F at X_0 and denote it by both $DF(X_0)$ and $F'(X_0)$. If f is a scalar-valued function we also write $\nabla f(X)$ in place of $f'(X)$ and call ∇f the *gradient* of f. Gradient is just another word for *slope* and, in one variable calculus the derivative is the slope of the tangent line to the graph of the function. Note that if $F\colon \mathbb{R}^n \to \mathbb{R}^m$ then $F'(X)$ is an $m \times n$ matrix (i.e. the order is reversed) and is identified with a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$. The term

$$F(X_0) + F'(X_0)(X - X_0)$$

is our *linear approximation* to $F(X)$ near X_0.

If F is differentiable then F is continuous. The sum, difference and scalar multiple of differentiable functions are differentiable and the product of scalar-valued and vector-valued differentiable functions are differentiable. We have (see also Exercise 1.7) the following formulae whenever $F, G : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ are differentiable, $a, b \in \mathbb{R}$, $A \in \mathbb{R}^m$, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear and $(A + T)(X) = A + T(X)$ for all $X \in \mathbb{R}^m$:

(i) $(aF + bG)'(X) = aF'(X) + bG'(X)$,

(ii) $(f \cdot G)'(X) = f(X) \cdot G'(X) + G(X) \circ \nabla f(X)$,

(iii) $(A + T)'(X) = T$.

Part (ii) above is the product rule for differentiation. Note that $\cdot$ denotes scalar multiplication, while $\circ$ denotes matrix multiplication of the $m \times 1$ matrix $G(X)$ and the $1 \times n$ matrix $\nabla f(X)$. Part (iii) tells us that the derivative of a linear function is constant and equal to itself at all points while the derivative of a constant function is 0 (in the appropriate space).

The composition of differentiable functions is again differentiable and the *chain rule*, (1.2), gives the derivative of the composition in an elegant form. Let

$$F : U \text{ (open)} \subset \mathbb{R}^n \to \mathbb{R}^m, \;\; G : V \text{ (open)} \subset \mathbb{R}^m \to \mathbb{R}^p$$

denote differentiable functions. If $X \in U$ and $F(X) \in V$ then $(G \circ F)'(X)$ exists and

$$D(G\circ F)(X) = (G\circ F)'(X) = G'(F(X))\circ F'(X) = DG(F(X))\circ DF(X). \quad (1.2)$$

If $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $X = (x_1, \ldots, x_n) \in U$ then $\nabla f(X)$ is a $1 \times n$ matrix, that is a row matrix with n entries. If $\mathbf{e}_i \in \mathbb{R}^n$ and $g_i(x) = X + x\mathbf{e}_i$ for all $x \in \mathbb{R}$ then g_i is differentiable and $g_i'(x) = [g_i'(x)](\mathbf{e}_1) = \mathbf{e}_i$ at all $x \in \mathbb{R}$. Since

$$f \circ g_i(x) = f(X + x\mathbf{e}_i) = f(x_1, \ldots, x_{i-1}, x_i + x, \ldots, x_n)$$

the composition $f \circ g_i : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable and

$$\begin{aligned}\frac{d(f\circ g_i)}{dx}(X) &= \lim_{x\to 0}\frac{f(x_1,\ldots,x_{i-1},x_i+x,\ldots,x_n) - f(x_1,\ldots,x_{i-1},x_i,\ldots,x_n)}{x}\\ &= \frac{\partial f}{\partial x_i}(X) = \nabla f(g(0))(g_i'(0)) = \nabla f(X)(\mathbf{e}_i).\end{aligned}$$

This shows that

$$\nabla f(X) = Df(X) = f'(X) = \left(\frac{\partial f}{\partial x_1}(X), \ldots, \frac{\partial f}{\partial x_n}(X)\right)$$

and, if $\omega = \sum_{j=1}^n \omega_j \mathbf{e}_j \in \mathbb{R}^n$,

$$\nabla f(X)(\omega) = \nabla f(X)\Big(\sum_{j=1}^n \omega_j \mathbf{e}_j\Big) = \sum_{j=1}^n \omega_j \frac{\partial f}{\partial x_j}(X).$$

If $F{:}U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ then $F(X)$ has m coordinates $f_1(X), \ldots, f_m(X)$ and we often write $F = (f_1, \ldots, f_m)$ where each f_i is a real-valued function of n variables. Hence, $F = \sum_{i=1}^m f_i\mathbf{e}_i$ and F is differentiable if and only if each f_i is differentiable. If $\omega = \sum_{j=1}^n \omega_j \mathbf{e}_j \in \mathbb{R}^n$ then,

$$F'(X)\Big(\sum_{j=1}^n \omega_j\mathbf{e}_j\Big) = \sum_{i=1}^m \Big\{\nabla f_i(X)\Big(\sum_{j=1}^n \omega_j\mathbf{e}_j\Big)\Big\}\mathbf{e}_i = \sum_{i=1}^m \Big\{\sum_{j=1}^n \omega_j \frac{\partial f_i}{\partial x_j}(X)\Big\}\mathbf{e}_i$$

and hence, for $1 \le j \le n, 1 \le i \le n$, we have $\langle F'(X)\mathbf{e}_j, \mathbf{e}_i\rangle = \dfrac{\partial f_i}{\partial x_j}(X)$ and $\dfrac{\partial f_i}{\partial x_j}(X)$ is the (i, j) entry in the $m \times n$ matrix $DF(X) = F'(X)$.

If $\mathbf{v} \in \mathbb{R}^n$ then the function $G : x \in \mathbb{R} \longrightarrow X + x\mathbf{v} \in \mathbb{R}^n$ is differentiable at any $X \in \mathbb{R}^n$ and, as above, $G'(x) = G'(x)(\mathbf{e}_1) = \mathbf{v}$ for all $x \in \mathbb{R}$. If $F : U\ (\text{open}) \in \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at X then $(F\circ G)'(0) = F'(G(0))\circ G'(0) =$

$F'(X)(\mathbf{v})$. We write $D_{\mathbf{v}}F(X)$ and $F_{\mathbf{v}}(X)$ in place of $D(F \circ G)(0)$ and call $D_{\mathbf{v}}F(X)$ the *directional derivative* of F at X in the direction $\mathbf{v}$. Since

$$F(X + x\mathbf{v}) = F(G(x)) = F(G(0)) + A(x)x = F(X) + A(x)x$$

where $\lim_{x\to 0} A(x) = D_{\mathbf{v}}F(X)$ we have

$$D_{\mathbf{v}}F(X) = \lim_{x\to 0} \frac{F(X + x\mathbf{v}) - F(X)}{x}. \tag{1.3}$$

If $\mathbf{v} = \mathbf{e}_i$ we write $\dfrac{\partial F}{\partial x_i}(X)$ and $F_{x_i}(X)$ in place of $D_{\mathbf{e}_i}F(X)$ and call these the (first-order) *partial derivatives* of F.

If F is differentiable at X then all (first-order) directional and partial derivatives of F exist and we have shown

$$F'(X) = \begin{pmatrix} \nabla f_1(X) \\ \vdots \\ \nabla f_m(X) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(X) & \dots & \dfrac{\partial f_1}{\partial x_n}(X) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(X) & \dots & \dfrac{\partial f_m}{\partial x_n}(X) \end{pmatrix} = \big(F_{x_1}(X), \dots, F_{x_n}(X)\big).$$

If $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ this shows that

$$D_{\mathbf{v}}F(X) = F'(X)(\mathbf{v}) = \sum_{i=1}^{n} v_i F_{x_i}(X) = \sum_{i=1}^{n} v_i \frac{\partial F}{\partial x_i}(X).$$

If all first-order partial derivatives of F exist and are continuous then F is differentiable. This criterion enables us to see, literally at a glance, the sets on which most functions have derivatives, partial derivatives and directional derivatives. Consider, for instance, the function

$$F(x, y, z) = \big(\sin(xyz), x^2 - y^2, \exp(xy)\big).$$

We have

$$F(x, y, z) = \sin(xyz)\mathbf{e}_1 + (x^2 - y^2)\mathbf{e}_2 + \exp(xy)\mathbf{e}_3$$

and it suffices to consider separately the three $\mathbb{R}$-valued functions $\sin(xyz)$, $x^2 - y^2$ and $\exp(xy)$. Since linear functions are continuous and the composition and product of continuous functions are continuous we see, on calculating the partial derivatives, as in Example 1.1, that all the first order partial derivatives exist and are continuous. Hence F is differentiable.

Higher-order directional and partial derivatives are defined in the usual way, i.e. by repeated differentiation. If $F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable then each *row* in $F'(X)$ corresponds to a *coordinate function* of F while each *column* corresponds to an (independent) *coordinate variable*.

Example 1.5 Let $F:\mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$F(x, y, z, w) = (x^2 y, xyz, x^2 + y^2 + zw^2).$$

Then $F = (f_1, f_2, f_3)$ where $f_1(x, y, z, w) = x^2 y$, $f_2(x, y, z, w) = xyz$ and $f_3(x, y, z, w) = x^2 + y^2 + zw^2$. Moreover, $\nabla f_1(x, y, z, w) = (2xy, x^2, 0, 0)$, $\nabla f_2(x, y, z, w) = (yz, xz, xy, 0)$ and $\nabla f_3(x, y, z, w) = (2x, 2y, w^2, 2zw)$. Hence

$$F'(x, y, z, w) = \begin{pmatrix} 2xy & x^2 & 0 & 0 \\ yz & xz & xy & 0 \\ 2x & 2y & w^2 & 2zw \end{pmatrix}.$$

If $X = (1, 2, -1, -2)$ and $\mathbf{v} = (0, 1, 2, -2)$ then

$$D_{\mathbf{v}} F(X) = F'(X) \circ {}^t\mathbf{v} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

Associated with any function $F:U \subset \mathbb{R}^n \to \mathbb{R}^m$ we have two types of sets which play a special role in the development of the subject—graphs and level sets. The *graph* of F is the subset of $\mathbb{R}^{n+m}$ defined as follows

$$\begin{aligned} \text{graph}\,(F) &= \big\{(X, Y) : X \in U \text{ and } Y = F(X)\big\} \\ &= \big\{(X, F(X)) : X \in U\big\}. \end{aligned}$$

If $C = (c_1, \ldots, c_m)$ is a point in $\mathbb{R}^m$ we define the *level set* of F, $F^{-1}(C)$, by the formula

$$F^{-1}(C) = \big\{X \in U : F(X) = C\big\}.$$

In terms of the coordinate functions $f_1, \ldots, f_m$ of F we have

$$F(X) = C \iff f_i(X) = c_i \quad \text{for } i = 1, \ldots, m$$

and hence

$$F^{-1}(C) = \bigcap_{i=1}^{m} \big\{X \in U : f_i(X) = c_i\big\} = \bigcap_{i=1}^{m} f_i^{-1}(c_i).$$

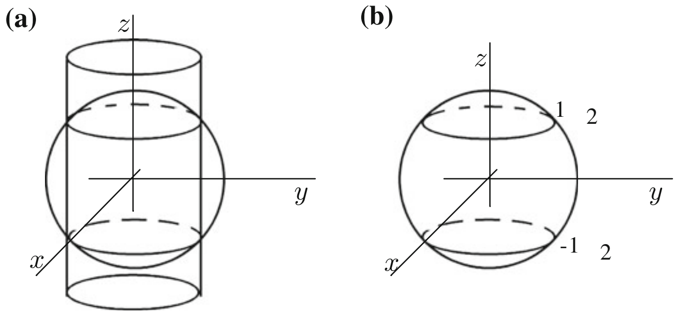


Fig. 1.1

Thus a level set of a vector-valued function is the finite intersection of level sets of real-valued functions. This is frequently useful in arriving at a geometrical interpretation of level sets as the following example shows.

Example 1.6 Let $F(x, y, z) = (x^2 + y^2 + z^2 - 1, 2x^2 + 2y^2 - 1)$. We have $F = (f_1, f_2)$ where $f_1(x, y, z) = x^2+y^2+z^2-1$ and $f_2(x, y, z) = 2x^2+2y^2-1$. The set $f_1^{-1}(0)$ is a *sphere* of radius 1 while $f_2^{-1}(0)$ is a *cylinder* parallel to the z–axis built on a circle with centre the origin and radius $1/\sqrt{2}$. If $\mathbf{0} = (0, 0)$ is the origin in $\mathbb{R}^2$ then

$$F^{-1}(\mathbf{0}) = f_1^{-1}(0) \cap f_2^{-1}(0)$$

is the intersection of a sphere and a cylinder in $\mathbb{R}^3$ (Fig. 1.1a).

For this particular example we obtain more information by solving the equations $f_1(x, y, z) = f_2(x, y, z) = 0$. We have $x^2 + y^2 = 1 - z^2 = 1/2$. Hence $z^2 = 1/2$, $z = \pm 1/\sqrt{2}$ and the level set consists of two circles on the unit sphere (Fig. 1.1b).

The relationship between graphs and level sets plays an important role in our study. The easy part of this relationship—every graph is a level set—is given in the next example while the difficult part—every (regular) level set is locally a graph—is the *implicit function theorem* (Chap. 2).

Example 1.7 Let $F{:}U \subset \mathbb{R}^n \to \mathbb{R}^m$. We define $G{:}U \times \mathbb{R}^m \to \mathbb{R}^m$ by $G(X, Y) = F(X) - Y$. If $\mathbf{0}$ is the origin in $\mathbb{R}^m$ then

$$\begin{aligned}(X, Y) \in G^{-1}(\mathbf{0}) &\iff G(X, Y) = \mathbf{0}\\ &\iff F(X) - Y = \mathbf{0}\\ &\iff (X, Y) \in \text{graph}\,(F).\end{aligned}$$

Hence $G^{-1}(\mathbf{0}) = \text{graph}\,(F)$ and every graph is a level set.

Exercises

1.1. Sketch the following sets

(a) $\left\{(x, y, z) : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} < 1\right\}$
(b) $\{(x, y, z) : x^2 + y^2 = z^2\}$
(c) $\{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0\}$
(d) $\{(x, y, z) : x^2 + y^2 + z^2 - 2z = 0\}$
(e) $\{(x, y, z) : x^2 + y^2 + z^2 - 4z = 0,\ x^2 + y^2 = 4\}$.

From your sketches determine which of the sets are: open, closed, bounded, compact.

1.2. Find all first-order partial derivatives of

(a) $f(x, y, z) = (z^2 + x^2)\log(1 + x^2y^2)$
(b) $g(x, y, z) = xy\tan^{-1}(xz)$
(c) $h(x, y, z, w) = \dfrac{\sin(x^2 + y^2 + z^2 + w^2)}{1 + (x - y)^2}$.

1.3. If $F(x, y, z, w) = (x^2 - y^2, 2xy, zx, z^2w^2x^2)$ and $\mathbf{v} = (2, 1, -2, -1)$ find $F'(1, 2, -1, -2)$ and $D_{\mathbf{v}}F(1, 2, -1, -2)$.
1.4. Let $f(x, y, z) = x^2 - xy + yz^3 - 6z$. Find all points (x, y, z) such that $\nabla f(x, y, z) = (0, 0, 0)$.
1.5. If $f(x, y, z) = x^2e^y$ and $g(x, y, z) = y^2e^{xz}$ find ∇f, ∇g and $\nabla(fg)$. Verify that $\nabla(fg) = f\nabla g + g\nabla f$.
1.6. Let $P\colon (a, b) \subset \mathbb{R} \to \mathbb{R}^n$. Show that

$$\frac{d}{dt}\left(\|P(t)\|^2\right) = 2P(t) \circ P'(t) = 2\langle P(t), P'(t)\rangle$$

and deduce that if $\|P(t)\|$ does not depend on t then $P'(t) \perp P(t)$.
1.7. If $F(x, y, z) = (x^2, y^2 + z^2, xyz)$ and $G(x, y, z) = (e^x, y^2 - z^2, xyz)$ find $F'(x, y, z)$ and $G'(x, y, z)$. Let $H(x, y, z) = \langle F(x, y, z), G(x, y, z)\rangle$ where $\langle\ ,\ \rangle$ is the inner product in $\mathbb{R}^3$. Find $\nabla H(x, y, z)$ and verify that

$$\nabla H(x, y, z) = G(x, y, z) \circ F'(x, y, z) + F(x, y, z) \circ G'(x, y, z).$$

1.8. If $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ show that

$$\nabla f(x, y, z) = -\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

1.9. If $F:U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and $F(P) \neq 0$ show that $\|F\|$ is differentiable at the point P. If $\mathbf{v} \in \mathbb{R}^n$ show that

$$\nabla_{\mathbf{v}}(\|F\|)(P) = \frac{\langle D_{\mathbf{v}}F, F\rangle}{\|F\|}(P).$$

1.10. Use Exercise 1.9 to give another proof of Exercise 1.8, i.e. first find $D_{e_i}\left(\frac{1}{\|x\|}\right)$ for $i = 1, 2, 3$.

1.11. If

$$\begin{array}{ccc} (x, y, z) & \xrightarrow{F} & (xyz, x^2 + y^2, x^2 - y^2, z^2) \\ & & \| \\ (u, v, w, t) & \xrightarrow{G} & (u^2 + v^2, u^2 - v^2, w^2 - t^2, w^2 + t^2) \end{array}$$

$G = (G_1, G_2, G_3, G_4)$ and $H = (H_1, H_2, H_3, H_4) = G \circ F$ verify that

$$\frac{\partial H_2}{\partial x} = \frac{\partial G_2}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial G_2}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial G_2}{\partial w} \cdot \frac{\partial w}{\partial x} + \frac{\partial G_2}{\partial t} \cdot \frac{\partial t}{\partial x}$$

directly and also by using $H' = G' \circ F'$.

1.12. If the function $f:\mathbb{R}^n \to \mathbb{R}$ satisfies

$$\sum_{i=1}^{n} x_i^2 \frac{\partial^2 f}{\partial {x_i}^2} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = 0$$

show that $h(x_1, \ldots, x_n) = f(e^{x_1}, \ldots, e^{x_n})$ satisfies

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial {x_i}^2} = 0.$$

1.13. Let $f:\mathbb{R}^n \to \mathbb{R}$ be differentiable, let $X \in \mathbb{R}^n$ and $\Delta X = (\Delta x_1, \ldots, \Delta x_n) \in \mathbb{R}^n$. Show that there exists $g:\mathbb{R}^{2n} \to \mathbb{R}$ such that

$$f(X + \Delta X) = f(X) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X)\Delta x_i + g(X, \Delta X)\|\Delta X\|$$

and $g(X, \Delta X) \to 0$ as $\Delta X \to 0$.

1.14. Let $f(x, y, z) = x^2y^2 + y^2z^2 + xyz$. By using the previous exercise and the values of f and its first-order derivatives at $(1, 1, 1)$ estimate $f(1.1, 1.05, 0.95)$. Find the error in your approximation and the error as a percentage of $f(1, 1, 1)$.

1.15. Identify geometrically and sketch the level set $F^{-1}(C)$ where $F(x, y, z) = (z^2 - x^2 - y^2, 2x - y)$ and $C = (1, 2)$.

1.16. If A is a subset of $\mathbb{R}^n$ show that A is closed if and only if for all $X_0 \in \mathbb{R}^n$ and $(X_k)_{k=1}^{\infty} \subset A$, $\lim_{k\to\infty} X_k = X_0$ implies $X_0 \in A$.

1.17. If A is a subset of $\mathbb{R}^n$ show that A is open if and only if for each $X_0 \in A$ and $(X_k)_{k=1}^{\infty} \subset \mathbb{R}^n$, $\lim_{k\to\infty} X_k = X_0$ there exists a positive integer k_0 such that $X_k \in A$ for all $k \geq k_0$.

1.18. Show that the sum, product and quotient of continuous (respectively differentiable) functions are continuous (respectively differentiable) and that differentiable functions are continuous.

1.19. Show that the level set

$$y^6 + y^2x^2 + x^2 + y^2 + 16z^2 - 8xyz - 2xy = 51$$

is a compact subset of $\mathbb{R}^3$.

1.20. If $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is differentiable and $F : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $F(x, y, z) = (x, y, z, \phi(x, y, z))$ find F'.

1.21. If A is a symmetric $m \times n$ matrix and X and Y are eigenvectors corresponding to different eigenvalues show that $\langle X, Y \rangle = 0$.

Chapter 2
Level Sets and Tangent Spaces

Summary *Using systems of linear equations as a guide we discuss the significance of the implicit function theorem for level sets. We define the tangent space and the normal space at a point on a level set.*

We shall be concerned with many different aspects of surfaces, level sets and graphs in this book. In this chapter we consider the role of differentiability in the local structure of level sets. By considering the linear approximation of differentiable functions and standard results on solving systems of linear equations we see that level sets are locally graphs.

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, U open, $F = (f_1, \ldots, f_m)$, $C = (c_1, \ldots, c_m) \in \mathbb{R}^m$. We suppose that F is differentiable and consider the level set $F^{-1}(C) = \bigcap_{i=1}^m f_i^{-1}(c_i)$, i.e. the points $(x_1, \ldots, x_n) \in U$ which satisfy the equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= c_1 \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= c_m. \end{aligned} \tag{2.1}$$

We have n unknowns, $x_1, \ldots, x_n$ and m equations. If each f_i is linear we have a *system of linear equations* and the rank of the matrix of coefficients gives information on the number of linearly independent solutions and procedures on how to identify a complete set of independent variables. The *implicit function theorem* says that this process is also valid *locally* for differentiable functions. The key, of course, is the fact that differentiable functions, by definition, enjoy a good local *linear* approximation.

Fix a point $P \in F^{-1}(C)$ and suppose $X \in \mathbb{R}^n$ is close to 0 and $P + X \in F^{-1}(C)$. Since F is differentiable,

$$F(P + X) = F(P) + F'(P)(X) + G(X)(X)$$

where $G : U \longrightarrow M_{m,n}$ and $G(X) \to 0$ as $X \to 0$. Since $F(P + X) = F(X) = C$,

$$F'(P)(X) \approx 0$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_2,

(where $\approx$ denotes approximately equal).

We assume from now on that $n \geq m$. We thus have something very close to the following system of *linear* equations

$$\begin{aligned}\frac{\partial f_1}{\partial x_1}(P)x_1 + \quad \cdots + \frac{\partial f_1}{\partial x_n}(P)x_n &= 0 \\ &\vdots \\ \frac{\partial f_m}{\partial x_1}(P)x_1 + \quad \cdots + \frac{\partial f_m}{\partial x_n}(P)x_n &= 0.\end{aligned} \tag{2.2}$$

The matrix of coefficients of this system of linear equations is

$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$$

i.e. $F'(P)$. From linear algebra we have

$$\begin{aligned}\text{Rank}\left(F'(P)\right) = m &\iff \text{the } m \text{ rows of } F'(P) \text{ are linearly independent} \\ &\iff \text{there exist } m \text{ linearly independent columns} \\ &\qquad \text{in } F'(P) \\ &\iff F'(P) \text{ contains } m \text{ columns such that the} \\ &\qquad \text{associated } m \times m \text{ matrix has non-zero} \\ &\qquad \text{determinant} \\ &\iff \text{the space of solutions for the system (2.2)} \\ &\qquad \text{is } n-m \text{ dimensional.}\end{aligned}$$

Moreover, if any of these conditions are satisfied and we choose m columns which are linearly independent then the variables corresponding to the *remaining columns* can be taken as a *complete set of independent variables* and the full set of solutions coincides with the graph of a function of the independent variables. If the above conditions are satisfied we say that F has *full* or *maximum rank* at P.

As a simple example consider

$$\begin{aligned}2x - y + z &= 0 \\ y - w &= 0\end{aligned}$$

with matrix of coefficients

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The 2×2 matrix obtained by using the first two columns is $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ and this has determinant $2 \neq 0$. Hence A has rank 2 and the two rows are linearly independent. Since the first two columns are linearly independent we can take the remaining two variables, z and w, as the independent variables. We have $y = w$, $2x = y - z = w - z$ and so $\big\{\big((w-z)/2, w, z, w\big) : z \in \mathbb{R},\ w \in \mathbb{R}\big\}$ is the solution set for this system of equations. We can write this in the form

$$\big\{\big(\phi(z, w), z, w\big) : (z, w) \in \mathbb{R}^2\big\}$$

where $\phi(z, w) = \big((w - z)/2, w\big)$ and in this format the solution space is the *graph* of the function ϕ (see Example 1.7).

Note that columns 1 and 3 are not linearly independent, since the corresponding 2×2 matrix $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ has zero determinant, and we cannot choose y and w (the remaining variables) as the independent variables.

Assuming that the rows of $F'(P)$ are linearly independent is equivalent to requiring that $\{\nabla f_1(P), \ldots, \nabla f_m(P)\}$ are linearly independent vectors in $\mathbb{R}^n$. The implicit function theorem says that with this condition we can solve the *non-linear* system of Eq. (2.1) near P and use the same method to identify a set of independent variables. The hypothesis of a *good* linear approximation in the definition of differentiable functions implies that the systems of Eqs. (2.1) and (2.2), are very close to one another.

We now state without proof the Implicit Function Theorem.

Theorem 2.1 (Implicit Function Theorem) *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, denote a differentiable function, let $P \in U$ and suppose $F(P) = C$ and rank $\big(F'(P)\big) = m$ (for convenience we suppose that the final m columns of $F'(P)$ are linearly independent). If $P = (p_1, \ldots, p_n)$ let $P_1 = (p_1, \ldots, p_{n-m})$ and $P_2 = (p_{n-m+1}, \ldots, p_n)$ so that $P = (P_1, P_2)$. Then there exists an open set V in $\mathbb{R}^{n-m}$ containing P_1, a differentiable function $\phi : V \to \mathbb{R}^m$, an open subset W of U containing P such that $\phi(P_1) = P_2$ and*

$$F^{-1}(C) \cap W = \Big\{\big(X, \phi(X)\big) : X \in V\Big\} = graph\ (\phi).$$

Thus locally every level set is a graph (Fig. 2.1).

Example 2.2 Let $F(x_1, x_2, x_3, x_4) = (x_1x_2, x_3x_4^2)$. We have

$$F'(x_1, x_2, x_3, x_4) = \begin{pmatrix} \nabla f_1(x_1, x_2, x_3, x_4) \\ \nabla f_2(x_1, x_2, x_3, x_4) \end{pmatrix} = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_4^2 & 2x_3x_4 \end{pmatrix}$$

where $f_1(x_1, x_2, x_3, x_4) = x_1x_2$ and $f_2(x_1, x_2, x_3, x_4) = x_3x_4^2$. Consider the level set $F^{-1}(2, -4)$ and the point $P = (1, 2, -1, 2) \in F^{-1}(2, -4)$. We have

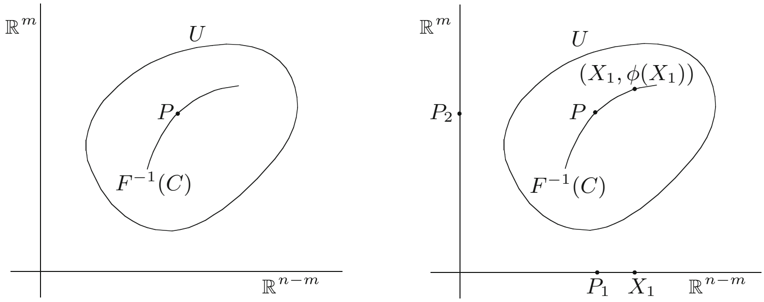


Fig. 2.1

$$F'(1, 2, -1, 2) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & -4 \end{pmatrix}.$$

If $\alpha(2, 1, 0, 0) + \beta(0, 0, 4, -1) = (0, 0, 0, 0)$ then $(2\alpha, \alpha, 4\beta, -\beta) = (0, 0, 0, 0)$ and hence $\alpha = \beta = 0$. This implies that the rows of $F'(1, 2, -1, 2)$ are linearly independent and $F'(P)$ has rank 2. This can be seen even more rapidly by finding a 2×2 submatrix with non-zero determinant, e.g. if we use columns 2 and 3 we get the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ with determinant $4 \neq 0$. It is easily checked that the following pairs of columns are linearly independent $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(2, 4)$ while the pairs $(1, 2)$ and $(3, 4)$ are not linearly independent. Since columns 1 and 4 are linearly independent we know that the variables x_2 and x_3 can be chosen as a complete set of independent variables. Thus *we know* that x_1 and x_4 can be expressed as functions of x_2 and x_3 near the point $(1, 2, -1, 2)$. The implicit function theorem is *important* because it tells us that certain functions *exist* even though it does not show how to find them. In general, we would have to solve the system of Eq. (2.1) to find these functions and this may often be extremely difficult or even impossible in any reasonable fashion. In our rather simple situation we have the two equations

$$x_1 x_2 = 2$$
$$x_3 x_4^2 = -4$$

and can find a solution. We have $x_1 = 2/x_2$ and since (x_1, x_2, x_3, x_4) is close to $(1, 2, -1, 2)$ we have x_2 close to 2 and the natural domain for x_2 from this equation is $x_2 > 0$. We have $x_4^2 = -4/x_3$ and since x_3 is close to -1 we take $x_3 < 0$. Hence $-4/x_3$ is positive and $x_4 = \pm\sqrt{-4/x_3}$. Since x_4 is close to 2 we take the positive square root. Thus the function ϕ, whose *existence is foretold* by the Implicit Function Theorem, has the form

$$\phi(x_2, x_3) = \left(\frac{2}{x_2}, +\sqrt{\frac{-4}{x_3}} \right)$$

on the open set $V = \{(x_2, x_3) : x_2 > 0,\ x_3 < 0\}$. After a rearrangement of the variables we get

$$\text{graph}\,(\phi) = \left\{\left(\frac{2}{x_2}, x_2, x_3, +\sqrt{\frac{-4}{x_3}}\right) : x_2 > 0,\ x_3 < 0\right\}$$

and, as expected, we have

$$\begin{aligned} F\left(\frac{2}{x_2}, x_2, x_3, +\sqrt{\frac{-4}{x_3}}\right) &= \left(\frac{2}{x_2}\cdot x_2, x_3\left(+\sqrt{\frac{-4}{x_3}}\right)^2\right) \\ &= \left(2, \frac{x_3(-4)}{x_3}\right) = (2, -4). \end{aligned}$$

An examination of the equations $x_1x_2 = 2$ and $x_3x_4^2 = -4$ shows that it is not possible to find, say x_3, as a function of x_1 and x_2 and thus we cannot, as expected, use x_1 and x_2 as the independent variables.

Example 2.3 Given the equations

$$x^2 - y^2 + u^2 + 2v^2 = 1 \tag{2.3}$$

$$x^2 + y^2 - u^2 - v^2 = 2 \tag{2.4}$$

we wish to find all (x, y, u, v) such that near (x, y, u, v), u and v can be expressed as differentiable functions of x and y and we also wish to find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ in terms of $x,\ y,\ u$ and v.

Let $F(x, y, u, v) = (x^2 - y^2 + u^2 + 2v^2, x^2 + y^2 - u^2 - v^2)$. Then (x, y, u, v) satisfies (2.3) and (2.4) if and only if $F(x, y, u, v) = (1, 2)$, i.e. if and only if $(x, y, u, v) \in F^{-1}((1, 2))$. We have

$$F'(x, y, u, v) = \begin{pmatrix} 2x & -2y & 2u & 4v \\ 2x & 2y & -2u & -2v \end{pmatrix}.$$

We require x and y as a complete set of independent variables and so must have linear independence of the third and fourth columns. Hence we wish to find the points (x, y, u, v) such that $\det\begin{pmatrix} 2u & 4v \\ -2u & -2v \end{pmatrix} = 4uv \neq 0$. This implies that any point (x, y, u, v) satisfying (2.3) and (2.4) with u and v both non-zero will be suitable. To compute $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ we could apply the chain rule to the equation

$$F\big(x, y, u(x, y), v(x, y)\big) = (1, 2)$$

or just use (2.3) and (2.4) which now read

$$x^2 - y^2 + u(x, y)^2 + 2v(x, y)^2 = 1$$
$$x^2 + y^2 - u(x, y)^2 - v(x, y)^2 = 2.$$

On differentiating we get

$$2x + 2u(x, y)\frac{\partial u}{\partial x} + 4v(x, y)\frac{\partial v}{\partial x} = 0 \tag{2.5}$$
$$2x - 2u(x, y)\frac{\partial u}{\partial x} - 2v(x, y)\frac{\partial v}{\partial x} = 0. \tag{2.6}$$

The method of differentiation used to obtain Eqs. (2.5) and (2.6) is often called *"implicit differentiation"* especially in one-variable calculus where level sets of the form $f(x, y) = 0$ are considered.

These are just two linear equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ which are easily solved to give

$$\frac{\partial u}{\partial x} = \frac{3x}{u} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{-2x}{v}.$$

Notice how we need u and v to be both non-zero. In most cases of this type it is not possible to find *explicit* formulae for u and v by solving equations similar to (2.3) and (2.4)—hence the *implicit* in the implicit function theorem. So although in general we cannot find explicit formulae for the dependent variables in terms of the independent variables we can find the partial derivatives in terms of the independent and dependent variables. We choose this particular example because we are able to verify our formulae for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. Adding (2.3) and (2.4) gives $2x^2 + v^2 = 3$ and hence $v = \pm(3 - 2x^2)^{1/2}$ where we take the appropriate sign depending on the value of v. We have

$$\frac{\partial v}{\partial x} = \pm\frac{1}{2}(3 - 2x^2)^{-1/2}(-4x) = \frac{-2x}{\pm(3 - 2x^2)^{1/2}} = \frac{-2x}{v}.$$

Subtracting (2.4) from (2.3) gives

$$-2y^2 + 2u^2 + 3v^2 = -1 = -2y^2 + 2u^2 + 3(3 - 2x^2)$$

i.e. $2u^2 = 2y^2 + 6x^2 - 10$ and $u = \pm(3x^2 + y^2 - 5)^{1/2}$. Hence

$$\frac{\partial u}{\partial x} = \pm\frac{1}{2}(3x^2 + y^2 - 5)^{-1/2} \cdot 6x = \frac{3x}{\pm(3x^2 + y^2 - 5)^{1/2}} = \frac{3x}{u}$$

and this agrees with our earlier calculation.

We now return to the general situation and let

$$F = (f_1, \ldots, f_m) \colon U \text{ (open) } \subset \mathbb{R}^n \to \mathbb{R}^m$$

denote a differentiable function which has full rank at the point P in U. Let $F(P) = C$. We have

$$\begin{aligned} F^{-1}(C) &= \{X : F(X) = C\} = \{X : F(P + X - P) = C\} \\ &= \{X : F(P) + F'(P)(X - P) + G(X)(X - P) = C\} \\ &\approx \{X : F'(P)(X - P)\} = 0\} \\ &= \big\{X : \langle \nabla f_i(P), X - P \rangle = 0,\ i = 1, \ldots, m\big\} \end{aligned}$$

where $G : U \longrightarrow M_{m,n}$ and $G(X) \longrightarrow 0$ as $X \longrightarrow P$.

The set

$$\big\{X \in \mathbb{R}^n : F'(P)(X - P) = 0\big\} = P + \big\{X \in \mathbb{R}^n : F'(P)(X) = 0\big\}$$

is the closest linear approximation to $F^{-1}(C)$ near P and we call it the *tangent space* to $F^{-1}(C)$ at P. Since

$$F'(P) = \left(\frac{\partial f_i}{\partial x_j}(P)\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

the set of all X satisfying the equation $F'(P)(X) = 0$ is precisely the set of all solutions $\{x_1, \ldots, x_n\}$ of the system of *homogeneous* linear equations (2.2) that we encountered earlier and, as F has full rank at P, this space of solutions forms an $(n - m)$–dimensional subspace of $\mathbb{R}^n$. Since

$$\big\{X \in \mathbb{R}^n : F'(P)(X) = 0\big\} = \big\{X \in \mathbb{R}^n : \langle \nabla f_i(P), X \rangle = 0,\ i = 1, \ldots, m\big\}$$

the tangent space consists of the vectors which are perpendicular to the gradients of the component functions *transferred* to the point P (see Fig. 2.2). Normal vectors are perpendicular to tangent vectors and it is thus natural to define the *normal space* to the level set at P as

$$P + \left\{\sum_{i=1}^{m} \lambda_i \nabla f_i(P) : \lambda_i \in \mathbb{R}\right\}.$$

If the tangent space or normal space is two-dimensional we use the term *tangent plane* and *normal plane* respectively and if it is one-dimensional we use *tangent line* and *normal line* respectively. The tangent space and normal space are both translates of vector subspaces of $\mathbb{R}^n$ to the point P. The tangent space is the subspace which fits closest to the level set of F at P while the normal space is the set of directions which are—roughly speaking—perpendicular to the surface near P.

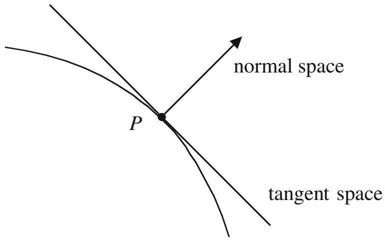


Fig. 2.2

In $\mathbb{R}^n$ there are various ways of presenting lines, planes, etc. The *normal form* consists of a description as the set of points satisfying a set of equations while the *parametric form* is in terms of independent variables and this, as we shall see in Chaps. 7 and 10, is almost a parametrization of the space.

Example 2.4 Let S denote the set of all points in $\mathbb{R}^3$ which satisfy the equation $x^2 + 2y^2 - 5z^2 = 1$. We wish to find the tangent space and the normal line at the point $(2, -1, 1)$ on S. Let $f(x, y, z) = x^2 + 2y^2 - 5z^2$. Then $S = f^{-1}(1)$. The tangent plane at $P = (x_0, y_0, z_0)$ in normal form is

$$\Big\{(x, y, z) : \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0\Big\}$$
$$= \Big\{(x, y, z) : (x - x_0)\frac{\partial f}{\partial x}(P) + (y - y_0)\frac{\partial f}{\partial y}(P) + (z - z_0)\frac{\partial f}{\partial z}(P) = 0\Big\}.$$

Now $\nabla f(x, y, z) = (2x, 4y, -10z)$ and $\nabla f(2, -1, 1) = (4, -4, -10)$ and so the tangent plane at $(2, -1, 1)$ is

$$\Big\{(x, y, z) : (x - 2)4 + (y + 1)(-4) + (z - 1)(-10) = 0\Big\}$$
$$= \Big\{(x, y, z) : 2x - 2y - 5z = 1\Big\}.$$

The normal line at (x_0, y_0, z_0) is the line through (x_0, y_0, z_0) in the direction of $\nabla f(x_0, y_0, z_0)$. In our case we have, in parametric form, the normal line

$$\Big\{(2, -1, 1) + t(4, -4, -10) : t \in \mathbb{R}\Big\} = \Big\{(2 + 4t, -1 - 4t, 1 - 10t) : t \in \mathbb{R}\Big\}.$$

To change this into normal form let $x = 2 + 4t$, $y = -1 - 4t$ and $z = 1 - 10t$. Solving for t we get

$$\frac{x - 2}{4} = \frac{y + 1}{-4} = \frac{z - 1}{-10} = t$$

and obtain the normal form

$$\Big\{(x, y, z) : \frac{x - 2}{4} = \frac{y + 1}{-4} = \frac{z - 1}{-10}\Big\}.$$

To find the tangent plane in parametric form we must find two linearly independent vectors which are perpendicular to $\nabla f(x_0, y_0, z_0)$. Applying the observation that (x, y) and $(-y, x)$ are perpendicular vectors in $\mathbb{R}^2$ to different pairs of coordinates in $\mathbb{R}^3$ we see easily that

$$(4, -4, -10) \cdot (4, 4, 0) = 0$$

and

$$(4, -4, -10) \cdot (10, 0, 4) = 0$$

and, moreover, $(4, 4, 0)$ and $(10, 0, 4)$ are linearly independent. In parametric form the tangent space at $(2, -1, 1)$ is

$$\begin{aligned}&(2, -1, 1) + \Big\{x(4, 4, 0) + y(10, 0, 4) : x, y \in \mathbb{R}\Big\}\\ &= \Big\{(2 + 4x + 10y, -1 + 4x, 1 + 4y) : x, y \in \mathbb{R}\Big\}.\end{aligned}$$

Example 2.5 We wish to find the normal plane and the tangent line to the set of points satisfying

$$x^2 + y^2 - 2z^2 = 2 \quad \text{and} \quad xyz = 2$$

at the point $(\sqrt{2}, \sqrt{2}, 1)$.

Let $F(x, y, z) = (x^2 + y^2 - 2z^2, xyz)$. Then the set of points which satisfy the above equations form the level set $F^{-1}(2, 2)$. We have

$$F'\left(\sqrt{2}, \sqrt{2}, 1\right) = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} & -4 \\ \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

The final two columns form a 2×2 matrix with non-zero determinant and hence F has full rank at $(\sqrt{2}, \sqrt{2}, 1)$. The normal plane at $(\sqrt{2}, \sqrt{2}, 1)$ has parametric form

$$\begin{aligned}&\left(\sqrt{2}, \sqrt{2}, 1\right) + \left\{x\left(2\sqrt{2}, 2\sqrt{2}, -4\right) + y\left(\sqrt{2}, \sqrt{2}, 2\right) : x,\ y \in \mathbb{R}\right\}\\ &= \left\{\left(\sqrt{2} + 2\sqrt{2}x + \sqrt{2}y, \sqrt{2} + 2\sqrt{2}x + \sqrt{2}y, 1 - 4x + 2y\right) : x,\ y \in \mathbb{R}\right\}.\end{aligned}$$

To find the normal plane in normal form we must find a non-zero vector perpendicular to both $\left(2\sqrt{2}, 2\sqrt{2}, -4\right)$ and $\left(\sqrt{2}, \sqrt{2}, 2\right)$. The cross product (see Chaps. 6 and 7) of the two given vectors is of the required type but we take a first-principles approach here. This amounts to finding (a, b, c) such that $\left(\sqrt{2}, \sqrt{2}, 2\right) \cdot (a, b, c) = 0$ and $\left(2\sqrt{2}, 2\sqrt{2}, -4\right) \cdot (a, b, c) = 0$. We thus have to solve the system of equations

$$\sqrt{2}a + \sqrt{2}b + 2c = 0$$
$$\sqrt{2}a + \sqrt{2}b - 2c = 0.$$

Subtracting we see that $c = 0$ and we can take $a = -b = 1$. Hence $(1, -1, 0)$ is a suitable vector. The normal plane (in normal form) is

$$\begin{aligned}&\Big\{(x, y, z) : \Big(x - \sqrt{2}, y - \sqrt{2}, z - 1\Big) \cdot (1, -1, 0) = 0\Big\}\\ &= \Big\{(x, y, z) : x - \sqrt{2} - y + \sqrt{2} = 0\Big\}\\ &= \Big\{(x, y, z) : x - y = 0\Big\}.\end{aligned}$$

The tangent line in normal form is

$$\begin{aligned}&\Big\{(x, y, z) : 2\sqrt{2}x + 2\sqrt{2}y - 4z = 4,\ \sqrt{2}x + \sqrt{2}y + 2z = 6\Big\}\\ &= \Big\{(x, y, z) : x + y = 2\sqrt{2},\ z = 1\Big\}.\end{aligned}$$

From this we see that the tangent line in parametric form is

$$\Big\{\Big(t, 2\sqrt{2} - t, 1\Big) : t \in \mathbb{R}\Big\}.$$

Exercises

2.1. Let $F_i : \mathbb{R}^4 \to \mathbb{R}^i$

$$\begin{aligned}F_1(x_1, x_2, x_3, x_4) &= x_1^2 - x_2^2,\ P_1 = (1, 2, 0, -1)\\ F_2(x_1, x_2, x_3, x_4) &= (x_1^2 - x_2^2, x_3^2 - x_4^2),\ P_2 = (1, 0, 2, -1)\\ F_3(x_1, x_2, x_3, x_4) &= (x_1^2 - x_2^2, x_3^2 - x_4^2, x_4^2 - x_1^2),\ P_3 = (1, 2, 3, 4).\end{aligned}$$

Calculate $F_i'(X)$ for $X \in \mathbb{R}^4$ and find all X such that F_i has full rank at X. When F_i has full rank find all subsets of $\{x_1, x_2, x_3, x_4\}$ which can be taken as complete sets of independent variables. If $F_i(P_i) = C_i$ and F_i has full rank at P_i find a function $\phi_i : \mathbb{R}^{4-i} \to \mathbb{R}^i$ such that $F_i^{-1}(C_i) = \text{graph}(\phi_i)$ near P_i.

2.2. If $u(x, y)$ and $v(x, y)$ are defined by the equations $u \cos v = x$ and $u \sin v = y$ find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ by

(i) finding explicit formulae for u and v
(ii) using implicit differentiation.

2.3. Let $F(x_1, x_2, x_3, x_4) = (x_1^2 x_2^2, x_1 x_2 x_3, x_4^2)$. Find $F'(1, 2, 3, 4)$. Let $A = F'(1, 2, 3, 4)$. Display the system of equations $AX = 0$. Solve this system of equations and find a basis for the space of solutions. Using your set of solutions find the tangent space to the level set of F at $(1, 2, 3, 4)$.

2.4. (a) Find in normal and parametric form the normal line and the tangent plane to the surface $z = xe^y$ at the point $(1, 0, 1)$.

(b) The surfaces $x^2 + y^2 - z^2 = 1$ and $x + y + z = 5$ intersect in a curve Γ. Find the equation in parametric form of the tangent line to Γ at the point $(1, 2, 2)$.

2.5. Find the equation of the plane passing through the points $(1, 2, 3)$ and $(4, 5, 6)$ which is perpendicular to the plane $7x + 8y + 9z = 10$.

2.6. Find the equation of the tangent plane to $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at the point $(1, 4, 1)$.

2.7. Find the tangent planes at $(1/\sqrt{2}, 1/4, 1/4)$ and $(\sqrt{3}/2, 0, 1/4)$ to the ellipsoid $x^2 + 4y^2 + 4z^2 = 1$. Find the line of intersection of these two planes. Show that this line is tangent to the sphere $x^2 + y^2 + z^2 = k$ for exactly one value of k and find this value.

2.8. Find the coordinates of the four points where the hyperbola $x^2 - y^2 = 1$ and the ellipse $x^2 + 2y^2 = 4$ intersect. Sketch to scale both curves and their tangents at the points of intersection of these tangent lines. If (a, b), $a > b > 0$, is one of these points show that the tangent line to the hyperbola at (a, b) coincides with the normal line to the ellipse at (a, b). Show that the tangent lines to the hyperbola at the four points of intersection enclose a parallelogram and find its area.

2.9. Find the direction of the normal line at the point $(1, 1, 4)$ to the paraboloid $z = x^2 + y^2 + 2$. Find the tangent plane in normal form at this point. Show that the normal line meets the paraboloid again at the point $(-5/4, -5/4, 41/8)$. If θ is the angle between the normal line through the point $(1, 1, 4)$ and the normal line through the point $(-5/4, -5/4, 41/8)$ show that $\sin\theta = 1/\sqrt{3}$.

2.10. Consider the subset S of $\mathbb{R}^3$ which lies above the (x, y)–plane and which is characterised by the property:

$p \in S \iff$ the distance from p to the xy-plane is the logarithm of its distance to the z-axis.

Describe S as a level set and as a graph. Find the normal line and the tangent plane to S at the point $(1, -1, \log 2/2)$.

2.11. Let $S_1 = \{(x, y, z) \in \mathbb{R}^3 : y = f(x)\}$ denote a cylinder and let S_2 denote the level set $z^2 + 2zx + y = 0$. If S_1 is tangent to S_2 at all points of contact find f.

2.12. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ denote two non-zero vectors in $\mathbb{R}^2$ making angles θ_1 and θ_2, respectively, with the positive x-axis. Show that $\cos\theta_1 = a_1/\|\mathbf{a}\|$ and $\sin\theta_1 = a_2/\|\mathbf{a}\|$. Show that $\theta_2 - \theta_1$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and, by expanding $\cos(\theta_2 - \theta_1)$, prove that $\cos(\theta_2 - \theta_1) = \mathbf{a} \cdot \mathbf{b}/\|\mathbf{a}\| \cdot \|\mathbf{b}\|$. Prove the same result for arbitrary vectors in $\mathbb{R}^n$.

Chapter 3
Lagrange Multipliers

Summary *We develop the method of Lagrange multipliers to find the maximum and minimum of a function with constraints.*

We consider the problem of finding the maximum and minimum of a sufficiently regular function g of n variables subject to the constraints

$$\left.\begin{aligned} f_1(x_1, \ldots, x_n) &= c_1 \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= c_m. \end{aligned}\right\} \tag{3.1}$$

To apply differential calculus we suppose there exists an open subset U of $\mathbb{R}^n$ such that each f_i is defined and differentiable on U and that all points which satisfy (3.1), that we wish to consider, lie in U. We also require g to be smooth in some fashion and it is convenient to suppose now that g is also differentiable on U.

If $F = (f_1, \ldots, f_m)$ then $F{:}U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and our problem can be restated as that of finding the maximum and minimum of g on the level set $F^{-1}(C)$ where $C = (c_1, \ldots, c_m)$. To apply the methods of the previous chapter we suppose that F has full rank on $F^{-1}(C)$ or equivalently that $\{\nabla f_1(X), \ldots, \nabla f_m(X)\}$ are linearly independent vectors for all X in $F^{-1}(C)$.

Suppose g has a local maximum on $F^{-1}(C)$ at the point P. Since we only need to examine g near P we may suppose that $F^{-1}(C)$ is the graph of a function ϕ of $n - m$ variables. By rearranging the variables, if necessary, we can assume that ϕ: $V \subset \mathbb{R}^{n-m} \to \mathbb{R}^m$, V open in $\mathbb{R}^{n-m}$, that P_1 (the first $n - m$ coordinates of P) lies in V and $P = (P_1, \phi(P_1))$.

Let $X_1 = (x_1, \ldots, x_{n-m}) \in \mathbb{R}^{n-m}$. Since g has a local maximum on $F^{-1}(C)$ at P the function

$$X_1 \to g\big(X_1, \phi(X_1)\big) \tag{3.2}$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_3,

has a local maximum on the open set V in R^{n-m} at P_1. Moreover, since the graph of ϕ coincides with the level set $F^{-1}(C)$ near P we have $F(X_1, \phi(X_1)) = C$ for all $X_1 \in V$. In terms of the coordinate functions this implies

$$f_i(X_1, \phi(X_1)) = c_i \tag{3.3}$$

for $i = 1, \ldots, m$. To apply the chain rule to the functions in (3.2) and (3.3) we first differentiate the function

$$X_1 \in V \subset \mathbb{R}^{n-m} \longrightarrow \big(X_1, \phi(X_1)\big) \in \mathbb{R}^n$$

and obtain the $n \times (n-m)$ matrix

$$\left(\begin{array}{c} \nabla(x_1) \\ \vdots \\ \nabla(x_{n-m}) \\ \\ \phi'(X_1) \\ \\ \end{array}\right) = \left(\begin{array}{ccc} 1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & 1 \\ & & \\ & \phi'(X_1) & \\ & & \end{array}\right). \tag{3.4}$$

Let A denote the matrix in (3.4) evaluated at $X_1 = P_1$. Since the first $n-m$ rows of A form a matrix with determinant 1 the matrix A has maximum rank $n-m$. Hence the space $S = \{X \in \mathbb{R}^n : X \circ A = 0\}$, i.e. the solution space for the system of $n-m$ linear equations in n variables, is an $n-(n-m) = m$ dimensional subspace of $\mathbb{R}^n$. Applying the chain rule to (3.3) and letting $X_1 = P_1$ we obtain

$$\nabla f_i(P) \circ A = 0$$

for $i = 1, \ldots, m$. Hence each $\nabla f_i(P) \in S$. Since $F'(P)$ has maximum rank, $\{\nabla f_1(P), \ldots, \nabla f_m(P)\}$ is a set of m linearly independent vectors and hence a *basis* for S. Since the function in (3.2) has a local maximum on the open set V at P_1 one variable calculus shows that all its partial derivatives at P_1 are zero. A further application of the chain rule implies

$$\nabla g(P) \circ A = 0.$$

Hence $\nabla g(P) \in S$ and there exist m scalars, called *Lagrange multipliers*, $\lambda_1, \ldots, \lambda_m$, such that

$$\nabla g(P) = \lambda_1 \nabla f_1(P) + \cdots + \lambda_m \nabla f_m(P). \tag{3.5}$$

Equation (3.5), in coordinates, is a set of n equations, and these together with the system of m equations in (3.1) gives $n+m$ equations in the $n+m$ unknowns $\{x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m\}$. The *method of Lagrange multipliers* consists in solving these equations. The solutions are the critical points of g on $F^{-1}(C)$ and contain the

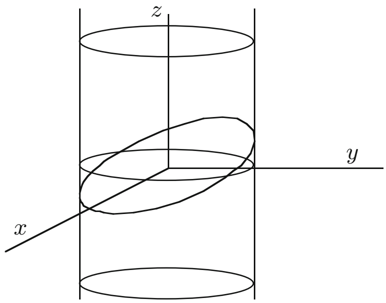


Fig. 3.1

local maxima and minima of g. To determine if g has a maximum and minimum on $F^{-1}(C)$ requires further on-the-spot investigation as we shall see in examples.

Frequently we are interested in finding the maximum and minimum on $\overline{U} \cap F^{-1}(C)$ where U is open in $\mathbb{R}^n$ and $\overline{U}$ is the closure of U in $\mathbb{R}^n$. The set $\overline{U}$ consists of all points which can be reached from U, i.e.

$$\overline{U} = \{X \in \mathbb{R}^n; \text{ there exists } (X_n)_n \in U \text{ with } \|X_n - X\| \to 0 \text{ as } n \to \infty\}.$$

In this case we apply the method of Lagrange multipliers to $U \cap F^{-1}(C)$ and examine separately the values of g on $(\overline{U} \backslash U) \cap F^{-1}(C)$—these points are often easily identified since they lie on the *boundary* of U.

Example 3.1 To find the maximum and minimum of $x + y + z$ subject to the constraints $x^2 + y^2 = 2$ and $x + z = 1$. Let $F = (f_1, f_2)$ where $f_1(x, y, z) = x^2 + y^2$ and $f_2(x, y, z) = x + z$ and let $g(x, y, z) = x + y + z$. We wish to find the maximum and minimum of g on the set $F^{-1}(2, 1)$. We have

$$F'(x, y, z) = \begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

On $F^{-1}(2, 1)$ we have $x^2 + y^2 = 2$ and so x and y cannot both be zero at the same time. Thus the rows of $F'(x, y, z)$ are linearly independent when $(x, y, z) \in F^{-1}(2, 1)$. Hence ∇f_1 and ∇f_2 are linearly independent on $F^{-1}(2, 1)$ and we may apply the method of Lagrange multipliers. The level set

$$F^{-1}(2, 1) = \{(x, y, z) : x^2 + y^2 = 2\} \cap \{(x, y, z) : x + z = 1\}$$

is the intersection of a cylinder parallel to the z-axis and a plane (why?) (see Fig. 3.1). This gives us a closed ellipse and compact set in $\mathbb{R}^3$. By the fundamental existence theorem g has a maximum and a minimum on $F^{-1}(2, 1)$. This apparently rather theoretical information has practical implications since we now know, before doing *any* calculations, that the method of Lagrange multipliers will yield *at least* two solutions and the maximum (minimum) solution will be the maximum (minimum) of g on $F^{-1}(2, 1)$. By the method of Lagrange multipliers there exist, at local maxima and minima, scalars λ_1 and λ_2 such that

$$\nabla g(x, y, z) = \lambda_1 \nabla f_1(x, y, z) + \lambda_2 \nabla f_2(x, y, z)$$

i.e.

$$(1, 1, 1) = \lambda_1(2x, 2y, 0) + \lambda_2(1, 0, 1).$$

In terms of coordinates this is equivalent to the following system of equations

$$\left.\begin{array}{l} 1 = 2\lambda_1 x + \lambda_2 \\ 1 = 2\lambda_1 y \\ 1 = \lambda_2 \end{array}\right\} \Longrightarrow 1 = 2\lambda_1 x + 1 \Longrightarrow \lambda_1 x = 0.$$

Hence $\lambda_1 = 0$ or $x = 0$. However, the second equation implies $1 = 2\lambda_1 y$ so $\lambda_1 \neq 0$. If $x = 0$ then $f_1(x, y, z) = y^2 = 2$ and $y = \pm\sqrt{2}$, while $f_2(x, y, z) = x + z = 1$ implies $z = 1$. Our only solutions are $(0, \sqrt{2}, 1)$ and $(0, -\sqrt{2}, 1)$. Since $g(0, \sqrt{2}, 1) = \sqrt{2} + 1$ and $g(0, -\sqrt{2}, 1) = -\sqrt{2} + 1$ it follows that $\sqrt{2} + 1$ and $-\sqrt{2} + 1$ are the maximum and minimum of g on $F^{-1}(2, 1)$.

Of course, substituting the constraint $x + z = 1$ into $g = 0$ implies $g(x, y, z) = g(x, y, 1 - x) = 1 + y$ and our problem reduces to finding the maximum and minimum of $1 + y$ on the set $x^2 + y^2 = 2$. On this set $-\sqrt{2} \leq y \leq \sqrt{2}$ and hence the maximum is $1 + \sqrt{2}$ and the minimum $1 - \sqrt{2}$. This verifies the solution obtained using Lagrange multipliers and also reminds us that it is always worthwhile thinking about a problem before attempting any solution.

Example 3.2 The *geometric mean* of n positive numbers $x_1, \ldots, x_n$, $(x_1 \cdots x_n)^{1/n}$ is always less than or equal to the *arithmetic mean* $\dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

In this example (see also Example 3.4) we standardise one of the quantities and generate a Lagrange multiplier type problem. Let $g(x_1, \ldots, x_n) = x_1 \cdots x_n$ and $f(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$. We begin by finding the maximum of g on the set

$$f^{-1}(1) \cap \big\{(x_1, \ldots, x_n) : x_i \geq 0 \quad \text{for all } i\big\}.$$

It is easily checked that this is a compact set and, by the fundamental existence theorem, the maximum exists. On the set $f^{-1}(1) \cap \big\{(x_1, \ldots, x_n) : x_i > 0\big\}$ we may apply the method of Lagrange multipliers. We have

$$\begin{aligned} \nabla g(x_1, \ldots, x_n) &= (x_2 \cdots x_n, x_1 x_3 \cdots x_n, \ldots, x_1 x_2 \cdots x_{n-1}) \\ &= x_1 \cdots x_n \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) \end{aligned}$$

and

$$\nabla f(x_1, \ldots, x_n) = (1, 1, \ldots, 1).$$

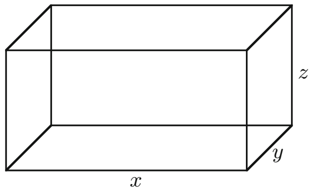


Fig. 3.2

If $\nabla g(x_1,\ldots,x_n)=\lambda\nabla f(x_1,\ldots,x_n)$ then $x_1\cdots x_n/x_i=\lambda$ and $x_i=x_1\cdots x_n/\lambda$ for all i. This shows $x_1=x_2=\cdots=x_n$. Since $x_1+x_2+\cdots+x_n=1$ we have $x_i=1/n$ for all i and $g(1/n,1/n,\ldots,1/n)=n^{-n}$. As $g(x_1,\ldots,x_n)=0$ whenever one of the x_i's is equal to zero it follows that the maximum of g, on the set $f(x_1,\ldots,x_n)=1$ and $x_i\geq 0$ all i, is $(1/n)^n$.

If x_i, $i=1,\ldots,n$, are arbitrary positive numbers let $y_i=x_i\Big/\sum_{j=1}^n x_j$ for each i. We have

$$\sum_{i=1}^n y_i=\sum_{i=1}^n x_i\Big/\sum_{j=1}^n x_j=1,$$

and, by the first part,

$$\frac{x_1\cdots x_n}{\left(\sum\limits_{j=1}^n x_j\right)^n}=y_1\cdots y_n\leq\left(\frac{1}{n}\right)^n.$$

Hence

$$x_1\cdots x_n\leq\left(\frac{1}{n}\sum_{j=1}^n x_j\right)^n$$

and, as required,

$$(x_1\cdots x_n)^{1/n}\leq\frac{1}{n}(x_1+x_2+\cdots+x_n).$$

Notice that although the number of variables, n, could be large symmetry led to rather simple equations. In mathematics symmetry compensates for size.

Example 3.3 To find the dimensions of the box (Fig. 3.2) of maximum volume V given that the surface area S is $10\,\mathrm{m}^2$.

We wish to maximise $V(x,y,z)=xyz$ subject to the constraint $S(x,y,z)=2(xy+yz+xz)=10$. Physical constraints imply that $x\geq 0,\ y\geq 0$ and $z\geq 0$. The set $U=\{(x,y,z):x>0,\ y>0,\ z>0\}$ is open and our problem is to determine the maximum of V on $\overline{U}\cap S$. The method of Lagrange multipliers will identify a set which contains all local maxima and minima in $U\cap S$. We proceed to do this now. We have

$$\nabla V(x, y, z) = (yz, xz, xy)$$

and

$$\nabla S(x, y, z) = \big(2(y+z), 2(x+z), 2(x+y)\big).$$

Since $x,\ y$ and z are all positive on U we have $\nabla V(x, y, z) \neq (0, 0, 0)$. Applying the method of Lagrange multipliers we consider

$$\nabla V = \lambda \nabla S$$

and obtain the following system of equations

$$\begin{aligned} yz &= 2\lambda(y+z) \implies xyz = 2\lambda(xy+xz)\\ xz &= 2\lambda(x+z) \implies xyz = 2\lambda(xy+yz)\\ xy &= 2\lambda(x+y) \implies xyz = 2\lambda(xz+yz). \end{aligned}$$

Hence $\lambda \neq 0$ and, on dividing by 2λ, we get $xy + xz = xy + yz = xz + yz$ and afterwards $xy = yz = zx$. Since $x \neq 0$, $y \neq 0$ and $z \neq 0$ this implies $x = y = z$ and $S(x, y, z) = 10$ now shows that $6x^2 = 10$. Hence $x = y = z = (5/3)^{1/2}$. Let $P = \big((5/3)^{1/2},\ (5/3)^{1/2},\ (5/3)^{1/2}\big)$. We have $V(P) = (5/3)^{3/2}$.

We now wish to show that V takes its *maximum* value, subject to the constraint $S = 10$, at P. The set $\overline{U} \cap S^{-1}(10)$ is closed but not bounded and so the fundamental existence theorem for maxima and minima cannot be applied directly. In such cases ad hoc methods are necessary and the following approach is sometimes useful. For $r > 0$ let

$$U_r = \big\{(x, y, z) : 0 \leq x \leq r,\ 0 \leq y \leq r,\ 0 \leq z \leq r\big\}.$$

The set U_r is compact (i.e. closed and bounded) and as r increases to infinity U_r expands, inside $\overline{U}$, and in the limit, i.e. at infinity, it covers $\overline{U}$. If V does have a maximum on $\overline{U} \cap S^{-1}(10)$ then it must lie in some U_r. Since $V(P) > 1$ our strategy is to show that for r sufficiently large we have $V(x, y, z) \leq 1$ whenever $(x, y, z) \notin U_r$. This will imply that the maximum of V on U_r, which exists by the fundamental existence theorem, equals the maximum of V on U. If $(x, y, z) \notin U_r$ then one of $x,\ y,\ z$, say x, is greater than r. If $S(x, y, z) = 10$ then $2xy \leq 10$ and $2xz \leq 10$. Hence $y \leq 10/2x = 5/x$ and $z \leq 10/2x = 5/x \leq 5/r$ and

$$V(x, y, z) = xyz \leq x \cdot \frac{5}{x} \cdot \frac{5}{r} = \frac{25}{r}. \tag{3.6}$$

In particular, if $r = 25$ then $V(x, y, z) \leq 1$ for all $(x, y, z) \notin U_{25}$ satisfying $S(x, y, z) = 10$. Since $P \in U_{25}$ and $V(P) > 1$ this implies

$$\begin{aligned}&\text{maximum}\big\{V(x,y,z) : (x,y,z) \in \overline{U} \cap S^{-1}(10)\big\}\\&= \text{maximum}\big\{V(x,y,z) : (x,y,z) \in U_{25} \cap S^{-1}(10)\big\}.\end{aligned}$$

Since $U_{25} \cap S^{-1}(10)$ is compact and V is continuous the fundamental existence theorem implies that V has a maximum on $U_{25} \cap S^{-1}(10)$. The maximum occurs either inside or on the boundary. If it occurs inside then our use of Lagrange multipliers implies that it occurs at the point P. On the boundary of U_{25} we have either at least one coordinate zero, in which case $V = 0$, or at least one coordinate equal to 25 in which case (3.6) implies $V \leq 1$ and we conclude that the maximum cannot occur on the boundary. We have thus shown that the absolute maximum of V occurs at the point $P = \big((5/3)^{1/2},\ (5/3)^{1/2}, (5/3)^{1/2}\big)$ and equals $(5/3)^{3/2}$.

Example 3.4 In this example we prove the Cauchy–Schwarz inequality

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

where a_i and b_i are arbitrary real numbers. This is one of the best known and most widely used inequalities in mathematics. If we use vector notation, i.e. let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, and take square roots we can rewrite the inequality as

$$\big|\langle \mathbf{a}, \mathbf{b}\rangle\big| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|. \tag{3.7}$$

The advantages of vector notation in this proof are obvious. Let $g\colon \mathbb{R}^{2n} \to \mathbb{R}$ be defined by $g(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b}\rangle$. Motivated by Example 3.2 and the form of the inequality given in (3.7) we first find the maximum and minimum of g on the set $\|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = 1$.

Let $f_1{:}\mathbb{R}^{2n} \to \mathbb{R}$ and $f_2{:}\mathbb{R}^{2n} \to \mathbb{R}$ be defined by

$$f_1(\mathbf{a}, \mathbf{b}) = \|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a}\rangle = \sum_{i=1}^{n} a_i^2$$

and

$$f_2(\mathbf{a}, \mathbf{b}) = \|\mathbf{b}\|^2 = \langle \mathbf{b}, \mathbf{b}\rangle = \sum_{i=1}^{n} b_i^2.$$

We wish to maximise $g(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} a_i b_i = \langle \mathbf{a}, \mathbf{b}\rangle$ on the set $S = \big\{(\mathbf{a}, \mathbf{b}) : \|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = 1\big\}$. The set S is easily seen to be compact and hence g has a maximum and minimum on S. We have

$$\nabla f_1(\mathbf{a}, \mathbf{b}) = 2(a_1, \ldots, a_n, 0, \ldots, 0) = 2(\mathbf{a}, \mathbf{0})$$

and

$$\nabla f_2(\mathbf{a}, \mathbf{b}) = 2(0, \ldots, 0, b_1, \ldots, b_n) = 2(\mathbf{0}, \mathbf{b})$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^n$. An inspection of the matrix

$$\begin{pmatrix} \nabla f_1(\mathbf{a}, \mathbf{b}) \\ \nabla f_2(\mathbf{a}, \mathbf{b}) \end{pmatrix} = \begin{pmatrix} 2\mathbf{a} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{b} \end{pmatrix}$$

shows immediately that $\nabla f_1(\mathbf{a}, \mathbf{b})$ and $\nabla f_2(\mathbf{a}, \mathbf{b})$ are linearly independent whenever $\mathbf{a} \neq 0$ and $\mathbf{b} \neq 0$ and implies that we may apply the method of Lagrange multipliers. From $\nabla g = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2$ and

$$\nabla g(\mathbf{a}, \mathbf{b}) = (b_1, \ldots, b_n, a_1, \ldots, a_n) = (\mathbf{b}, \mathbf{a})$$

we see that

$$(\mathbf{b}, \mathbf{a}) = \lambda_1(2\mathbf{a}, \mathbf{0}) + \lambda_2(\mathbf{0}, 2\mathbf{b}) = (2\lambda_1\mathbf{a}, 2\lambda_2\mathbf{b}).$$

Hence $\mathbf{b} = 2\lambda_1\mathbf{a}$ and since $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ this implies $2\lambda_1 = \pm 1$ and $\mathbf{b} = \pm\mathbf{a}$. If $\mathbf{b} = \mathbf{a}$ then $g(\mathbf{a}, \mathbf{a}) = \sum_{i=1}^{n} a_i^2 = 1$, while if $\mathbf{b} = -\mathbf{a}$ then $g(\mathbf{a}, -\mathbf{a}) = -\sum_{i=1}^{n} a_i^2 = -1$. Hence, if $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$, then

$$-1 \leq g(\mathbf{a}, \mathbf{b}) \leq 1$$

and

$$\big|g(\mathbf{a}, \mathbf{b})\big| \leq 1.$$

If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ then

$$\left|g\left(\frac{\mathbf{a}}{\|\mathbf{a}\|}, \frac{\mathbf{b}}{\|\mathbf{b}\|}\right)\right|^2 = \frac{\big|g(\mathbf{a}, \mathbf{b})\big|^2}{\|\mathbf{a}\|^2\|\mathbf{b}\|^2} \leq 1$$

and

$$\big|g(\mathbf{a}, \mathbf{b})\big|^2 \leq \|\mathbf{a}\|^2 \cdot \|\mathbf{b}\|^2.$$

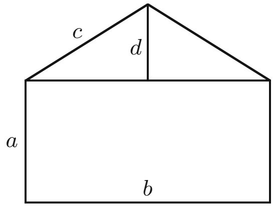
This is trivially verified if either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$. Since $g(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle$, we have proved the Cauchy–Schwarz inequality. The above also shows that we have *equality* in the Cauchy–Schwarz inequality if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors. In particular, for unit vectors $\mathbf{a}$ and $\mathbf{b}$, we have $\langle \mathbf{a}, \mathbf{b} \rangle = 1$ if and only if $\mathbf{a} = \mathbf{b}$ (see Example 8.5).

Exercises

3.1. Find the maximum and minimum of $xy + yz$ on the set of points which satisfy $x^2 + y^2 = 1$ and $yz = x$.

3.2. Find the highest and lowest points on the ellipse of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

3.3. Suppose a pentagon is composed of a rectangle surmounted by an isosceles triangle. If the length of the outer perimeter P is fixed find the maximum possible area.



(Hint: you must maximise $ab + \frac{1}{2}bd$ subject to the constraints $b+2a+2c = P$ and $\left(b/2\right)^2 + d^2 = c^2$.)

3.4. Find the minimum value of $x^2+y^2+z^2$ subject to the constraints $x+y-z = 0$ and $x + 3y + z = 2$. Give a *geometrical* interpretation of your answer and using your interpretation explain why $x^2 + y^2 + z^2$ has no maximum subject to these constraints.

3.5. Explain why the function $f(x, y, z) = x^2 + y^2 + 2z^2$ has a minimum on the surface S defined by the equations $x + y + z = 3$ and $x - y + 3z = 2$. Use Lagrange multipliers with two constraints to find the minimum value. Using the first constraint find ϕ such that $z = \phi(x, y)$ on S. Now use Lagrange multipliers to find the minimum of $f\big(x, y, \phi(x, y)\big)$ on the level set $x - y + 3\phi(x, y) = 2$. Using both constraints express y and z as functions ϕ_1 and ϕ_2 of x for all points on S. Find the minimum of $f\big(x, \phi_1(x), \phi_2(x)\big)$ using one-variable calculus.

3.6. Show that the maximum and minimum of $f(x, y, z) = x/a + y/b + z/c$ on the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ are $\sqrt{3}$ and $-\sqrt{3}$ respectively where a, b and c are positive constants.

3.7. Find the minimum value of xyz on the level set $F^{-1}(1)$, $x > 0$, $y > 0$, $z > 0$, where

$$F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,.$$

3.8. Use Lagrange multipliers to find the maximum volume of the rectangular solid in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) with one vertex at the origin and the opposite vertex lying in the plane $x/a + y/b + z/c = 1$ where a, b and c are positive constants.

3.9. Show that of all triangles inscribed in a fixed circle the equilateral triangle maximises; (a) the product of the lengths of the sides, (b) the sum of the squares of the lengths of the sides.

3.10. Find the minimum of $f(x, y) = 2y$ on the set $3x^2 - y^5 = 0$.

3.11. If $f : U(\text{open}) \subset \mathbb{R}^n \to \mathbb{R}$, $P \in U$ and $\nabla f(P) \neq 0$ find the direction, at P, in which f increases most rapidly.

3.12. Show that the maximum of $x_1 \cdots x_n$ on $\sum_{i=1}^{n} \frac{x_i^2}{i^2} = 1$ is $\frac{n!}{n^{n/2}}$.

3.13. Within a triangle there is a point P such that the sum of the squares of the distances to the sides is a minimum. Find this minimum in terms of the lengths of the sides and the area.

3.14. Find the distance from the point $(10, 1, -6)$ to the intersection of the planes

$$x + y + 2z = 5$$
$$2x - 3y + z = 12.$$

3.15. What is the relationship between Exercise 2.12 and the Cauchy–Schwarz inequality (Example 3.4)?

3.16. If a and b are positive numbers find the maximum and minimum values of $(xv - yu)^2$ subject to the constraints $x^2 + y^2 = a^2$ and $u^2 + v^2 = b^2$.

Chapter 4
Maxima and Minima on Open Sets

Summary *We derive, using critical points and the Hessian, a method of locating local maxima, local minima and saddle points of a real-valued function defined on an open subset of* $\mathbb{R}^n$.

We turn to the problem of finding local maxima and local minima of a real-valued function f defined on an open subset U of $\mathbb{R}^n$. The set of critical points of f on U, $\{X; \nabla f(X) = 0\}$, will include all points where f achieves a local maximum or minimum but may contain additional points such as saddle points. A critical point P is a *saddle point* if f, restricted to some curve passing through P, has a local maximum at P, while it has a local minimum at P along some other curve passing through P.

The *Hessian* of f at $P \in U$, $H_{f(P)}$, is defined as the $n \times n$ matrix $\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(P)\right)_{1 \le i,j \le n}$. To define the Hessian[1] we are, of course, assuming that all first- and second-order partial derivatives of f exist. We use the convention that the order of differentiation is from right to left, i.e.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

If all these second-order partial derivatives exist and are continuous then the order of differentiation is immaterial and

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(P) = \frac{\partial^2 f}{\partial x_j \partial x_i}(P)$$

for all i and j. In this case $H_{f(P)}$ is a symmetric $n \times n$ matrix. We will not prove this result but provide, in Exercise 4.6, a practical method which proves that all functions you will probably ever encounter have this property. If $\mathbf{v} = (v_1, \ldots, v_n)$ is a row

[1] The notation $\nabla^2(f)$ is also used for the Hessian of f.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_4,

vector and ${}^t\mathbf{v}$ is the corresponding column vector then

$$\frac{\partial f}{\partial \mathbf{v}}(P) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(P)$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial \mathbf{v}^2}(P) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{\partial f}{\partial \mathbf{v}}\right)(P) &= \sum_{i=1}^{n} v_i \frac{\partial}{\partial \mathbf{v}}\left(\frac{\partial f}{\partial x_i}\right)(P) \\
&= \sum_{i=1}^{n} v_i \Big(\sum_{j=1}^{n} v_j \frac{\partial^2 f}{\partial x_j \partial x_i}(P)\Big) \\
&= \sum_{i,j=1}^{n} v_i v_j \frac{\partial^2 f}{\partial x_i \partial x_j}(P) \\
&= \mathbf{v} H_{f(P)} {}^t\mathbf{v}.
\end{aligned}$$

The main theoretical result on the existence of local maxima, local minima and saddle points is the following theorem.

Theorem 4.1 *If U is an open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is a twice continuously differentiable function on U and P is a critical point of f, i.e.*

$$\nabla f(P) = \left(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P)\right) = 0$$

then

(1) *f has a strict local maximum at P if $\dfrac{\partial^2 f}{\partial \mathbf{v}^2}(P) < 0$ for all $\mathbf{v} \neq 0$,*
(2) *f has a strict local minimum at P if $\dfrac{\partial^2 f}{\partial \mathbf{v}^2} > 0$ for all $\mathbf{v} \neq 0$,*
(3) *f has a saddle point at P if there exist $\mathbf{v}$ and $\mathbf{w}$ such that*

$$\frac{\partial^2 f}{\partial \mathbf{v}^2}(P) < 0 < \frac{\partial^2 f}{\partial \mathbf{w}^2}(P).$$

To derive a practical test from this result we use linear algebra and Lagrange multipliers. To simplify matters we change our notation and let $\mathbf{v} = X = (x_1, \ldots, x_n)$, $a_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}(P)$ and $A = (a_{ij})_{1 \le i,j \le n}$. With this notation

$$\frac{\partial^2 f}{\partial \mathbf{v}^2}(P) = \mathbf{v} H_{f(P)} {}^t\mathbf{v} = X A {}^t X.$$

Since

$$\frac{X}{\|X\|}A^t\Big(\frac{X}{\|X\|}\Big) = \frac{1}{\|X\|^2}XA^tX$$

for $X \neq 0$ we have

$$XA^tX > 0 \quad \text{for all } X \neq 0 \iff \min_{\|X\|=1} XA^tX > 0$$
$$XA^tX < 0 \quad \text{for all } X \neq 0 \iff \max_{\|X\|=1} XA^tX < 0$$

there exists $X,\ Y \in \mathbb{R}^n$ such that

$$XA^tX < 0 < YA^tY \iff \min_{\|X\|=1} XA^tX < 0 < \max_{\|X\|=1} XA^tX.$$

We thus need to examine the extreme values of XA^tX on the set

$$\|X\|^2 = \langle X, X\rangle = \sum_{i=1}^{n} x_i^2 = 1.$$

Let

$$h(X) = h(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_i x_j = XA^tX$$

and

$$g(X) = g(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i^2 = \langle X, X\rangle.$$

Since the set $g^{-1}(1)$ is compact and h is continuous the fundamental existence theorem implies that h has a maximum and minimum on $g^{-1}(1)$. Using the coordinate expansion of g we see that $\nabla g(X) = (2x_1, \ldots, 2x_n) = 2X$ and $\nabla g(X) \neq 0$ on $g^{-1}(1)$. Hence we may apply the method of Lagrange multipliers to find the maximum and the minimum of h on the set $g^{-1}(1)$. We have

$$\nabla h(X) = \left(\sum_{j=1}^{n} 2a_{1j}x_j, \ldots, \sum_{j=1}^{n} 2a_{nj}x_j\right) = 2XA.$$

By the method of Lagrange multipliers there exists at the maximum and minimum points of h on $g^{-1}(1)$ a real number λ such that

$$2XA = 2\lambda X.$$

Taking the transpose we get

$$A^tX = \lambda^tX = \lambda I^tX$$

i.e.

$$(A - \lambda I)^tX = 0. \tag{4.1}$$

Since $\|X\| = 1$, any λ which satisfies (4.1) is an *eigenvalue*[2] of A and, moreover,

$$h(X) = XA^tX = \lambda X^tX = \lambda\langle X, X\rangle = \lambda.$$

Thus the maximum and minimum values of h are eigenvalues of A and are achieved at the corresponding unit eigenvectors. If all eigenvalues are positive then h is always positive and f has a local minimum at P, if all eigenvalues are negative then h is always negative and f has local maximum at P and if some are positive and some negative then h takes positive and negative values and f has a saddle point at P. If λ is an eigenvalue of A the set

$$E_\lambda := \{X \in \mathbb{R}^n : A^tX = \lambda^tX\}$$

is a subspace of $\mathbb{R}^n$, called the λ-*eigenspace* of A, and the dimension of E_λ is called the *multiplicity* of the eigenvalue λ. An $n \times n$ symmetric matrix has n eigenvalues when eigenvalues are counted according to multiplicity, i.e. if E_λ is j–dimensional then λ is counted j times, and $\det(A) = \lambda_1 \cdots \lambda_n$. Since a particular case of the above result will play an important role in our study of Gaussian curvature in Chap. 16 we display it separately.

Proposition 4.2 *If $A = \big(a_{ij}\big)_{1\le i,j\le 2}$ is a symmetric 2×2 matrix with eigenvalues λ_1 and λ_2, $\lambda_1 \ge \lambda_2$, then*

$$\max.\{a_{11}x^2 + 2a_{12}xy + a_{22}y^2 : x^2 + y^2 = 1\} = \lambda_1$$

and

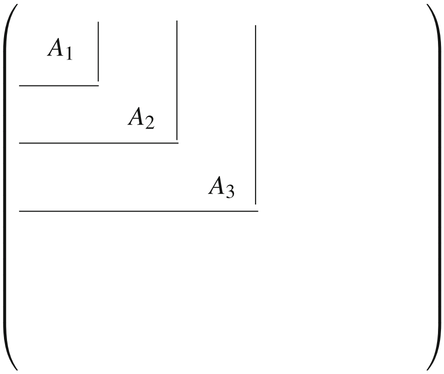
$$\min.\{a_{11}x^2 + 2a_{12}xy + a_{22}y^2 : x^2 + y^2 = 1\} = \lambda_2.$$

Moreover, the maximum and minimum are achieved at eigenvectors of A and

$$\det(A) = \lambda_1\lambda_2.$$

For our test we just need to know the sign of the largest and smallest eigenvalues. Since eigenvalues may be difficult to calculate we will use a reasonably well-known result from linear algebra. If A is a square matrix then the $k \times k$ matrix A_k obtained by deleting all except the first k rows and k columns of A is called the $k \times k$ *principal minor* of A. We have $A_n = A$.

[2] We could refocus this analysis to show that any symmetric $n \times n$ matrix with real entries admits eigenvalues.



We require the following result:

if A is a symmetric $n \times n$ *matrix then all eigenvalues of A are positive if and only if* $\det(A_k) > 0$ *for all k.*

Proposition 4.3 *If* $f : U \subset \mathbb{R}^n \to \mathbb{R}$, *U open, has continuous first- and second-order partial derivatives, P is a critical point of f, and* $A = H_{f(P)}$ *is the Hessian of f at P then the following hold:*

(a) *if* $\det(A_{2k}) < 0$ *for some k then P is a saddle point of f,*
(b) *if* $\det(A_n) \neq 0$ *then*

 (b1) *if* $\det(A_k) > 0$ *for all k then f has a strict local minimum at P,*
 (b2) *if* $(-1)^k \det(A_k) > 0$ *for all k then f has a strict local maximum at P,*
 (b3) *if the conditions on the determinants in (b1) and (b2) do not apply then f has a saddle point at P.*

Proof (a) Suppose $\det(A_{2k}) < 0$ for some positive integer k, $2k \leq n$. If $P = (p_1, \ldots, p_n)$ let $Q = (p_1, \ldots, p_{2k})$. Consider the function $g : V \subset \mathbb{R}^{2k} \to \mathbb{R}$ defined by

$$g(x_1, \ldots, x_{2k}) = f(x_1, \ldots, x_{2k}, p_{2k+1}, \ldots, p_n)$$

where V is the open set in $\mathbb{R}^{2k}$ consisting of all $(x_1, \ldots, x_{2k})$ such that

$$(x_1, \ldots, x_{2k}, p_{2k+1}, \ldots, p_n) \in U.$$

It is easily checked that

$$H_{g(Q)} = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(Q)\right)_{1 \leq i, j \leq 2k} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(P)\right)_{1 \leq i, j \leq 2k} = A_{2k}.$$

Let $\beta_1, \ldots, \beta_{2k}$ denote the $2k$ eigenvalues of the symmetric $2k \times 2k$ matrix A_{2k} counted according to multiplicity. We have

$$\beta_1 \cdots \beta_{2k} = \det(A_{2k}) < 0.$$

Since $2k$ is an even integer it follows that A_{2k} has positive and negative eigenvalues. Hence there exist $\mathbf{u} = (u_1, \ldots, u_{2k})$ and $\mathbf{v} = (v_1, \ldots, v_{2k})$ such that

$$\mathbf{u}A_{2k}{}^t\mathbf{u} < 0 < \mathbf{v}A_{2k}{}^t\mathbf{v}.$$

Let $\mathbf{w}_1 = (u_1, \ldots, u_{2k}, 0, \ldots, 0)$ and $\mathbf{w}_2 = (v_1, \ldots, v_{2k}, 0, \ldots, 0)$. Then

$$\begin{aligned}\frac{\partial^2 f}{\partial \mathbf{w}_1{}^2}(P) &= \mathbf{w}_1 H_{f(P)}{}^t\mathbf{w}_1 \\ &= \mathbf{u}A_{2k}{}^t\mathbf{u} < 0 < \mathbf{v}A_{2k}{}^t\mathbf{v} \\ &= \mathbf{w}_2 H_{f(P)}{}^t\mathbf{w}_2 = \frac{\partial^2 f}{\partial \mathbf{w}_2{}^2}(P)\end{aligned}$$

and f has a saddle point at P. This proves (a).
(b) The first part, (b1), follows directly from the linear algebra result quoted above. Since f has a local maximum at P if and only if $(-f)$ has a local minimum at the same point and $H_{-f(P)} = -H_{f(P)}$ it follows, from (b1), that f has a local maximum at P if $\det(-A_k) > 0$ for all k. Hence (b2) follows since $\det(-A_k) = (-1)^k \det(A_k)$. If (b1) and (b2) do not apply then, since all eigenvalues of A are non-zero, A has positive and negative eigenvalues and hence f has a saddle point at P. This completes the proof. □

If $\det(H_{f(P)}) = 0$ we call P a *degenerate critical point* of f (all other critical points are called *non-degenerate*) and higher order derivatives may be required to test the nature of the critical point.

Proposition 4.3 enables us to classify all non-degenerate and some degenerate critical points. When P is a non-degenerate critical point of f and f has a local maximum or minimum at P then the determinants of the *odd* principal minors all have the same sign. This leads to the second test for saddle points given below and a little reflection shows that it may be applied at any critical point. In practice the determinants, $\det(A_i)$, are calculated in the order $i = 1, 2, 3, \ldots$ and testing for *saddle points* is carried out as the calculations proceed. The critical point is a saddle point and the calculations stop when for the first time either of the following is observed:

$$\det(A_{2k}) < 0$$

$$\det(A_{2k-1})\det(A_{2k+1}) < 0.$$

If (b1) or (b2) are used to find a local maximum or minimum then all determinants of the Hessian must be calculated.

Example 4.4 Let $f(x, y, z) = x^2y^2 + z^2 + 2x - 4y + z$. We have

$$\nabla f(x, y, z) = (2xy^2 + 2, 2x^2y - 4, 2z + 1).$$

If P is a critical point of f then

$$\begin{aligned} 2xy^2 + 2 &= 0 \\ 2x^2y - 4 &= 0 \\ 2z + 1 &= 0 \end{aligned}$$

Hence $z = -1/2$ from the third equation. From the first two equations we see that x and y are non-zero. Hence $xy^2 = -1$ and $x^2y = 2$ imply $xy^2/x^2y = -1/2 = y/x$ and $x = -2y$. We have $-2y \cdot y^2 = -1$, i.e. $y^3 = 1/2$ and $y = 2^{-1/3}$. From $x = -2y$ we obtain $x = -2^{2/3}$ and conclude that $(-2^{2/3}, 2^{-1/3}, -1/2)$ is the only critical point of f. A simple calculation shows that

$$H_{f(x,y,z)} = \begin{pmatrix} 2y^2 & 4xy & 0 \\ 4xy & 2x^2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and hence

$$H_{f(-2^{2/3},2^{-1/3},1/2)} = \begin{pmatrix} 2^{1/3} & -4 \cdot 2^{1/3} & 0 \\ -4 \cdot 2^{1/3} & 2 \cdot 2^{4/3} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $\det(2^{1/3}) > 0$ and

$$\begin{aligned} \det\begin{pmatrix} 2^{1/3} & -4 \cdot 2^{1/3} \\ -4 \cdot 2^{1/3} & 2 \cdot 2^{4/3} \end{pmatrix} &= 2 \cdot 2^{5/3} - 16 \cdot 2^{2/3} \\ &= 4 \cdot 2^{2/3} - 16 \cdot 2^{2/3} < 0 \end{aligned}$$

the critical point $(-2^{2/3}, 2^{-1/3}, -1/2)$ is a saddle point of f.

Example 4.5 We wish to find and classify the non-degenerate critical points of $f(x, y, z) = x^2y + y^2z + z^2 - 2x$. We have

$$\nabla f(x, y, z) = (2xy - 2, x^2 + 2yz, y^2 + 2z)$$

and the critical points satisfy the equations

$$2xy - 2 = 0, \quad x^2 + 2yz = 0 \quad \text{and} \quad y^2 + 2z = 0.$$

Substituting $z = -y^2/2$ into the second equation implies $y^3 = x^2$. Hence, the first equation shows $y^{5/2} = 1$ and we have $y = 1$ and $z = -1/2$. From $xy = -1$ we get

$x = 1$ and $(1, 1, -1/2)$ is the only critical point of f. We have

$$H_{f(x,y,z)} = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{pmatrix}$$

and

$$H_{f(1,1,-1/2)} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

Since $\det(2) > 0$ and

$$\det\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} = -2 - 4 < 0$$

the point $(1, 1, -1/2)$ is a saddle point of f.

A number of simple initial checks may be carried out to detect saddle points. These are based on rearranging the variables, a procedure which does not alter the nature of a critical point. Thus we may first interchange row i and row j and afterwards column i and column j in the Hessian and then apply our standard test. In this way any 2×2 sub-matrix of the Hessian, which is symmetric about the diagonal, can be moved to become the 2×2 principal minor. Since a 2×2 symmetric matrix with positive and negative diagonal entries has negative determinant this means that a critical point is a saddle point if the diagonal contains positive and negative terms. This applies, for instance, to Example 4.5. Another useful indicator is the presence of a relatively large non-diagonal term; if a_{ij}, the ijth term in the Hessian, satisfies $|a_{ij}| > |a_{kk}|$ for all k then, as we see in our next example, this also implies that the critical point is a saddle point.

It is *not necessary* to physically interchange these rows and columns as the appropriate sub-matrix can be isolated by inspection but to convince you that this process is valid we include this step in our next example.

Example 4.6 Suppose the Hessian at a critical point P is

$$A = \begin{pmatrix} 2 & 2 & 9 & 1 \\ 2 & 3 & 2 & 0 \\ 9 & 2 & 4 & 2 \\ 1 & 0 & 2 & -1 \end{pmatrix}.$$

Since the diagonal contains positive and negative terms P is a saddle point. We see this more clearly if we interchange row 2 and row 4 and column 2 and column 4. We obtain the matrix B below and $\det(B_2) < 0$.

The presence of 9 in the third column suggests we interchange the second and third rows and the second and third columns. This gives the matrix C below with $\det(C_2) < 0$ and confirms that P is a saddle point.

$$B = \begin{pmatrix} 2 & 1 & 9 & 2 \\ 1 & -1 & 2 & 0 \\ 9 & 2 & 4 & 2 \\ 2 & 0 & 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & 9 & 2 & 1 \\ 9 & 4 & 2 & 1 \\ 2 & 2 & 3 & 0 \\ 1 & 2 & 0 & -1 \end{pmatrix}.$$

The standard approach gives $\det(A_1) = 2$, $\det(A_2) = 2$, $\det(A_3) = -171$ and $\det(A_1)\det(A_3) < 0$ implies that P is a saddle point.

The above method does not identify absolute or global maxima and minima. We encountered a similar problem in Example 3.3. We now describe a useful method which can be applied to certain functions on convex open sets. A subset $U \subset \mathbb{R}^n$ is *convex* if the straight line joining any two points in U is contained in U. The interior of a circle, sphere, box, polygon, the first quadrant or octant, and the upper half-plane are typical examples of convex open sets. The exterior of a circle or polygon is not convex.

Suppose $f : U(\text{open, convex}) \longrightarrow \mathbb{R}$ has continuous first and second order partial derivatives at all points. If $P, Q \in U$ and $\mathbf{v} = Q - P$ then, by convexity, $g(x) = f(P + x\mathbf{v})$ is defined on an *open interval* in $\mathbb{R}$ and is the restriction of f to that part of the straight line through P and Q which lies in U. Moreover, $g'(x) = \frac{\partial f}{\partial \mathbf{v}}(P + x\mathbf{v})$ and $\frac{\partial^2 f}{\partial \mathbf{v}^2}(P + x\mathbf{v}) = g''(x)$. Suppose $\frac{\partial^2 f}{\partial \mathbf{v}^2}(X) \neq 0$ for all X in U and all $\mathbf{v} \neq 0$. By the Intermediate Value Theorem $g''(x)$ is either always positive or always negative. This implies that g' is either strictly increasing or strictly decreasing and in either case g has at most one critical point on every line in U. Since we can carry out this analysis for any pair of points P and Q in U this shows that f has at most one critical point in U. Suppose f has a local maximum at P. If $f(P) < f(Q)$ then the function g considered above must have a local minimum at some point on the line joining P and Q. This contradicts the fact that g has at most one critical point and shows that f has an absolute maximum over U at P. Similarly if f has a local minimum at P then it has an absolute minimum over U at P. To verify that $\frac{\partial^2 f}{\partial \mathbf{v}^2}(X) \neq 0$ when U is a convex open subset in $\mathbb{R}^n$ it suffices to show that $\det(H_{f(x,y)})$ is never zero.

Example 4.7 Let

$$f(x, y) = x + y - e^x - e^y - e^{x+y}$$

for all (x, y) in $\mathbb{R}^2$. We have

$$\nabla f(x, y) = (1 - e^x - e^{x+y}, 1 - e^y - e^{x+y}).$$

If (x, y) is a critical point of f then $e^x = 1 - e^{x+y} = e^y$ and $x = y$. Hence $e^{2x} + e^x - 1 = 0$. If $w = e^x$ then $w^2 + w - 1 = 0$ and $w = (-1 \pm \sqrt{5})/2$. Since $w = e^x > 0$ we have only one solution $w = (-1 + \sqrt{5})/2$ and f has just one critical point at $P = ((-1 + \sqrt{5})/2, (-1 + \sqrt{5})/2)$. The existence of a single critical point suggests that we consider the Hessian at all points. We have

$$H_{f(x,y)} = \begin{pmatrix} -e^x - e^{x+y} & -e^{x+y} \\ -e^{x+y} & -e^y - e^{x+y} \end{pmatrix}.$$

Hence $\det(H_{f(x,y)}) = (e^x + e^{x+y})(e^y + e^{x+y}) - e^{2x+2y} = e^{2x+y} + e^{y+2x} + e^{x+y}$ is positive at all points $(x, y) \in \mathbb{R}^2$. Since $\dfrac{\partial^2 f}{\partial x^2}(P) < 0$, f has a local and hence a global maximum at P.

Exercises

4.1 Classify the non-degenerate critical points of

(a) $x^2 + xy + 2x + 2y + 1$	(f) $xyze^{-x^2-y^2-z^2}$
(b) $x^3 + y^3 - 3xy$	(g) $xy + x^2z - x^2 - y - z^2$
(c) $x^3z - 192x + y^2 - yz$	(h) $x^2y + y^2z + z^2 - 8\sqrt{2}x$
(d) $(2-x)(4-y)(x+y-3)$	(i) $2x^2y^2 - z^2 + x - 2y + z$
(e) $4xyz - x^4 - y^4 - z^4$	(j) $x^2 - xy + yz^3 - 6z$.

Show, using convexity, that the function in (b) has an absolute minimum over the set $U = \{(x, y) : x > 1/2, y > 1/2\}$. Show using the exhaustion method outlined in Example 3.3 that the function in (e) has an absolute maximum and no absolute minimum over $\mathbb{R}^3$. Show that the function in (f) has an absolute maximum and an absolute minimum over $\mathbb{R}^3$.

4.2 If $f(x, y, z) = (ax^2 + by^2 + cz^2)e^{-x^2-y^2-z^2}$ and $a > b > c > 0$ show that the function has two local maxima, one local minimum and four saddle points. Find the maximum and minimum of f over $\mathbb{R}^3$.

4.3 Show that the function $xyz(x+y+z-1)$ has one non-degenerate critical point and an infinite set of degenerate critical points. Show that the non-degenerate critical point is a local minimum.

4.4 Show that every critical point of $\dfrac{x^3 + y^3 + z^3}{xyz}$ is degenerate.

4.5 Find the distance from the point $(-1, 1, 1)$ to the level set $z = xy$.

4.6 Let $U = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$ and let

$$M(U) = \Big\{ f : U \to \mathbb{R} : \text{ all first- and second-order partial derivatives of } f \text{ exist and } \frac{\partial^2 f}{\partial x_i \partial x_j}(P) = \frac{\partial^2 f}{\partial x_j \partial x_i}(P) \text{ for all } P \in U \text{ and all } i,\ j, 1 \le i, j \le n \Big\}.$$

Show that $M(U)$ has the following properties:

(i) if $f,\ g \in M(U)$ and $c \in \mathbb{R}$ then $f \pm g$, $f \cdot g$ and $c \cdot f \in M(U)$ and if $g \neq 0$ then $f/g \in M(U)$

(ii) if $f \in M(U)$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then $\phi \circ f \in M(U)$.

Using (i) and (ii) show that

$$h(x, y, z) = \sin^2\left(\frac{e^{xyz}}{y^2 + z^2 + 1}\right)$$

lies in $M(U)$. Verify this result by calculating directly the appropriate second-order partial derivatives of h.

4.7 Let $Y_1, \ldots, Y_m$ be m points in $\mathbb{R}^n$. Show that $\sum_{i=1}^{m} \|X - Y_i\|^2$ achieves its absolute minimum at $X = \frac{1}{m}\sum_{i=1}^{m} Y_i$. Interpret your result geometrically.

4.8 If $z = \phi(x, y)$ satisfies the equation

$$x^2 + 2y^2 + 3z^2 - 2xy - 2yz = 2$$

find the points (x, y) at which ϕ has a local maximum or a local minimum.

4.9 If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has continuous first and second order partial derivatives show that $f''(x_1, \ldots, x_n) = (f')'(x_1, \ldots, x_n) = H_{f(x_1, \ldots, x_n)}$.

4.10 Show directly that the set $F^{-1}(2, 1)$ in Example 3.1 is compact.

Chapter 5
Curves in $\mathbb{R}^n$

Summary *We introduce and discuss the concept of directed curve in $\mathbb{R}^n$. We obtain a formula for the length of a curve, prove the existence of unit speed parametrizations and define piecewise smooth curves.*

Directed and parametrized curves play a role in many of the topics discussed in the remaining chapters of this book, e.g. line integrals, existence of a potential, Stokes' theorem and the geometry of surfaces in $\mathbb{R}^3$, and, furthermore, in a simple fashion introduce us to concepts such as parametrizations and orientations that are later developed and generalised in more involved settings.

We begin by giving a rigorous definition of directed curve. This may appear complicated and unnecessarily cumbersome at first glance and so we feel it proper to elaborate on why each condition is included. It is always important in mathematics to understand the basic *definition* and to refer to it until one appreciates each part separately and the totality of parts collectively. As progress is achieved there is usually less need to refer to the definition but in case of ambiguity the definition is the book of *rules*. The only requirement in a definition is that it be consistent, i.e. that the various conditions do not contradict one another or the rules of mathematics. Apart from this there is freedom in the choice of conditions and, indeed, many books would give a slightly different definition of directed curve. The differences depend on the degree of generality sought, the results aimed at and the methods used. However, all definitions of curve, directed curve and parametrized curve contain the same essential features.

Definition 5.1 A *directed* (or *oriented*) curve in $\mathbb{R}^n$ is a quadruple $\{\Gamma, A, B, \mathbf{v}\}$ where Γ is a set of points in $\mathbb{R}^n$; A and B are points in Γ, called respectively the *initial* and *final points* of Γ; $\mathbf{v}$ is a unit vector in $\mathbb{R}^n$ called the *initial direction*, for which there exists a mapping $P\colon [a, b] \to \mathbb{R}^n$, called a *parametrization* of Γ, such that the following conditions hold:

(a) there exists an *open* interval I, containing $[a, b]$ and a mapping from I into $\mathbb{R}^n$ which has derivatives of all orders and which coincides with P on $[a, b]$
(b) $P([a, b]) = \Gamma$, $P(a) = A$, $P(b) = B$ and $P'(a) = \alpha\mathbf{v}$ for some $\alpha > 0$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_5,

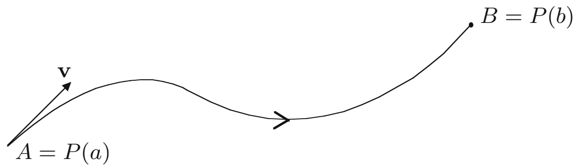


Fig. 5.1

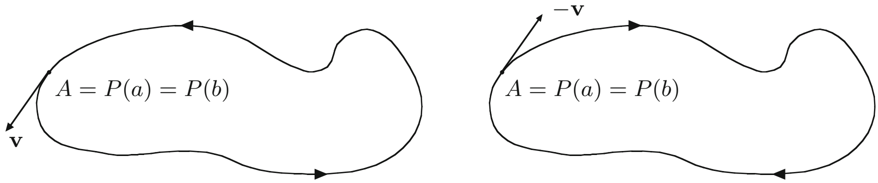


Fig. 5.2

(c) $P'(t) \neq 0$ for all $t \in [a, b]$
(d) P is injective (i.e. one to one) on $[a, b)$ and $(a, b]$.

Condition (a) is a rather strong regularity condition—we use the first derivative to define the tangent, the second to define curvature, the third to define torsion and the fourth to obtain the Frenet–Serret equations, and at this stage we felt it was just as easy to assume that we had derivatives of all orders. We also wished to have derivatives at the end points of the interval $[a, b]$ and for this reason we assumed that P has an extension as a *smooth* function (i.e. as a function with derivatives of all orders) to an open interval containing $[a, b]$. We could achieve precisely the same degree of smoothness by using one-sided derivatives at a and b but felt this *appears* even more complicated. All definitions of a curve will include, as an essential feature a *continuous* mapping P from an interval I in $\mathbb{R}$ *onto* Γ. The degree of differentiability, whether the interval I is open, closed, finite or infinite may be regarded as options that are available. We have chosen the options that suit our purposes.

The essential feature of condition (b), as we have just noted, is $P([a, b]) = \Gamma$. The remaining parts endow the set Γ with a sense of direction. If $A \neq B$ then the conditions $P(a) = A$ and $P(b) = B$ define a direction along Γ and in this case the condition $P'(a) = \alpha\mathbf{v}$ is redundant as $\mathbf{v}$ is determined by $\{\Gamma, A, B\}$ (Fig. 5.1).

If, however, $A = B$ then we have a *closed curve*, and it is necessary to distinguish between the two directions we may travel around Γ. In the case of curves in $\mathbb{R}^2$ we have clockwise and anticlockwise or counterclockwise directions. In $\mathbb{R}^n$ we do not have such a concept and instead specify the direction along the curve by giving an initial direction $\mathbf{v}$ (Fig. 5.2).

Note that when we know Γ and A then $\mathbf{v} = \pm P'(a) \big/ \|P'(a)\|$ and the condition $\alpha > 0$ distinguishes between the two signs and fixes the direction. Condition (c) is necessary to obtain a unit speed parametrization and we have already used $P'(a) \neq 0$ in (b). This condition excludes curves with corners but we get around this problem

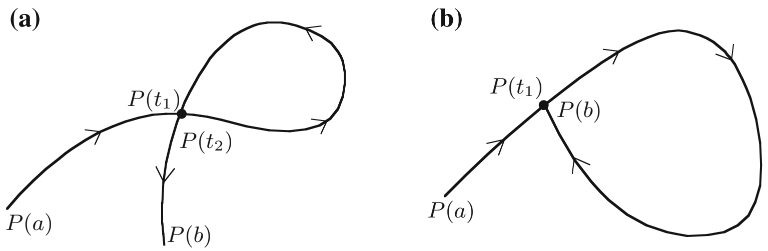


Fig. 5.3

by defining piecewise smooth curves. These are obtained by placing end to end a finite number of directed curves. The transition from directed curves to piecewise smooth curves is painless. Later we will require $P''(t) \neq 0$ in order to define a unit normal to a directed curve in $\mathbb{R}^3$.

Since we allow $A = B$ it follows that the mapping P may not be injective on $[a, b]$. However, we do not wish the curve to cross itself (Fig. 5.3a) or to half cross itself (Fig. 5.3b) as these lead to unnecessary complications and we have included condition (d) to exclude such possibilities.

A continuous mapping $P\colon [a, b] \to \mathbb{R}^n$ which satisfies (a), (c) and (d) is called a *parametrized curve*. A parametrized curve determines precisely one directed curve

$$\left\{P([a, b]),\ P(a),\ P(b),\ \frac{P'(a)}{\|P'(a)\|}\right\}$$

for which it is a parametrization.

Since the terminology "directed curve" and the notation $\{\Gamma, A, B, \mathbf{v}\}$ are rather cumbersome we will use the term *curve* and the notation Γ in all cases where there is little danger of confusion. If $A \neq B$ we sometimes write $\{\Gamma, A, B\}$. If we need to use coordinates we usually let $P(t) = \big(x_1(t), \ldots x_n(t)\big)$ if $\Gamma \in \mathbb{R}^n$ and, when $n = 3$, we let $P(t) = \big(x(t), y(t), z(t)\big)$. It is helpful to think of $[a, b]$ as an interval of time and $P(t)$ as the position of a particle at time t as it travels along the route Γ from A to B.

We call $P'(t)$ the *velocity* and $\|P'(t)\|$ the *speed* at time t. Since *distance* = *speed* × *time* the formula

$$l(\Gamma) = \int_a^b \|P'(t)\| dt$$

where $l(\Gamma)$ is the length of Γ, is not surprising. We shall, however, pause to prove this formula in order to show the usefulness of vector notation. Since P is differentiable we have for all t and $t + \Delta t$ in $[a, b]$

$$P(t + \Delta t) = P(t) + P'(t)\Delta t + g(t, \Delta t) \cdot \Delta t$$

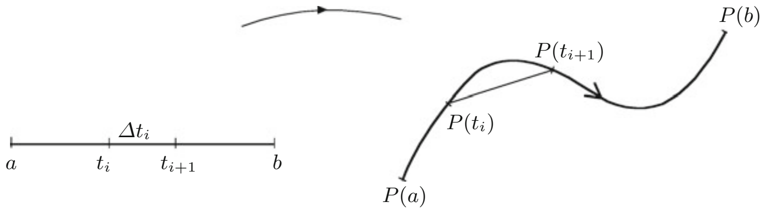


Fig. 5.4

where $g(t, \Delta t) \to 0$ as $\Delta t \to 0$ for any fixed t. Hence $P(t+\Delta t) - P(t) \approx P'(t)\Delta t$ for Δt close to zero. If we partition $[a, b]$ we get a corresponding partition of Γ and an approximation of the length of Γ (Fig. 5.4).

We have

$$\begin{aligned} l(\Gamma) &\approx \sum_i \|P(t_{i+1}) - P(t_i)\| \\ &\approx \sum_i \|P'(t_i)\|\Delta t_i \\ &\longrightarrow \int_a^b \|P'(t)\|dt \end{aligned}$$

as we take finer and finer partitions of $[a, b]$. In terms of coordinates we have $P(t) = \big(x_1(t), \ldots, x_n(t)\big)$, $P'(t) = \big(x_1'(t), \ldots, x_n'(t)\big)$ and

$$l(\Gamma) = \int_a^b \|P'(t)\|dt = \int_a^b \big(x_1'(t)^2 + \cdots + x_n'(t)^2\big)^{1/2} dt.$$

Example 5.2 Let $P(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, 2\pi]$, denote a parametrized curve Γ in $\mathbb{R}^3$, where **i**, **j** and **k** denote the standard unit vector basis in $\mathbb{R}^3$, i.e. $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The curve Γ is part of a *helix*—it *spirals* around a vertical *cylinder* (Fig. 5.5).

If we consider only the first two coordinates, this amounts to projecting onto the $\mathbb{R}^2$ plane in $\mathbb{R}^3$, we get the standard parametrization $t \to (\cos t, \sin t)$ of the unit circle. Hence, disregarding the final coordinate, which we take to be the height, the particle appears to move in a circle. We have

$$P'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}.$$

The rate of change of the height is given by the coefficient of the **k** term and so the particle is rising with constant speed. We have

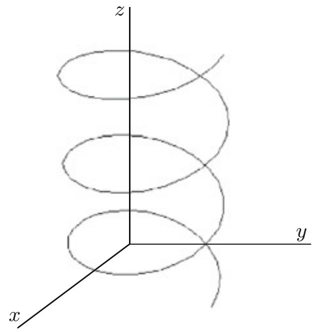


Fig. 5.5

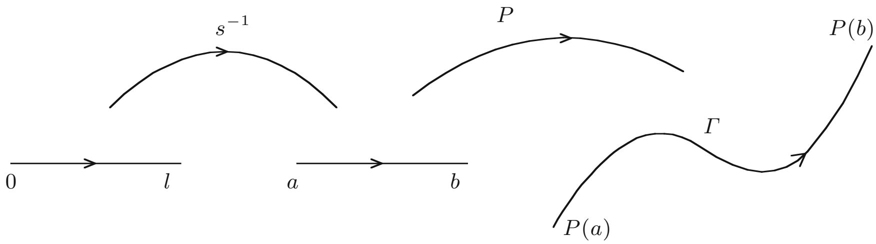


Fig. 5.6

$$\|P'(t)\| = \left((-\sin t)^2 + (\cos t)^2 + 1^2\right)^{1/2} = \sqrt{2}$$

and

$$l(\Gamma) = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi.$$

If $P\colon [a,b] \to \mathbb{R}^n$ is a parametrization of the directed curve Γ we define the *length function* by the formula

$$s(t) = \int_a^t \|P'(x)\| dx$$

for all $t \in [a,b]$. If $l = l(\Gamma)$ then $s\colon [a,b] \to [0,l]$ and, by the one-variable fundamental theorem of calculus, $s'(t) = \|P'(t)\| > 0$. Hence s is strictly increasing, $s^{-1}\colon [0,l] \to [a,b]$ has derivatives of all orders on $[0,l]$ and $P \circ s^{-1}$ maps $[0,l]$ onto Γ (Fig. 5.6). For the inverse function s^{-1} we have

$$(s^{-1})'(t) = \frac{1}{s'\left(s^{-1}(t)\right)} = \frac{1}{\|P'\left(s^{-1}(t)\right)\|}$$

and hence

$$\|(P \circ s^{-1})'(t)\| = \frac{\|P'(s^{-1}(t))\|}{s'(s^{-1}(t))} = \frac{\|P'(s^{-1}(t))\|}{\|P'(s^{-1}(t))\|} = 1.$$

Since the remaining conditions (for a parametrization) are easily checked it follows that $P \circ s^{-1}$ is a parametrization of Γ and we have proved the following result.

Proposition 5.3 *Directed curves admit unit speed parametrizations.*

If two particles start at the point A at time zero and both proceed along Γ towards B at unit speed then their positions on Γ at time t will always agree. This shows that a directed curve of length l admits a *unique* unit speed parametrization on $[0, l]$. Using our construction of unit speed curve we may restate this as follows: if $P_1 : [a, b] \to \Gamma$ and $P_2 : [c, d] \to \Gamma$ are any two parametrizations of the directed curve Γ, of length l, and

$$s_1(t) = \int_a^t \|P_1'(x)\| dx, \qquad s_2(t) = \int_c^t \|P_2'(x)\| dx$$

are the associated length functions then

$$P_1 \circ s_1^{-1}(t) = P_2 \circ s_2^{-1}(t)$$

for all $t \in [0, l]$.

To include curves with corners we extend the concept of directed curve to that of piecewise smooth directed curve. A finite collection of directed curves $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}_{i=1}^n$ is called a *piecewise smooth directed curve* if

(a) each $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}$ is a directed curve,
(b) $B_i = A_{i+1}$ for $i = 1, \ldots, n-1$ (i.e. the *final* point of Γ_i coincides with the *initial* point of Γ_{i+1}).

If $B_n = A_1$ we say that the piecewise smooth directed curve is *closed*. We use the notation Γ for a piecewise smooth directed curve and A and B for its initial and final points respectively. This definition is rather general and apart from curves with corners it also includes curves which cross one another. In general such curves do not admit a unit speed parametrization but it can be shown that there exists a *continuous* parametrization

$$P : [a, b] \longrightarrow \Gamma = \bigcup_{i=1}^n \Gamma_i$$

and a partition $\{a_0 = a, a_1, a_2, \ldots, a_n = b\}$ of $[a, b]$ such that $P\big([a_{i-1}, a_i]\big) = \Gamma_i$, $P(a_{i-1}) = A_i$, $P(a_i) = B_i$, $i = 1, \ldots, n$, P has derivatives of all orders on $[a, b]$ and $P'(t) \neq 0$ for $t \neq a_0, \ldots, a_n$.

A piecewise smooth directed curve $\Gamma = \{\Gamma_i, A_i, B_i, \mathbf{v_i}\}_{i=1}^n$ is studied and applied by considering each of the component sections $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}$ in turn (see for instance Exercise 5.6 and our method of finding a potential in Chap. 6).

Exercises

5.1 Find the length of the curve parametrized by

$$P(t) = (2\cosh 3t, -2\sinh 3t, 6t), \quad 0 \leq t \leq 5.$$

5.2 Show that the following parametrizations are unit speed

(a) $P_1(s) = \frac{1}{2}\Big(s + \sqrt{s^2+1}, (s + \sqrt{s^2+1})^{-1}, \sqrt{2}\log(s + \sqrt{s^2+1})\Big)$, $s \in [0, 1]$

(b) $P_2(s) = \Big(\dfrac{(1+s)^{3/2}}{3}, \dfrac{(1-s)^{3/2}}{3}, \dfrac{s}{\sqrt{2}}\Big), \quad s \in [-1, +1]$

(c) $P_3(s) = \dfrac{1}{2}\big(\cos^{-1}(s) - s\sqrt{1-s^2}, 1 - s^2, 0\big), \quad s \in [0, 1]$.

5.3 Let r and h denote positive numbers. Find a unit speed parametrization of the helix $P(t) = (r\cos t, r\sin t, ht), \quad 0 \leq t \leq 6\pi$.

5.4 Obtain unit speed parametrizations of the curves defined by

(a) $t \longrightarrow (e^t \cos t, e^t \sin t, e^t), \quad t \in [0, 1]$

(b) $t \longrightarrow (\cosh t, \sinh t, t), \quad t \in [0, 1]$.

5.5 Parametrize the curve of intersection of the sphere $x^2 + y^2 + z^2 = 16$ and the cylinder $x^2 + (y-2)^2 = 4$ which lies in the first octant.

5.6 Parametrize the anticlockwise directed triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(-1, -2)$ and $(4, 0)$ as a piecewise smooth curve.

5.7 Find the closest points on the curve $x^2 - y^2 = 1$ to $(a, 0)$ where (i) $a = 4$ (ii) $a = 2$ (iii) $a = \sqrt{2}$.

5.8 Let f denote a real-valued differentiable function defined on an open subset U of $\mathbb{R}^n$ and suppose $\nabla f(X) \neq \mathbf{0}$ for all X in U. Let P denote a parametrized curve in U. Use the chain rule to show that $\dfrac{d}{dt}(f \circ P)(t) = \langle \nabla f(P(t)), P'(t)\rangle$ and hence deduce, using the Cauchy–Schwarz inequality, that $\nabla f(X_0)$ gives the direction of maximum increase of f at X_0. Show that $\|\nabla f(X_0)\|$ is the maximum rate of increase.

5.9 If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping such that $\|TX\| = \|X\|$ for all $X \in \mathbb{R}^n$ show that T preserves the inner product, angles, area and the length of curves. When $n = 3$, show that T preserves the cross product.

Chapter 6
Line Integrals

Summary *We integrate vector-valued and scalar-valued functions along a directed curve in* $\mathbb{R}^n$*. We discuss scalar and vector potentials and define the curl of a vector field in* $\mathbb{R}^3$.

The differential calculus was developed to study extremal (i.e. maximal and minimal) values of functions. Since it is only possible to discuss the maximum and minimum of a real-valued function it is not surprising that such functions occupy a prominent role in several-variable differential calculus. However, in moving to integration theory it is more natural (and more natural in mathematics usually means more useful, more efficient and more elegant) to consider vector-valued functions where the domain and the range share, in perhaps a loose way, a *common dimension*. Formally, we have the following definition of a *vector field*.

Definition 6.1 *A function F which maps a subset U of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *is called a vector field on U.*

If U is an open subset of $\mathbb{R}^n$ and the vector field has derivatives of all orders we call it a *smooth vector field* and if U is arbitrary and the vector field is continuous we use the term *continuous vector field*. We shall also use the notation **F** to denote a vector field.

The gradient is an important example of a vector field, i.e. if U is open and $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable then $\nabla f: U \to \mathbb{R}^n$ is a vector field on U.

Another useful example occurs when Γ is a directed curve in $\mathbb{R}^n$ and F is a function which assigns a vector in $\mathbb{R}^n$ to each point on Γ—in this case we say that F is a vector field *along* Γ. For example, if P is a parametrization of a directed curve Γ in $\mathbb{R}^n$ then the mapping

$$P(t) \in \Gamma \to P'(t) \in \mathbb{R}^3$$

is a smooth vector field along Γ.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_6,

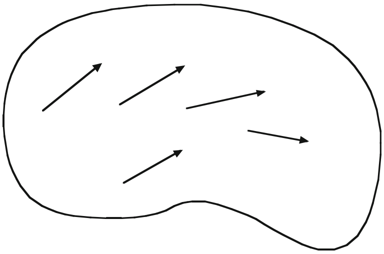


Fig. 6.1

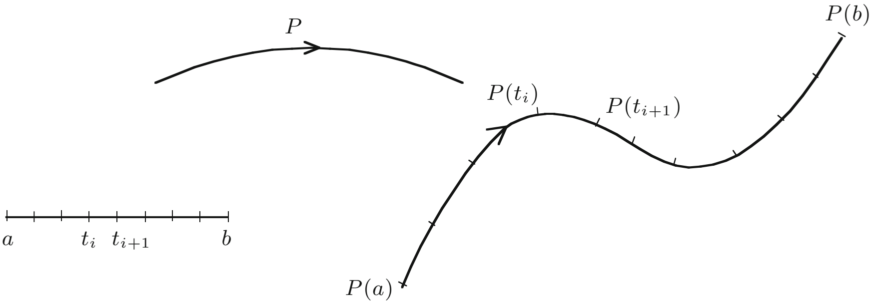


Fig. 6.2

Vector fields, which assign vectors to points in the domain of definition, are often represented as in Fig. 6.1.

This representation is useful in locating zeros and suggesting properties such as continuity and smoothness. Furthermore, it allows various physical interpretations of vector fields, e.g. as the velocity of a moving fluid and the flow of an electric current which lead, in turn, to important physical and engineering applications.

We begin our study of integration theory by defining the integral of a vector field F along a directed curve Γ. Let $P\colon [a,b] \to \Gamma$ denote a parametrization of Γ. To each partition of $[a,b]$ we obtain a partition of Γ (Fig. 6.2) and the Riemann sum

$$\sum_i F\big(P(t_i)\big) \cdot \big(P(t_{i+1}) - P(t_i)\big) \approx \sum_i F\big(P(t_i)\big) \cdot P'(t_i)\Delta t_i$$

where $\Delta t_i = t_{i+1} - t_i$ and $\cdot$ denotes the inner product in $\mathbb{R}^n$. Note that we are using, as usual, the linear approximation to $P(t+\Delta t)$, $P(t) + P'(t)\Delta t$. If F is a continuous vector field along Γ, i.e. if the mapping $t \in [a,b] \to F(P(t))$ is continuous, then as we take finer and finer partitions of $[a,b]$ the Riemann sums converge to the limit

$$\int_a^b F\big(P(t)\big) \cdot P'(t)\,dt.$$

We denote this integral by $\int_\Gamma F$, since we shall shortly prove that it is independent of the parametrization P, and call it the *line integral* of F over Γ. In terms of coordinates, for instance in the case $n = 3$, we have

$$F = (f, g, h), \quad P(t) = \big(x(t), y(t), z(t)\big), \quad P'(t) = \big(x'(t), y'(t), z'(t)\big)$$

and

$$\int_\Gamma F = \int_a^b \Big[f\big(x(t), y(t), z(t)\big) \cdot x'(t) \; + \; g\big(x(t), y(t), z(t)\big) \cdot y'(t) \\ + \; h\big(x(t), y(t), z(t)\big) \cdot z'(t)\Big]\, dt.$$

This is frequently written in the form

$$\int_\Gamma f\,dx + g\,dy + h\,dz.$$

If $(\Gamma, A, B, \mathbf{v})$ is a directed curve and $\tilde{\Gamma}$ is obtained by changing the direction along the curve Γ, i.e. $\tilde{\Gamma} = (\Gamma, B, A, -\mathbf{v})$, then clearly $\int_{\tilde{\Gamma}} F = -\int_\Gamma F$ for any continuous vector field F.

Example 6.2 We evaluate

$$\int_\Gamma xy\,dx + xz^2\,dy + xyz\,dz$$

where the curve Γ is parametrized by

$$P(t) = (t, t^2, t^3), \quad 0 \le t \le 1.$$

In coordinates we have

$$\begin{cases} x = x(t) = t \\ y = y(t) = t^2 \\ z = z(t) \ = t^3 \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = x'(t) = 1 \\ \dfrac{dy}{dt} = y'(t) = 2t \\ \dfrac{dz}{dt} = z'(t) = 3t^2 \end{cases} \Longrightarrow \begin{cases} dx = dt \\ dy = 2t\,dt \\ dz = 3t^2\,dt \end{cases}$$

and

$$\int_\Gamma xy\,dx + xz^2\,dy + xyz\,dz = \int_0^1 t^3\,dt + t^7 \cdot 2t\,dt + t^6 \cdot 3t^2\,dt \\ = \left[\frac{t^4}{4} + \frac{2t^9}{9} + \frac{3t^9}{9}\right]_0^1 = \frac{29}{36}.$$

Alternatively, changing to vector notation, we let $F(x, y, z) = (xy, xz^2, xyz)$. Then $F\big(P(t)\big) = (t^3, t^7, t^6)$ and $P'(t) = (1, 2t, 3t^2)$. Hence

$$\int_\Gamma F = \int_0^1 F\big(P(t)\big)\cdot P'(t)\,dt = \int_0^1 (t^3, t^7, t^6)\cdot(1, 2t, 3t^2)\,dt$$
$$= \int_0^1 (t^3 + 2t^8 + 3t^8)\,dt = \frac{29}{36}.$$

We return to the general situation. If $P\colon [a,b] \to \Gamma$ and $Q\colon [c,d] \to \Gamma$ are two parametrizations of Γ with length functions s and s_1 respectively, then, as we saw in Chap. 5,

$$P \circ s^{-1} = Q \circ s_1^{-1} \text{ on } [0,l], \quad l = \text{length of } \Gamma.$$

Using the one-variable change of variables, $y = s^{-1}(t)$ and $x = s_1^{-1}(t)$, we obtain

$$\int_0^l F\big(P\circ s^{-1}(t)\big)\cdot (P\circ s^{-1})'(t)\,dt$$
$$= \int_0^l F\big(P\circ s^{-1}(t)\big)\cdot P'\big(s^{-1}(t)\big)(s^{-1})'(t)\,dt$$
$$= \int_a^b F\big(P(y)\big)\cdot P'(y)\,dy.$$

Similarly

$$\int_0^l F\big(Q\circ s_1^{-1}(t)\big)\cdot (Q\circ s_1^{-1})'(t)\,dt = \int_c^d F\big(Q(x)\big)\cdot Q'(x)\,dx.$$

Since $P\circ s^{-1} = Q\circ s_1^{-1}$ this implies

$$\int_a^b F\big(P(y)\big)\cdot P'(y)\,dy = \int_c^d F\big(Q(x)\big)\cdot Q'(x)\,dx$$

and we get the same value no matter which parametrization is used. This justifies the notation $\int_\Gamma F$.

If $P\colon [a,b] \to \Gamma$ is a parametrization of the directed curve Γ in $\mathbb{R}^n$ and $t \in [a,b]$ let $T(t) = P'(t) \big/ \|P'(t)\|$. We call $T(t)$ the *unit tangent* to the (directed) curve at $P(t)$. It is easily seen that any two parametrizations define the same unit tangent vector at each point of Γ. This leads to another way of writing line integrals. If F is a vector field along Γ (always of course assumed to be continuous) then

$$\int_\Gamma F = \int_a^b F\big(P(t)\big)\cdot P'(t)\,dt$$
$$= \int_a^b F\big(P(t)\big)\cdot \frac{P'(t)}{\|P'(t)\|}\|P'(t)\|\,dt$$

$$= \int_a^b F\big(P(t)\big) \cdot T(t)\, \|P'(t)\|\, dt$$
$$= \int_a^b (F \cdot T)\, ds$$

and when written in this form one should remember, in applying a parametrization P, that F and T are both evaluated at $P(t)$ and $ds = \|P'(t)\|\, dt$.

Real-valued functions or *scalar fields*, such as the speed of a parametrization, can also be defined along a directed curve Γ. Since these are not endowed with a sense of direction we cannot apply directly our definition of integral. Fortunately, we have just observed a *special* or *privileged* direction associated with each point on a curve, the tangent direction, and by associating the continuous real-valued function $f : \Gamma \to \mathbb{R}$ with the vector field $fT : \Gamma \to \mathbb{R}^n$ we can define $\int_\Gamma f$. The privileged direction on an oriented surface in $\mathbb{R}^3$ is the *normal* direction and in this case it is also possible to consider scalar-valued integration as a special case of vector-valued integration.

If $P : [a, b] \to \Gamma$ is a parametrization of the directed curve Γ we define

$$\int_\Gamma f = \int_\Gamma fT = \int_a^b f\big(P(t)\big)T(t) \cdot T(t)\|P'(t)\|\, dt$$
$$= \int_a^b f\big(P(t)\big)\|P'(t)\|\, dt \tag{6.1}$$

Because of (6.1) we sometimes write $\int_\Gamma f\, ds$ in place of $\int_\Gamma f$.

We now seek to identify those vector fields in $\mathbb{R}^n$ which are the gradient of a scalar-valued function. Vector fields of this kind are called *conservative* and are said to have a scalar potential. If $\nabla f = F$ we call f a *scalar potential* of F. Our investigation of this problem leads to a generalisation of the *fundamental theorem of one-variable calculus*

$$\int_a^b g'(t)\, dt = g(b) - g(a)$$

where g is continuous on $[a, b]$ and differentiable on (a, b).

We begin by considering properties of the gradient of $f : U \subset \mathbb{R}^n \to \mathbb{R}$. If Γ is a directed curve in U parametrized by $P : [a, b] \to \mathbb{R}^n$ then

$$\int_\Gamma \nabla f = \int_a^b \nabla f\big(P(t)\big) \cdot P'(t)\, dt$$
$$= \int_a^b \frac{d}{dt}(f \circ P)(t)\, dt \quad \text{(chain rule)}$$
$$= \Big[f \circ P(t)\Big]_a^b \quad \text{(fundamental theorem of calculus in } \mathbb{R})$$
$$= f\big(P(b)\big) - f\big(P(a)\big).$$

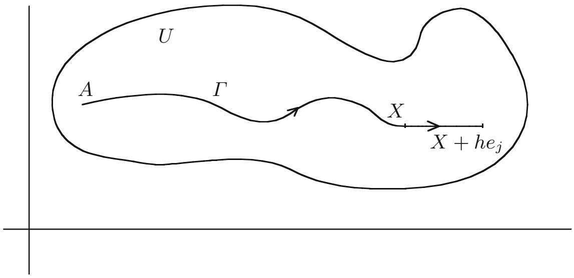


Fig. 6.3

Thus the line integral of ∇f along Γ depends only on the values of f at the initial and final points of the curve. This, as we shall see in Proposition 6.3, characterises vector fields which have a scalar potential. If Γ is a piecewise smooth directed curve in $\mathbb{R}^n$ which is the union of directed curves $(\Gamma_i)_{i=1}^k$ and F is either a vector field along Γ or a real-valued function on Γ we let

$$\int_\Gamma F = \sum_{i=1}^k \int_{\Gamma_i} F.$$

In the next proposition it is necessary to assume that any pair of points in the open set U can be joined by a piecewise smooth directed curve which lies in U; an open set of this kind is said to be *connected*.

Proposition 6.3 *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ denote a continuous vector field on the connected open subset U of $\mathbb{R}^n$. If for any two points A and B in $\mathbb{R}^n$ and any two piecewise smooth directed curves Γ_1 and Γ_2 joining A and B we have*

$$\int_{\Gamma_1} F = \int_{\Gamma_2} F$$

then F has a potential.

Proof Let A denote a fixed point in U. For any X in U let $f(X) = \int_\Gamma F$ where Γ is any piecewise smooth directed curve in U joining A to X. By our hypothesis f is well defined, i.e. there is no ambiguity in the definition. Let $F = (f_1, \ldots, f_n)$ and let Γ denote a curve joining A to X. Fix j, $1 \leq j \leq n$, and let $Y = he_j$ where h denotes a real number close to 0. Let Γ_1 denote the directed curve parametrized by $P(t) = X + tY$, $0 \leq t \leq 1$. Then Γ_1 joins X to $X + Y$ and $\Gamma \cup \Gamma_1$ is a piecewise smooth directed curve in U which joins A to $X + Y$ (Fig. 6.3).

Hence

$$f(X+Y)-f(X)=\int_{\Gamma\cup\Gamma_1}F-\int_\Gamma F$$
$$=\int_{\Gamma_1}F=\int_0^1 F\big(P(t)\big)\cdot P'(t)\,dt.$$

Since $P'(t)=Y=he_j$

$$f(X+Y)-f(X)=\int_0^1 F\big(P(t)\big)\cdot he_j\,dt$$
$$=h\int_0^1 f_j(X+tY)\,dt$$
$$=hf_j(X)+h\int_0^1\Big(f_j(X+tY)-f_j(X)\Big)\,dt.$$

As F is continuous, each component in F and, in particular, f_j is also continuous. Hence

$$\max_{0\le t\le 1}\big|f_j(X+tY)-f_j(X)\big|\longrightarrow 0 \text{ as } h\to 0$$

and

$$\Big|\int_0^1\Big(f_j(X+tY)-f_j(X)\Big)\,dt\Big|\le\max_{0\le t\le 1}\big|f_j(X+tY)-f_j(X)\big|.$$

We have shown

$$f(X+he_j)=f(X)+hf_j(X)+g(X,h)h$$

where $g(X,h)\to 0$ as $h\to 0$. Hence $\dfrac{\partial f}{\partial x_j}(X)=f_j(X)$ and this completes the proof. □

Proposition 6.3 can be used to *find* potentials but is not very practical for *showing* existence. We need a simpler method which follows from the observation:

if $F=(f_1,\ldots,f_n)$ is a continuously differentiable vector field with potential f, i.e. $f_j=\dfrac{\partial f}{\partial x_j}$ for all j, then for all i and j

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big)=\frac{\partial^2 f}{\partial x_j\partial x_i}=\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_j}\Big)=\frac{\partial f_j}{\partial x_i}.$$

The converse is true for *suitable* open sets—the whole space $\mathbb{R}^n$ is always suitable and so also is any convex set in $\mathbb{R}^n$. In $\mathbb{R}^2$ an open set U is suitable if and only if the "interior" of any closed curve in U also lies in U; roughly speaking this means that U contains no holes. In particular the open set $\mathbb{R}^2\backslash\{(0,0)\}$ *is not* suitable.

However, in $\mathbb{R}^3$ the whole space with a finite number of points removed *is* suitable. These examples show that the concept of suitability is rather subtle. The proof of the following result involves Green's theorem (Chap. 9) and Proposition 6.3.

Proposition 6.4 *If $F = (f_1, \ldots, f_n)$ is a continuously differentiable vector field on a suitable open set in $\mathbb{R}^n$ then F has a potential if and only if*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \tag{6.2}$$

for all i and j.

Example 6.5 We wish to show that $F(x, y, z) = (ye^z, xe^z, xye^z)$ has a potential. Let $F = (f_1, f_2, f_3)$. To apply Proposition 6.4 we must verify (6.2), that is we must show

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}.$$

We have

$$\frac{\partial f_1}{\partial y} = e^z = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = ye^z = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = xe^z = \frac{\partial f_3}{\partial y}$$

and hence F has a potential on $\mathbb{R}^3$.

The following is probably the simplest way to find a potential. If f is a potential for F then

$$\frac{\partial f}{\partial x} = ye^z, \quad \frac{\partial f}{\partial y} = xe^z, \quad \frac{\partial f}{\partial z} = xye^z. \tag{6.3}$$

Hence

$$f = \int \frac{\partial f}{\partial x}\, dx = \int ye^z\, dx = xye^z + \phi(y, z)$$

where ϕ is the constant of integration *with respect to x* which may, however, depend on y and z. Differentiating with respect to y we get

$$\frac{\partial f}{\partial y} = xe^z + \frac{\partial \phi}{\partial y}$$

and comparing this with the formula for $\frac{\partial f}{\partial y}$ in (6.3) we have

$$xe^z + \frac{\partial \phi}{\partial y} = xe^z$$

and

$$\frac{\partial \phi}{\partial y} = 0.$$

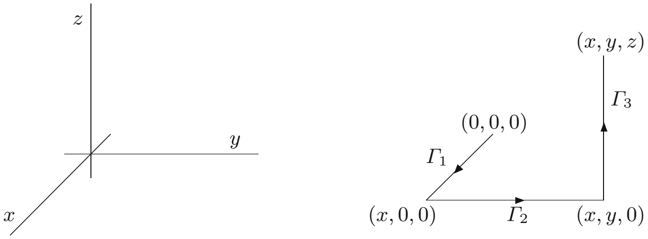


Fig. 6.4

Hence ϕ does not depend on y and we let $\phi(y, z) = \psi(z)$. Now differentiating

$$f(x, y, z) = xye^z + \psi(z)$$

and comparing this with (6.3) gives

$$\frac{\partial f}{\partial z} = xye^z + \psi'(z) = xye^z$$

and $\psi'(z) = 0$. This implies that ψ is a constant and we have shown

$$f(x, y, z) = xye^z + c$$

for some constant c.

We can also use Proposition 6.3 to find a potential f. This proposition tells us that

$$f(x, y, z) = \int_\Gamma F$$

where Γ is any piecewise smooth directed curve in $\mathbb{R}^3$, joining a fixed point to (x, y, z), is a potential of F. We take the fixed point to be the origin in $\mathbb{R}^3$ and a piecewise smooth curve Γ consisting of three straight lines parallel to the axis Γ_1, Γ_2, Γ_3. Specifically we use the following:

Γ_1 joins $(0, 0, 0)$ to $(x, 0, 0)$, $P_1(t) = (t, 0, 0)$, $P_1'(t) = (1, 0, 0)$, $0 \leq t \leq x$
Γ_2 joins $(x, 0, 0)$ to $(x, y, 0)$, $P_2(t) = (x, t, 0)$, $P_2'(t) = (0, 1, 0)$, $0 \leq t \leq y$
Γ_3 joins $(x, y, 0)$ to (x, y, z), $P_3(t) = (x, y, t)$, $P_3'(t) = (0, 0, 1)$, $0 \leq t \leq z$

(see Fig. 6.4).

Then

$$\begin{aligned}
f(x,y,z) &= \int_{\Gamma_1} F + \int_{\Gamma_2} F + \int_{\Gamma_3} F \\
&= \int_0^x f_1\big(P_1(t)\big)\,dt + \int_0^y f_2\big(P_2(t)\big)\,dt + \int_0^z f_3\big(P_3(t)\big)\,dt \\
&= \int_0^x f_1(t,0,0)\,dt + \int_0^y f_2(x,t,0)\,dt + \int_0^z f_3(x,y,t)\,dt \\
&= \int_0^x 0\cdot dt + \int_0^y x\mathrm{e}^0\,dt + \int_0^z xy\mathrm{e}^t\,dt \\
&= 0 + xy + \big[xy\cdot \mathrm{e}^t\big]_0^z \\
&= xy + xy\mathrm{e}^z - xy \\
&= xy\mathrm{e}^z.
\end{aligned}$$

We verify this result by noting

$$\frac{\partial}{\partial x}(xy\mathrm{e}^z) = y\mathrm{e}^z = f_1, \quad \frac{\partial}{\partial y}(xy\mathrm{e}^z) = x\mathrm{e}^z = f_2, \quad \frac{\partial}{\partial z}(xy\mathrm{e}^z) = xy\mathrm{e}^z = f_3.$$

We now define the *cross-product* of two vectors in $\mathbb{R}^3$. This allows us to present the classical notation for what are essentially vector derivatives. We shall also need this product when we discuss the Frenet-Serret equations in Chap. 7. If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbb{R}^3$ then the *cross product* of $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}\times\mathbf{w}$, is defined as

$$\begin{aligned}
\mathbf{v}\times\mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k} \\
&= (v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1).
\end{aligned}$$

Since $\nabla f = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(f)$ we define $\mathrm{curl}(F)$ or $\nabla \times F$, $F = (f_1, f_2, f_3)$ a vector field, by the formula

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)\mathbf{i} - \Big(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\mathbf{k}.$$

One easily sees that $\mathrm{curl}(\nabla f) = 0$ for any real-valued function with continuous first- and second-order partial derivatives while Proposition 6.4 says the converse is true on suitable domains, that is

$$F = \nabla f \Longleftrightarrow \mathrm{curl}(F) = 0.$$

If F is the velocity of a fluid then curl (F) measures the tendency of the fluid to curl or rotate about an axis. Curl(F) gives the direction of the axis of rotation and $\|\mathrm{curl}(F)\|$ measures the speed of rotation.

Since the symbolism $\nabla \times F$ has proved useful we consider the analogous symbol $\nabla \cdot F$ where the dot replaces the cross product. This makes sense for a vector field on $\mathbb{R}^n$. If $F = (f_1, f_2, \ldots, f_n)$, we let

$$\nabla \cdot F = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big) \cdot (f_1, \ldots, f_n) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} := \mathrm{div}(\mathbf{F}).$$

This is called the *divergence* of F and written div(F) (see Chap. 15). A vector field G is a *vector potential* for the vector field F on the open subset U of $\mathbb{R}^3$ if on U we have

$$\mathrm{curl}(G) = \nabla \times G = F.$$

On "suitable" open sets a vector field F has a vector potential if and only if $\nabla \cdot F = \mathrm{div}(F) = 0$.

Example 6.6 We show that $\mathbf{F}(X) = X\big/\|X\|^3$ has a vector potential on $\mathbb{R}^3\backslash\{$ the z-axis$\}$. Let

$$\mathbf{G}(x, y, z) = \frac{(yz, -xz, 0)}{(x^2 + y^2)(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{\|X\|^2 - z^2} \cdot \frac{1}{\|X\|}(y, -x, 0)$$

on $\mathbb{R}^3\backslash\{$ the z-axis$\} = \mathbb{R}^3\backslash\{(x, y, z) : x = y = 0\}$. We calculate curl$(\mathbf{G})$. To simplify our calculations we use symmetry and the following result which follows immediately from Exercise 1.10:

$$\frac{\partial}{\partial x}\big(\|X\|\big) = \frac{x}{\|X\|}.$$

We have

$$\mathrm{curl}\,(\mathbf{G}) = (g_1, g_2, g_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{(x^2 + y^2)\|X\|} & \dfrac{-xz}{(x^2 + y^2)\|X\|} & 0 \end{vmatrix}$$

which implies

$$g_1 = \frac{\partial}{\partial z}\Big(\frac{xz}{(x^2+y^2)\|X\|}\Big) = \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\Big(\frac{z}{\|X\|}\Big)$$
$$= \frac{x}{x^2+y^2}\cdot\frac{\Big(\|X\| - \frac{z^2}{\|X\|}\Big)}{\|X\|^2}$$
$$= \frac{x}{x^2+y^2}\cdot\frac{\big(\|X\|^2 - z^2\big)}{\|X\|^3}$$
$$= \frac{x}{\|X\|^3},$$

since $x^2+y^2+z^2 = \|X\|^2$. By symmetry $g_2 = y \,/\, \|X\|^3$. Finally

$$g_3 = \frac{\partial}{\partial x}\Big(\frac{-xz}{(x^2+y^2)\|X\|}\Big) - \frac{\partial}{\partial y}\Big(\frac{yz}{(x^2+y^2)\|X\|}\Big)$$
$$= -z\Bigg[\frac{(x^2+y^2)\|X\| - 2x^2\|X\| - \frac{x^2(x^2+y^2)}{\|X\|}}{(x^2+y^2)^2\|X\|^2} + \frac{(x^2+y^2)\|X\| - 2y^2\|X\| - \frac{y^2(x^2+y^2)}{\|X\|}}{(x^2+y^2)^2\|X\|^2}\Bigg]$$
$$= \frac{-z(-1)}{\|X\|^3} = \frac{z}{\|X\|^3}$$

and we have shown

$$\mathrm{curl}(\mathbf{G}) = \frac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}} = \frac{X}{\|X\|^3} = \mathbf{F}.$$

Exercises

6.1 Evaluate $\int_\Gamma F$ where F is a vector field and P is a parametrization of the directed curve Γ

(a) $F(x, y, z) = (x, y, z)$, $P(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$,
(b) $F(x, y, z) = x^2\,dx + xyz\,dy + xz\,dz$, $P(t) = (t, t^2, t^3)$, $0 \le t \le 1$,
(c) $F(x, y, z) = \cos z\mathbf{i} + e^x\mathbf{j} + e^y\mathbf{k}$, $P(t) = (1, t, e^t)$, $0 \le t \le 4$.

6.2 Find, if they exist, scalar potentials for the following vector fields

(a) $F(x, y, z) = (2x + ze^{xz},\ z,\ y + xe^{zx})$,
(b) $G(x, y, z) = (y,\ z\cos yz,\ y\cos yz)$,
(c) $H(x, y, z) = \big(y + yz\cos(xyz),\ x + xz\cos(xyz),\ 2z + xy\cos(xyz)\big)$,
(d) $K(x, y, z) = \big(2x\cos(x^2+yz),\ z\cos(x^2+yz),\ y\cos(x^2+yz)\big)$.

6.3 Let $f(x, y, z) = x^2y^2 + y^2z^2$. Verify directly that $\nabla \times \nabla f = 0$.

6.4 Compute the curl of each of the following vector fields:

(a) $F_1(x, y, z) = \dfrac{(3, 1, 2)}{\|X\|}$

(b) $F_2(x, y, z) = \dfrac{(yz,\ zx,\ xy)}{\|X\|}$

(c) $F_3(X) = \dfrac{\langle X, X\rangle X}{\|X\|^4}$

where $X = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$.

6.5 If **a**, **b**, **c** and **d** are vectors in $\mathbb{R}^3$ show that

(a) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.

6.6 Let $f,\ g\colon \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{F},\ \mathbf{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote smooth functions. Prove

(a) $\mathrm{div}(f\mathbf{F}) = f\,\mathrm{div}(\mathbf{F}) + \nabla f \cdot \mathbf{F}$

(b) $\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathrm{curl}(\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \mathrm{curl}(\mathbf{G})$

(c) $\mathrm{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f\,\mathrm{curl}(\mathbf{F})$.

6.7 If $f\colon U \text{ (open)} \subset \mathbb{R}^3 \to \mathbb{R}$ has continuous second-order derivatives show that

$$\mathrm{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Symbolically the left-hand side has the form $\nabla \cdot \nabla f$ and is written (for this reason) $\nabla^2 f$. ($\nabla^2 f$ is called the *Laplacian* of f and if $\nabla^2 f = 0$ then f is called *harmonic*).

6.8 Show that $\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ is harmonic on $\mathbb{R}^3 \setminus \{0, 0, 0\}$.

6.9 If $f\colon \mathbb{R}^+ \to \mathbb{R}$ and $g(X) = f(\|X\|)$ for $X \in \mathbb{R}^3 \setminus \{0, 0, 0\}$ show that

$$\nabla g(X) = f'(\|X\|)\frac{X}{\|X\|} \quad \text{and} \quad \nabla^2 g(X) = f''(\|X\|) + \frac{2}{\|X\|} f'(\|X\|).$$

Show that g is harmonic on $\mathbb{R}^3 \setminus \{0, 0, 0\}$ if and only if

$$f(r) = \frac{A}{r} + B$$

for all $r \neq 0$ in $\mathbb{R}$.

Chapter 7
The Frenet–Serret Equations

Summary *We discuss curvature and torsion of directed curves and derive the Frenet–Serret equations. Vector-valued differentiation and orthonormal bases are the main tools used.*

In this chapter we define geometric concepts associated with a directed curve and derive a set of equations—the *Frenet–Serret equations*—which capture the fundamental relationships between them.

We begin with directed curves in $\mathbb{R}^2$ since this particular case exhibits special features not present in higher dimensions. These are due to considering a *one-dimensional* object (the directed curve) in *two-dimensional* space, $\mathbb{R}^2$. The same phenomena appear in Chap. 12 when we examine a *two-dimensional* object (an oriented surface) in *three-dimensional* space, $\mathbb{R}^3$, and the same underlying principles are present when we introduce *torsion* later in this chapter. Moreover, our motivation and interpretation of normal curvature (Chap. 16) and geodesic curvature (Chap. 18) are based on our study of curves in $\mathbb{R}^2$. This special straightforward case deserves particular attention because of the insight it provides into later developments.

Let $P\colon [a,b] \to \mathbb{R}^2$ denote a *unit speed* parametrization of the directed curve Γ and let $P(t) = \big(x(t), y(t)\big)$ for all t in $[a,b]$. At $P(t) \in \Gamma$ the unit tangent, $T(t)$, is given by

$$T(t) = P'(t) = \big(x'(t), y'(t)\big).$$

The special features, mentioned above, imply that there are just *two* unit vectors in $\mathbb{R}^2$ perpendicular to $T(t)$ and using the *anticlockwise* (or *counterclockwise*) *orientation* of $\mathbb{R}^2$ we can distinguish between them. If we rotate $T(t)$ through $+\pi/2$ in an anticlockwise direction we obtain a unit vector on the *left-hand side* of the direction of motion along Γ (Fig. 7.1). We call this unit vector the *unit normal* to Γ at $P(t)$ and denote it by $N(t)$. In coordinates

$$N(t) = \big(-y'(t), x'(t)\big).$$

We have $\langle T(t), T(t)\rangle = 1$ and differentiating we get, by the product rule,

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_7,

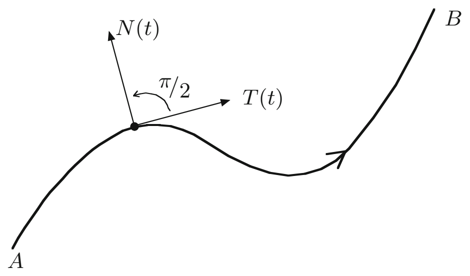


Fig. 7.1

$$\frac{d}{dt}\langle T(t), T(t)\rangle = 0 = \langle T'(t), T(t)\rangle + \langle T(t), T'(t)\rangle.$$

Since

$$\langle T'(t), T(t)\rangle = \langle T(t), T'(t)\rangle$$

this implies

$$\langle T'(t), T(t)\rangle = 0.$$

Hence $T'(t)$ is perpendicular to $T(t)$ and, using once more the fact that $\mathbb{R}^2$ is two-dimensional, we see that $T'(t)$ is parallel to $N(t)$. The *curvature* of Γ at $P(t)$ is defined as the unique scalar, $\kappa(t)$, satisfying

$$T'(t) = \kappa(t)N(t). \tag{7.1}$$

In terms of coordinates

$$\begin{aligned}\kappa(t) &= \langle \kappa(t)N(t), N(t)\rangle = \langle T'(t), N(t)\rangle \\ &= \big(x''(t), y''(t)\big)\cdot\big(-y'(t), x'(t)\big) \\ &= y''(t)x'(t) - x''(t)y'(t)\end{aligned} \tag{7.2}$$

for all $t \in [a, b]$. We call $|\kappa(t)|$ the *absolute curvature* of Γ at $P(t)$ and note that

$$|\kappa(t)| = \|T'(t)\| = \|P''(t)\|. \tag{7.3}$$

Example 7.1 Let $P: [a, b] \to \mathbb{R}^2$ denote an *arbitrary* parametrization of the directed curve Γ. We recall from Chap. 5 that $P \circ s^{-1}: [0, l] \to \Gamma$ is a unit speed parametrization of Γ where l is the length of Γ and s is the length function. If $P(t) = \big(x(t), y(t)\big)$, $t \in [a, b]$, then

$$P \circ s^{-1}(t) = \big(x \circ s^{-1}(t), y \circ s^{-1}(t)\big), \quad t \in [0, l]$$

and, moreover,

$$(s^{-1})'(t) = \frac{1}{\|P'\big(s^{-1}(t)\big)\|} = \frac{1}{\Big(x'\big(s^{-1}(t)\big)^2 + y'\big(s^{-1}(t)\big)^2\Big)^{1/2}}.$$

We have

$$(x \circ s^{-1})'(t) = x'\big(s^{-1}(t)\big) \cdot (s^{-1})'(t)$$

and

$$(x \circ s^{-1})''(t) = x''\big(s^{-1}(t)\big)\big((s^{-1})'(t)\big)^2 + x'\big(s^{-1}(t)\big) \cdot (s^{-1})''(t)$$

and analogous formulae for $(y \circ s^{-1})'(t)$ and $(y \circ s^{-1})''(t)$. Substituting these into (7.2) and simplifying we obtain the curvature at the point $P(t)$, $t \in [a, b]$,

$$\kappa(t) = \frac{y''(t)x'(t) - x''(t)y'(t)}{\Big(\big(x'(t)\big)^2 + \big(y'(t)\big)^2\Big)^{3/2}}. \tag{7.4}$$

If Γ is the graph of a smooth function $f : [a, b] \to \mathbb{R}$ directed from left to right then $P(t) = \big(t, f(t)\big)$, $t \in [a, b]$, is a parametrization of Γ. This parametrization is only unit speed in the trivial case of a horizontal line (why?). Since $x(t) = t$ we have $x'(t) = 1$ and $x''(t) = 0$ and as $y(t) = f(t)$ we obtain $y'(t) = f'(t)$ and $y''(t) = f''(t)$. Hence the curvature at $\big(t, f(t)\big)$ is

$$\kappa(t) = \frac{f''(t)}{\big(1 + f'(t)^2\big)^{3/2}}.$$

We now discuss the geometric significance of curvature. Let P denote a unit speed parametrization of the directed curve Γ. For simplicity we suppose $0 \in [a, b]$, the domain of definition of P. If t is close to zero then

$$P(t) = P(0) + P'(0)t + P''(0)\frac{t^2}{2} + g(t)t^2 \tag{7.5}$$

where $g(t) \to 0$ as $t \to 0$. Since $P'(0) = T(0)$ and $P''(0) = T'(0) = \kappa(0)N(0)$ we can rewrite this as

$$P(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} + g(t)t^2.$$

The function

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} \tag{7.6}$$

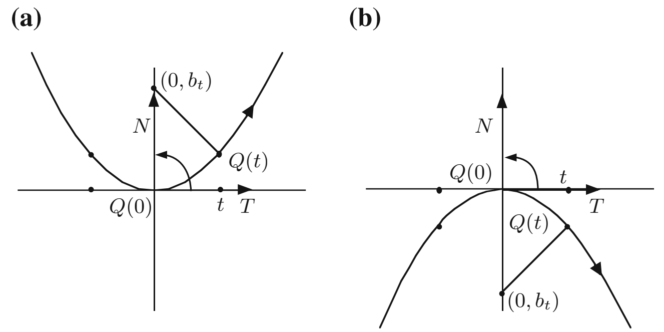


Fig. 7.2

is the best *quadratic approximation* to Γ near $P(0)$ and the curve, parametrized by Q, has the *same* tangent, the *same* normal and the *same* curvature as Γ at $P(0)$. By translating and rotating the plane, if necessary, we can suppose $P(0) = (0, 0)$, $T(0) = (1, 0)$ and $N(0) = (0, 1)$. This implies $Q(t) = (t, \kappa(0)t^2/2)$ and, if $\kappa(0) \neq 0$, then one of the two situations portrayed in Fig. 7.2 holds.

Let C_t denote the circle with centre (a_t, b_t) which passes through the points $Q(-t)$, $Q(0)$, $Q(t)$. By symmetry (a_t, b_t) lies on the y-axis, hence $a_t = 0$ and $|b_t|$ is the radius of C_t. As t tends to zero the circles C_t converge to a circle C with centre $(0, b)$ and radius $|b|$. This is the *circle of curvature* at $Q(0)$ to the directed curve parametrized by Q. Since $\frac{P(t)-Q(t)}{t^2} \to 0$ as $t \to 0$ it can easily be shown that C is also the circle of curvature to Γ at $P(0)$, i.e. the circle that fits closest to Γ near $P(0)$. The centre $(0, b)$ is called the *centre of curvature* of Γ at $P(0)$ and $|b|$ is the *radius of curvature*.

We have

$$b_t^2 = \|(0, b_t) - (t, \kappa(0)t^2/2)\|^2 = t^2 + (b_t - \kappa(0)t^2/2)^2.$$

Hence

$$b_t^2 = t^2 + b_t^2 - b_t\kappa(0)t^2 + \kappa(0)^2t^4/4$$

and

$$b_t\kappa(0) = 1 + \frac{\kappa(0)^2t^2}{4}.$$

Letting t tend to zero we get $b\kappa(0) = 1$. We interpret $|\kappa(0)|$ as

$$\frac{1}{|b|} = \frac{1}{\text{radius of circle of curvature}}.$$

Since the sign of b tells us on which side of Γ the circle of curvature lies and $b\kappa(0) = 1$ we have

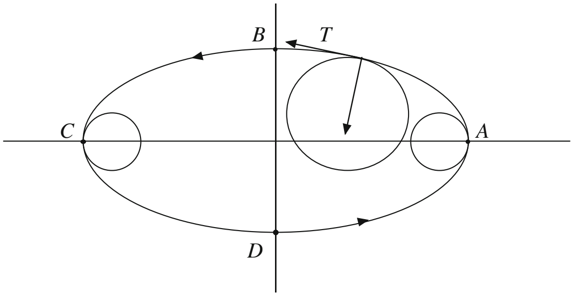


Fig. 7.3

$$\kappa(0) > 0 \iff \text{the circle of curvature and the normal are on the } \textit{same} \text{ side of } \Gamma \text{ (Fig.7.2a)}$$
$$\kappa(0) < 0 \iff \text{the circle of curvature and the normal lie on } \textit{opposite} \text{ sides of } \Gamma \text{(Fig.7.2b).}$$

If $\kappa(0) = 0$ then Γ is rather flat near $P(0)$ and the circle of curvature has infinite radius and thus is a straight line—in our case the x-axis.

We have thus established a geometric interpretation for both absolute curvature and the sign of curvature in $\mathbb{R}^2$ and this often yields immediate and useful information. For example, consider the ellipse in Fig. 7.3, oriented in an anticlockwise direction. The normal will always point into the ellipse and is called, for this reason, the *inner normal*. Since the circle of curvature at any point has finite radius and lies on the same side of the tangent line as the curve the curvature is always strictly positive.

At the points A and C the circles of closest fit to the ellipse have minimum radii among all points on the ellipse. Hence we have maximum curvature at A and C and, similarly, minimum curvature at B and D.

Now consider a directed curve Γ in $\mathbb{R}^n$, $n > 2$, with unit speed parametrization $P\colon [a, b] \to \Gamma$. As before the unit tangent to Γ at $P(t)$ is $P'(t) = T(t)$. We cannot, however, define the unit normal to Γ at $P(t)$ by rotating $T(t)$ since there are not just *two* but an *infinite* number of sides to Γ and thus an infinite number of ways of choosing a unit vector perpendicular to $T(t)$. Even if there were only two directions we would require some concept of *anticlockwise* direction in $\mathbb{R}^n$ to distinguish between them.

To define a normal we need to choose a direction perpendicular to T which is associated in some way to the curve Γ. We still have $\langle T(t), T(t)\rangle = 1$ for all t in $[a, b]$ and differentiating, as we did previously, we get $\langle T'(t), T(t)\rangle = 0$. We have found a special vector, $T'(t)$, perpendicular to $T(t)$ and if this is non-zero (or equivalently if $P''(t) \neq 0$) we define the *unit normal*, $N(t)$, by

$$N(t) = \frac{T'(t)}{\|T'(t)\|}.$$

We define the *curvature* $\kappa(t)$ by $\kappa(t) = \|T'(t)\|$. Thus the normal is *only* defined at points of non-zero curvature and at such points we obtain the equation

$$T'(t) = \kappa(t)N(t). \tag{7.1$'$}$$

Note that (7.1′) and (7.1) are the *same* equation. However, (7.1) applies to a curve in $\mathbb{R}^2$ while (7.1′), which is the first of the *Frenet–Serret equations* when $n = 3$, applies to a curve in $\mathbb{R}^n$. The definitions of curvature and normal are *different* in these two equations. The technique of using an equation in a simple setting to extend a definition to a more general setting is standard and useful in mathematics.

Now that we have defined κ and N in $\mathbb{R}^n$ we must investigate their properties as they could well be different to those in $\mathbb{R}^2$. Using the same terminology is an expression of our aspirations but does not qualify as a proof. We note first that curvature in $\mathbb{R}^n$ is *always* defined and always non-negative but certain curves, such as straight lines, do not have a normal.

Equation (7.5) and the approximation (7.6) are still valid for curves in $\mathbb{R}^n$ and the same analysis shows that $\kappa(t)$ can be interpreted as the reciprocal of the radius of the circle of curvature to Γ at $P(t)$. Hence curvature in $\mathbb{R}^n$, $n > 2$, has the *same* geometrical interpretation as absolute curvature in $\mathbb{R}^2$. We have seen that the sign of curvature in $\mathbb{R}^2$ was related to the different sides of a curve and, in view of our previous remarks, it is not surprising that it does not feature when $n > 2$.

From now on we restrict ourselves to curves in $\mathbb{R}^3$. The approximation

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2}$$

near $0 \in [a, b]$ is still valid and shows that the plane (or two-dimensional subspace) in $\mathbb{R}^3$ closest to the curve near $P(t)$ is the plane through $P(t)$ spanned by $\{T(t), N(t)\}$. We call this the *osculating plane* of the curve at $P(t)$. We may consider the osculating plane as the two-dimensional analogue of the *tangent line*. As we move along the curve the osculating plane will generally change and the more it changes the more twisted the curve. We measure this by defining a new concept—*torsion*—which we denote by τ. We define torsion and a third unit vector, the *binormal* in $\mathbb{R}^3$, in a fashion similar to the way we introduced curvature and the normal for curves in $\mathbb{R}^2$. At each point on the directed curve Γ in $\mathbb{R}^3$ we have obtained two perpendicular unit vectors T and N and these span a two-dimensional subspace of $\mathbb{R}^3$. Hence there are precisely *two* unit vectors perpendicular to T and N. To choose one of these *unambiguously* we require a sense of *direction* or *orientation* in $\mathbb{R}^3$. This will also be important in integration theory. The basis of this sense of direction is known as *"the right-hand rule"* and we describe it in the special case in which we are interested. Use the right thumb as the vector T and the first finger in place of N. Then the second finger will, when put perpendicular to T and N, give the direction of the *binormal*, B (Fig. 7.4).

Think of T and N as determining the flat plane of this page. This page has two sides and thus two unit vectors perpendicular to it. Since the vector N is obtained in

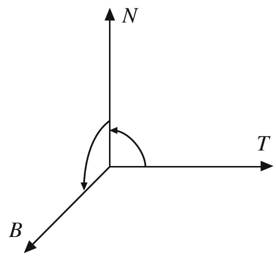


Fig. 7.4

Fig. 7.4 by rotating T in an anticlockwise direction, B will be the unit vector on *this* side of the page. Note that we placed T *before* N in this construction.

Mathematically, we can find the binormal by using the cross product in $\mathbb{R}^3$. For a directed curve Γ in $\mathbb{R}^3$ with parametrization P, unit tangent $T(t)$ and unit normal $N(t)$ the binormal at the point $P(t)$ is given by

$$B(t) = T(t) \times N(t).$$

To derive further results we list standard properties of the *cross product*—all of which follow easily from well-known results about *determinants*.

Let $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ be vectors in $\mathbb{R}^3$ then

(a) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
(b) $\mathbf{v} \times \mathbf{w} \neq 0 \iff \mathbf{v}$ and $\mathbf{w}$ are linearly independent
(c) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$
(d) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot |\sin\theta|$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$
(e)

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{w} \cdot \mathbf{u} \times \mathbf{v} = \mathbf{v} \cdot \mathbf{w} \times \mathbf{u}$$

(f) $\|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}\|$ = volume of the parallelepiped with adjacent sides $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$
(g) $\frac{1}{2}\|(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})\|$ = area of triangle with vertices $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$.

By (c), $B(t)$ is perpendicular to both $T(t)$ and $N(t)$ and, by (d),

$$\begin{aligned} \|B(t)\| &= \|T(t)\| \cdot \|N(t)\| \cdot |\sin(\pi/2)| \quad (\text{since} T(t) \perp N(t)) \\ &= 1. \end{aligned}$$

Hence $\{T(t), N(t), B(t)\}$ consists of three mutually perpendicular unit vectors in $\mathbb{R}^3$. In particular, they are linearly independent and so form a basis for $\mathbb{R}^3$. We call

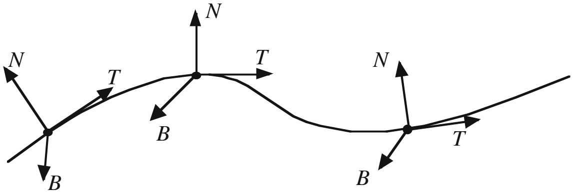


Fig. 7.5

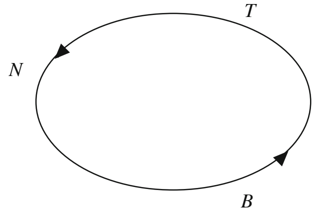


Fig. 7.6

them an *orthonormal basis* ("ortho" comes from orthogonal or perpendicular and normal comes from the fact that they are unit vectors). Another orthonormal basis for $\mathbb{R}^3$ is the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We consider $\{T(t), N(t), B(t)\}$ as a special basis which is *adapted* to studying the curve Γ near $P(t)$. As t varies the basis $\{T(t), N(t), B(t)\}$ changes and is called a *moving frame* along the curve (see Fig. 7.5).

Since $N(t) \times B(t)$ is easily seen to be a unit vector perpendicular to both N and B we must have

$$N(t) \times B(t) = \pm T(t).$$

Hence

$$\begin{aligned} T(t) \cdot N(t) \times B(t) &= \pm T(t) \cdot T(t) = \pm 1 \\ &\parallel \text{ by (e)} \\ B(t) \cdot T(t) \times N(t) &= B(t) \cdot B(t) = 1 \end{aligned}$$

and $N(t) \times B(t) = T(t)$. Similarly $B(t) \times T(t) = N(t)$ and, using (a), the remaining cross products involving T, N and B can be found.

The simplest way to remember these is to use the diagram shown in Fig. 7.6. The cross product of any two taken in an anticlockwise direction is the one that follows it, e.g. $N \times B = T$. If we work in a clockwise direction we get, from (a), the negative of the following one, e.g. $N \times T = -B$.

We are now in a position to make effective use of the orthonormal basis $\{T, N, B\}$. Since it is a basis any vector $\mathbf{v}$ can be written in the form

$$\mathbf{v} = \alpha T + \beta N + \gamma B$$

for some real numbers α, β and γ. If we take the inner product of both sides with respect to T then

$$\langle \mathbf{v}, T\rangle = \alpha\underset{\substack{\| \\ 1}}{\langle T, T\rangle} + \beta\underset{\substack{\| \\ 0}}{\langle N, T\rangle} + \gamma\underset{\substack{\| \\ 0}}{\langle B, T\rangle} = \alpha$$

Similarly $\beta = \langle \mathbf{v}, N\rangle$ and $\gamma = \langle \mathbf{v}, B\rangle$ and thus

$$\mathbf{v} = \langle \mathbf{v}, T\rangle T + \langle \mathbf{v}, N\rangle N + \langle \mathbf{v}, B\rangle B. \tag{7.7}$$

We also note, although we do not require it here, that Pythagoras' Theorem implies

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, T\rangle^2 + \langle \mathbf{v}, N\rangle^2 + \langle \mathbf{v}, B\rangle^2.$$

We use vector-valued differentiation to find $B'(t)$. Since $\|B(t)\| = 1$ we have $\langle B(t), B(t)\rangle = 1$ and hence

$$\begin{aligned} 0 = \frac{d}{dt}\langle B(t), B(t)\rangle &= \langle B'(t), B(t)\rangle + \langle B(t), B'(t)\rangle \\ &= 2\langle B'(t), B(t)\rangle. \end{aligned} \tag{7.8}$$

Since $\langle B(t), T(t)\rangle = 0$ we get, in the same way,

$$\begin{aligned} 0 &= \frac{d}{dt}\langle B(t), T(t)\rangle \\ &= \langle B'(t), T(t)\rangle + \langle B(t), T'(t)\rangle \\ &= \langle B'(t), T(t)\rangle + \langle B(t), \kappa(t)N(t)\rangle \quad \text{(by (7.1}'\text{))} \\ &= \langle B'(t), T(t)\rangle + \kappa(t)\langle B(t), N(t)\rangle \\ &= \langle B'(t), T(t)\rangle \quad (\text{since} B \perp N). \end{aligned} \tag{7.9}$$

Replacing $\mathbf{v}$ by $B'(t)$ in (7.7) and using (7.8) and (7.9) we have

$$B'(t) = \langle B'(t), N(t)\rangle N(t)$$

i.e. $B'(t)$ is parallel to $N(t)$. We use this equation to define the *torsion* of Γ at $P(t)$, $\tau(t)$, by letting[1]

$$\tau(t) = -\langle B'(t), N(t)\rangle$$

and obtain another of the Frenet–Serret equations

$$B'(t) = -\tau(t)N(t). \tag{7.10}$$

[1] By introducing one minus sign here we avoid many more minus signs later.

We discuss the geometric significance of torsion in Chap. 8 but note at this point that $B(t)$ is normal to the osculating plane and hence torsion is a measure of the rate of change of the osculating plane.

We have found T' and B' in Eqs. (7.1′) and (7.10) for a directed curve in $\mathbb{R}^3$ and the remaining Frenet–Serret equation is an expression for N' in terms of the basis $\{T, N, B\}$. To find N' we differentiate the equation $N = B \times T$ using the product rule. We have

$$N' = B' \times T + B \times T' = -\tau N \times T + B \times \kappa N = \tau T \times N - \kappa N \times B.$$

Since $T \times N = B$ and $N \times B = T$ we obtain for all t the equation

$$N'(t) = -\kappa(t)T(t) + \tau(t)B(t). \tag{7.11}$$

Equations (7.1′), (7.10) and (7.11) which express T', N' and B' in terms of T, N and B are known as the *Frenet–Serret equations* and contain, for all practical purposes, complete information on the curve (see also Example 8.5). The set $\{T, N, B, \kappa, \tau\}$ is known as the *Frenet–Serret apparatus* of the curve Γ. It is important to remember that whenever we discuss normals to curves in $\mathbb{R}^3$ or the Frenet–Serret equations for a curve we assuming the curvature is *strictly positive*.

The Frenet–Serret equations are easily remembered when expressed in matrix form

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Certain classical curves are not covered by Definition 5.1 and we encounter one of these—the helix—in our next example. In its natural state the helix does not have an initial or final point and is usually parametrized over $\mathbb{R}$. The reader should have little difficulty analysing such curves by the methods we have already developed. Essentially one examines different finite parts of the curve in turn—for example consider how we treated a part of the helix in Example 5.2. For the sake of completeness, however, we mention how our definition of parametrized curve can be extended in the following natural way to include such curves:

a continuous mapping $P : I \longrightarrow \mathbb{R}^n$, I an interval in $\mathbb{R}$, is a parametrized curve if for every closed interval $[a, b] \subset I$ the restriction of P to $[a, b]$ satisfies (a), (c) and (d) of Definition 5.1.

Example 7.2 Let Γ denote the helix parametrized by

$$P(t) = (r\cos\omega t, r\sin\omega t, h\omega t), \quad -\infty < t < +\infty$$

where $\omega = (r^2 + h^2)^{-1/2}$ and r, h and ω are all positive. Since

$$P'(t) = (-r\omega \sin \omega t, r\omega \cos \omega t, h\omega)$$

and

$$\begin{aligned} \|P'(t)\| &= (r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t + h^2\omega^2)^{1/2} \\ &= (r^2\omega^2 + h^2\omega^2)^{1/2} = \omega(r^2 + h^2)^{1/2} = 1 \end{aligned}$$

the parametrization is unit speed and $T(t) = P'(t)$. We have

$$P''(t) = T'(t) = (-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t, 0)$$

and

$$\kappa(t) = \|T'(t)\| = (r^2\omega^4 \cos^2 \omega t + r^2\omega^4 \sin^2 \omega t)^{1/2} = \omega^2 r.$$

Note that Γ is *not* a circle but has constant curvature. Hence $\kappa(t) > 0$ and

$$\begin{aligned} N(t) &= \frac{T'(t)}{\|T'(t)\|} = \frac{1}{\omega^2 r}(-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t, 0) \\ &= (-\cos \omega t, -\sin \omega t, 0). \end{aligned}$$

We have

$$\begin{aligned} B(t) = T(t) \times N(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\omega \sin \omega t & r\omega \cos \omega t & h\omega \\ -\cos \omega t & -\sin \omega t & 0 \end{vmatrix} \\ &= (h\omega \sin \omega t, -h\omega \cos \omega t, r\omega) \end{aligned}$$

and

$$B'(t) = (h\omega^2 \cos \omega t, h\omega^2 \sin \omega t, 0).$$

Since $\tau(t) = -\langle B'(t), N(t)\rangle$ this implies

$$\begin{aligned} \tau(t) &= -\Big\langle (h\omega^2 \cos \omega t, h\omega^2 \sin \omega t, 0),\ (-\cos \omega t, -\sin \omega t, 0) \Big\rangle \\ &= h\omega^2 \cos^2 \omega t + h\omega^2 \sin^2 \omega t = h\omega^2. \end{aligned}$$

We have calculated the Frenet–Serret apparatus for the helix. In doing so we used two of the Frenet–Serret equations. The third equation

$$N' = -\kappa T + \tau B$$

may be used to check our calculations. The Frenet–Serret apparatus was found in the following sequence: first check that P is unit speed then

$$\left\{ \begin{array}{ccccccccc} T(t) & , & \kappa(t) & , & N(t) & , & B(t) & , & \tau(t) \\ \| & & \| & & \| & & \| & & \| \\ P'(t) & & \|T'(t)\| & & T'(t)/\kappa(t) & & T(t)\times N(t) & & -\langle B'(t), N(t)\rangle \end{array} \right\}$$

Other sequences are also possible but the above appear to be generally more direct (for unit speed curves).

Exercises

7.1 Parametrize the curve $x^2 + (y/3)^2 = 9$ with an anticlockwise orientation and hence find its curvature. Find the points where the curvature is a maximum

(a) by inspecting a sketch;
(b) by differentiating the curvature function;
(c) by inspection of the curvature function.

7.2 Let $f: U$ (open) $\subset \mathbb{R}^2 \to \mathbb{R}$ and suppose $f^{-1}(0)$ has full rank at each point. If $f^{-1}(0)$ can be parametrized as a directed curve in $\mathbb{R}^2$ show that its absolute curvature equals

$$\frac{|f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2|}{(f_x^2 + f_y^2)^{3/2}}.$$

Using this result find the curvature at all points on the ellipse $(x/a)^2 + (y/b)^2 = 1$ directed so that the normal points outwards. Verify your answer when $a = 3$ and $b = 9$ using Exercise 7.1.

7.3 Let Γ denote the plane curve parametrized by

$$P(t) = (t, \log\cos t), \quad -\pi/4 \leq t \leq \pi/4.$$

Show that Γ has curvature $-\cos t$ at $P(t)$.

7.4 Show that a directed curve in $\mathbb{R}^3$ is a straight line if and only if all its tangent lines are parallel.

7.5 Show that each of the following gives a unit speed parametrization of a curve Γ in $\mathbb{R}^3$. Calculate the Frenet–Serret apparatus of the curve and verify that $N' = -\kappa T + \tau B$.

(a) $P(t) = \left(\frac{(1+t)^{3/2}}{3}, \frac{(1-t)^{3/2}}{3}, \frac{t}{\sqrt{2}}\right), \quad 0 \leq t \leq 1/2$

(b) $P(t) = \frac{1}{2}\left(\cos^{-1}(t) - t\sqrt{1-t^2},\ 1-t^2,\ 0\right), \quad 0 \leq t \leq 1/2$

(c) $P(t) = \left(\frac{(1+t^2)^{1/2}}{\sqrt{5}}, \frac{2t}{\sqrt{5}}, \frac{\log(t+\sqrt{1+t^2})}{\sqrt{5}}\right), \quad t \in \mathbb{R}.$

7.6 Let $P(t) = (a\cos t, a\sin t, at\tan\alpha)$ denote a parametrization of a helix where $0 < \cos\alpha < a < 1$. Find a unit speed parametrization. Show that the centre of curvature also moves on a helix and find the cylinder on which this helix lies.

7.7 If a and b are positive real numbers show that the curve parametrized by

$$P(t) = \left(a\cos t, a\sin t, b\cosh\frac{at}{b}\right), \quad t \in \mathbb{R}$$

lies on the cylinder $x^2 + y^2 = a^2$. Show that the osculating plane at any point on the curve makes a constant angle with the tangent plane to the cylinder at that point.

7.8 If $P\colon [a,b] \to \Gamma$ is a unit speed parametrized curve show that

$$\langle P' \times P'', P''' \rangle = \kappa^2\tau$$

and if $\tau \neq 0$ show that

$$\tau = \frac{\langle P' \times P'', P''' \rangle}{\langle P'', P'' \rangle}.$$

7.9 A plane in $\mathbb{R}^3$ (or a cross-section of $\mathbb{R}^3$) consists of all points (x, y, z) satisfying a linear equation $ax + by + cz = d$ where at least one of a, b, c is non-zero. Find $A \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ such that the plane above coincides with

$$\{A \in \mathbb{R}^3; A \cdot X = \alpha\}.$$

Find α when the plane goes through the origin.

7.10 Let P denote a unit speed parametrization of a directed curve in $\mathbb{R}^3$ with non-zero curvature at $P(0)$. Show that the equation of the osculating plane at $P(0)$ is

$$\{X \in \mathbb{R}^3 : (X - P(0)) \times P'(0) \cdot P''(0) = 0\}.$$

7.11 Let $P\colon [0,l] \to \mathbb{R}^3$ denote a unit speed parametrization of a directed curve Γ, with positive curvature and torsion. For $t \in [0,l]$ let

$$Q(t) = \int_0^t B(x)dx.$$

Show that Q defines a unit speed parametrization of a directed curve $\tilde{\Gamma}$. If $\{T, N, B, \kappa, \tau\}$ is the Frenet–Serret apparatus for Γ show that $\{B, -N, T, \tau, \kappa\}$ is the Frenet–Serret apparatus for $\tilde{\Gamma}$.

7.12 Let Γ denote a directed curve in $\mathbb{R}^3$ with positive curvature at all points and suppose $P : [a,b] \longrightarrow \mathbb{R}^3$ is a unit speed parametrization of Γ. Using the Frenet-Serret equations and the identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b}$ find a mapping $X : [a,b] \longrightarrow \mathbb{R}^3$ such that $T'(t) = X(t) \times T(t)$, $N'(t) = X(t) \times N(t)$ and $B'(t) = X(t) \times B(t)$ for all $t \in [a,b]$.

Chapter 8
Geometry of Curves in $\mathbb{R}^3$

Summary *We apply the Frenet–Serret equations to study the geometric significance of torsion, to analyse curves in spheres and to characterise generalised helices.*

We first provide a geometrical interpretation of *zero torsion*.

Proposition 8.1 *If Γ is a directed curve in $\mathbb{R}^3$ with positive curvature at all points then the following are equivalent*

(a) Γ *is a plane curve.*
(b) *the function* $t \to B(t)$ *is constant.*
(c) $\tau(t) = 0$ *for all* t.

Proof Since $\kappa(t) > 0$, $N(t)$ is defined. By the Frenet–Serret equations $B'(t) = -\tau(t)N(t)$ and since $\|N(t)\| = 1$ we have:

$$B(t) \text{ is independent of } t \iff B'(t) = 0 \iff \tau(t) = 0.$$

Hence (b) and (c) are equivalent. Now suppose Γ is a plane curve, i.e. there exists a plane in $\mathbb{R}^3$ which contains Γ. Then there exists a unit vector A in $\mathbb{R}^3$ and a real number c such that $\Gamma \subset \{X \in \mathbb{R}^3 : X \cdot A = c\}$. Let P denote a unit speed parametrization of Γ with domain $[a, b]$. For all $t \in [a, b]$, $P(t) \cdot A = c$. We have

$$\frac{d}{dt}(P(t) \cdot A) = T(t) \cdot A = 0$$

and

$$\frac{d^2}{dt^2}P(t) \cdot A = T'(t) \cdot A = \kappa(t)N(t) \cdot A = 0.$$

Hence A is perpendicular to both $T(t)$ and $N(t)$ and $B(t) = \pm A$ for all t. Suppose $B(t_1) = A$ and $B(t_2) = -A$. The mapping $g : t \longrightarrow B(t) \cdot A$ is a continuous real-valued function on $[a, b]$, $g(t) = \pm 1$ for all t, $g(t_1) = 1$ and $g(t_2) = -1$. This is impossible, since it would imply, by the Intermediate Value Theorem, that there exists $t_0 \in (a, b)$ such that $g(t_0) = 0$. Hence $g(t) = +1$ for all t or $g(t) = -1$ for

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_8,

all t and this implies $B(t) = A$ for all t or $B(t) = -A$ for all t. In either case B is a constant function and (a) $\Longrightarrow$ (b).

Now suppose (b) holds. Let P denote a unit speed parametrization of Γ with domain $[a, b]$. If $t_0 \in (a, b)$, $X_0 = P(t_0)$ and $B(t) = B$ for all t, then

$$\begin{aligned}\frac{d}{dt}\langle P(t) - X_0, B\rangle &= \langle P'(t), B(t)\rangle + \langle P(t) - X_0, 0\rangle \\ &= \langle T(t), B(t)\rangle + 0 \\ &= 0 \ \ (\text{since } T(t) \perp B(t)).\end{aligned}$$

Hence there exists a constant c such that

$$\langle P(t) - X_0, B\rangle = c$$

and P lies in the plane through X_0 perpendicular to B. This shows that (b) $\Longrightarrow$ (a) and completes the proof. □

Proposition 8.1 gives a precise geometric interpretation of *zero torsion*. To interpret *non-zero torsion* we look at an expansion of the parametrization P about a fixed point relative to the basis $\{T(t_0), N(t_0), B(t_0)\}$. For convenience we may suppose $t_0 = 0$.

In Chap. 7 we obtained, using orthogonality, the Taylor series expansion and the Frenet–Serret equations, the following three expansions of $P(t)$:

$$\begin{aligned}P(t) &= \langle P(t), T(0)\rangle T(0) + \langle P(t), N(0)\rangle N(0) + \langle P(t), B(0)\rangle B(0) &(8.1)\\ &= P(0) + P'(0)t + P''(0)\frac{t^2}{2} + g(t)t^2 &(8.2)\\ &= P(0) + T(0)t + \frac{\kappa(0)N(0)}{2}t^2 + g(t)t^2 &(8.3)\end{aligned}$$

where $g(t) \to 0$ in $\mathbb{R}^3$ as $t \to 0$. From (8.3) we can identify the main influence—the first non-constant term in the Taylor series expansion—on the shape of the curve in the $T(0)$ and $N(0)$ directions. Comparing (8.1–8.3) we see also that the first possible non-zero term in the $B(0)$ direction will be $\frac{d^3}{dt^3}\langle P(t), B(0)\rangle\Big|_{t=0}$. By repeated differentiation and use of the Frenet–Serret equation for T' we obtain

$$\begin{aligned}\langle P'(t), B(0)\rangle &= \langle T(t), B(0)\rangle \\ \langle P''(t), B(0)\rangle &= \langle T'(t), B(0)\rangle = \langle \kappa(t)N(t), B(0)\rangle\end{aligned}$$

and, using the Frenet–Serret equation for N',

$$\begin{aligned}\langle P'''(t), B(0)\rangle &= \langle \kappa'(t)N(t) + \kappa(t)N'(t), B(0)\rangle \\ &= \langle \kappa'(t)N(t) + \kappa(t)\big(-\kappa(t)T(t) + \tau(t)B(t)\big), B(0)\rangle.\end{aligned}$$

Letting $t = 0$ we get

$$\begin{aligned}\langle P'''(0), B(0)\rangle &= \kappa'(0)\langle N(0), B(0)\rangle - \kappa^2(0)\langle T(0), B(0)\rangle \\ &\quad + \kappa(0)\tau(0)\langle B(0), B(0)\rangle \\ &= \kappa(0)\tau(0)\end{aligned}$$

since $\{T, N, B\}$ are mutually perpendicular unit vectors. This gives us the approximation

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} + \kappa(0)\tau(0)B(0)\frac{t^3}{6}$$

called the *Frenet approximation* to the curve Γ at 0. The Frenet approximation is clearly a refinement of (8.3) which takes account of torsion. The function $t \to Q(t)$ defines a parametrized curve which has the *same* Frenet–Serret apparatus as the original curve at $P(0)$. From the Frenet approximation we see the influence of non-zero torsion on the shape of the curve. Torsion controls the motion of the curve *orthogonal to the osculating plane*. If $\tau(0) > 0$ then the curve twists towards the side of the osculating plane which contains $B(0)$ and the greater $\tau(0)$ the more dramatic the twist. If $\tau(0) < 0$ the curve twists towards $-B(0)$.

An everyday example of non-zero torsion is given by the curve on the edge of a screw. In tightening a screw one usually uses the right-hand and follows the right-hand rule while in loosening a screw one follows the left-hand rule. If you have any doubts about the difference change hands. This also illustrates the two different orientations of $\mathbb{R}^3$.

Example 8.2 In this example we study a directed curve Γ which lies in a sphere with centre c and radius r. Let P denote a unit speed parametrization of Γ. Our hypothesis states that

$$\|P(t) - c\|^2 = \langle P(t) - c, P(t) - c\rangle = r^2.$$

Consider the expansion of $P(t) - c$ relative to the orthonormal basis $\{T(t), N(t), B(t)\}$, i.e.

$$\begin{aligned}P(t) - c = \langle P(t) - c, T(t)\rangle T(t) + \langle P(t) - c, N(t)\rangle N(t) \\ + \langle P(t) - c, B(t)\rangle B(t).\end{aligned} \tag{8.4}$$

Differentiating we get

$$\frac{d}{dt}\langle P(t) - c, P(t) - c\rangle = 0 = 2\langle P(t) - c, P'(t)\rangle .$$

Since P is unit speed, $P'(t) = T(t)$, and we may restate this as follows:

$$\langle P(t) - c, T(t)\rangle = 0. \tag{8.5}$$

Differentiating again and using the Frenet–Serret equation for T' gives us

$$\begin{aligned} 0 = \frac{d}{dt}\langle P(t) - c, T(t)\rangle &= \langle T(t), T(t)\rangle + \langle P(t) - c, T'(t)\rangle \\ &= 1 + \kappa(t)\langle P(t) - c, N(t)\rangle. \end{aligned}$$

Hence $\kappa(t) \neq 0$ for all t, $N(t)$ is defined and

$$\langle P(t) - c, N(t)\rangle = -\frac{1}{\kappa(t)}. \tag{8.6}$$

Differentiating (8.6) and applying the Frenet–Serret equation for N' we obtain

$$\begin{aligned} -\left(\frac{1}{\kappa(t)}\right)' &= \frac{d}{dt}\langle P(t) - c, N(t)\rangle \\ &= \langle T(t), N(t)\rangle + \langle P(t) - c, N'(t)\rangle \\ &= \langle P(t) - c, -\kappa(t)T(t) + \tau(t)B(t)\rangle \text{ (since } N \perp T) \\ &= \tau(t)\langle P(t) - c, B(t)\rangle \text{ (by (8.5))}. \end{aligned}$$

If $\tau(t_0) \neq 0$ for some t_0 then, by continuity, $\tau(t) \neq 0$ for all t near t_0 and

$$\langle P(t) - c, B(t)\rangle = -\frac{1}{\tau(t)}\left(\frac{1}{\kappa(t)}\right)'. \tag{8.7}$$

Substituting (8.5–8.7) into (8.4) we get

$$P(t) - c = -\frac{1}{\kappa(t)}N(t) - \frac{1}{\tau(t)}\left(\frac{1}{\kappa(t)}\right)' B(t)$$

and we have found $P(t)$ in terms of the Frenet–Serret apparatus of Γ. By Pythagoras' theorem

$$r^2 = \|P(t) - c\|^2 = \frac{1}{\kappa(t)^2} + \left(\left(\frac{1}{\kappa(t)}\right)' \cdot \frac{1}{\tau(t)}\right)^2.$$

Hence we have recovered the radius of the sphere from the curvature and torsion *when we know* that the curve lies in a sphere. In particular, we see that $r^2 \geq 1/\kappa(t)^2$, or $\kappa(t) \geq 1/r$, which we may loosely rephrase as saying that a curve in a sphere is at least as curved as the sphere in which it lies.

If $\tau(t) = 0$ for all t in an open interval I then the above implies that $\kappa(t)$ is constant on I. Since a plane circle has constant curvature, Proposition 8.1 and Example 8.5 imply that the part of Γ parametrized by restricting P to I is part of a circle contained in a plane in $\mathbb{R}^3$.

Example 8.3 The helix in Example 7.2 satisfies $\langle T_p, (0, 0, 1)\rangle = h\omega$ for all p where T_p is the unit tangent at the point p. We generalise this by defining a *generalised* *helix* in $\mathbb{R}^3$, as a curve Γ of positive curvature (at all points) for which there exists a unit vector $\mathbf{u}$ in $\mathbb{R}^3$ such that

$$\langle T_p, \mathbf{u} \rangle = c \text{ (constant)}$$

for all $p \in \Gamma$. Let P denote a unit speed parametrization of the generalised helix Γ and let $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ denote the Frenet–Serret apparatus at $P(t)$. We prove the following characterisation:

$$\Gamma \text{ is a generalised helix} \iff \frac{\tau(t)}{\kappa(t)} \text{ is constant (i.e. independent of } t\text{).}$$

Since $\langle T(t), \mathbf{u} \rangle$ is constant it follows, by the Cauchy–Schwarz inequality (Example 3.4), that

$$|\langle T(t), \mathbf{u} \rangle| \leq \|T(t)\| \cdot \|\mathbf{u}\| \leq 1$$

and $\langle T(t), \mathbf{u} \rangle = \cos\theta$ for some θ. If $\theta = n\pi$ then, by the equality case in the Cauchy–Schwarz inequality, $T(t) = \pm\mathbf{u}$. By the Intermediate Value Theorem (see the proof of Proposition 8.1) this implies that $T(t)$ is a constant function of t. Hence $T'(t) = 0$ and this contradicts our hypothesis. We thus have $\langle T(t), \mathbf{u} \rangle = \cos\theta$ for some $\theta \neq n\pi$. Again using the orthonormal basis $\{T(t), N(t), B(t)\}$ we have

$$\mathbf{u} = \langle \mathbf{u}, T(t) \rangle T(t) + \langle \mathbf{u}, N(t) \rangle N(t) + \langle \mathbf{u}, B(t) \rangle B(t).$$

Since $\langle T(t), \mathbf{u} \rangle = \cos\theta$

$$\frac{d}{dt}\big(\langle T(t), \mathbf{u} \rangle\big) = 0 = \langle T'(t), \mathbf{u} \rangle + \langle T(t), (\mathbf{u})' \rangle = \langle \kappa(t) N(t), \mathbf{u} \rangle$$

(since $\mathbf{u}$ is a constant, $(\mathbf{u})' = 0$). By our hypothesis $\kappa(t) \neq 0$ and this implies $\langle \mathbf{u}, N(t) \rangle = 0$. Hence

$$\mathbf{u} = \cos\theta T(t) + \sin\theta B(t).$$

A further application of the Frenet–Serret equations implies

$$\begin{aligned} 0 = \frac{d}{dt}\mathbf{u} &= \cos\theta T'(t) + \sin\theta B'(t) \\ &= \cos\theta \kappa(t) N(t) - \sin\theta \tau(t) N(t) \\ &= \big(\cos\theta \kappa(t) - \sin\theta \tau(t)\big) N(t). \end{aligned}$$

Since $\|N(t)\| = 1$ this implies $\cos\theta\kappa(t) = \sin\theta\tau(t)$ and

$$\frac{\tau(t)}{\kappa(t)} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

We have shown $\tau(t)/\kappa(t)$ is constant for any generalised helix. In obtaining this result we obtained a formula for $\mathbf{u}$ and now use this to prove the converse. Let Γ denote a directed curve with non-zero curvature in $\mathbb{R}^3$ such that $\tau(t)/\kappa(t)$ is constant (i.e. independent of t). This hypothesis implies that there exists a real number θ, $0 < \theta < \pi$, such that $\cot\theta = \tau(t)/\kappa(t)$ for all t (note that θ does not depend on t). Let

$$\mathbf{u}(t) = \cos\theta\, T(t) + \sin\theta\, B(t).$$

By Pythagoras' theorem $\|\mathbf{u}\|^2 = \cos^2\theta + \sin^2\theta = 1$ and $\mathbf{u}$ is a unit vector. To show that $\mathbf{u}$ does not depend on t we prove $\frac{d}{dt}\big(\mathbf{u}(t)\big) = 0$. By the Frenet–Serret equations

$$\begin{aligned}\frac{d}{dt}\big(\mathbf{u}(t)\big) &= \cos\theta\, T'(t) + \sin\theta\, B'(t)\\ &= \big(\kappa(t)\cos\theta - \tau(t)\sin\theta\big)N(t) = 0.\end{aligned}$$

Hence $\mathbf{u}$ does not depend on t and so is a constant. Moreover, since $T \perp B$,

$$\langle T(t), \mathbf{u}\rangle = \langle T(t), \cos\theta\, T(t) + \sin\theta\, B(t)\rangle = \cos\theta.$$

This shows that Γ is a generalised helix and justifies our claim.

Our analysis so far applies to unit speed parametrizations of a directed curve. Unfortunately, many natural parametrizations of curves are not unit speed. It is thus useful to be able to calculate the Frenet–Serret apparatus directly from an arbitrary parametrization.

Let $P\colon [a,b] \to \mathbb{R}^3$ denote an *arbitrary* parametrization of the directed curve Γ. We suppose $P'(t)$ and $P''(t)$ are both non-zero for all t. Let $s\colon [a,b] \to [0,l]$ denote the length function associated with P (see Chap. 5). Then l is the length of Γ, $\|P'(t)\| = s'(t)$ and $Q := P \circ s^{-1}$ is a unit speed parametrization of Γ. Let $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ denote the Frenet–Serret apparatus *at the point* $P(t)$ on Γ. We have

$$Q \circ s(t) = P \circ s^{-1} \circ s(t) = P(t).$$

Hence

$$\frac{d}{dt}\Big(Q\big(s(t)\big)\Big) = Q'\big(s(t)\big)s'(t) = P'(t) = \|P'(t)\|T(t)$$

and

$$Q'\big(s(t)\big) = T(t) = \frac{P'(t)}{\|P'(t)\|}. \tag{8.8}$$

Differentiating again

$$\begin{aligned}\frac{d^2}{dt^2}\Big(Q\big(s(t)\big)\Big) &= \frac{d}{dt}\Big(Q'\big(s(t)\big)\cdot s'(t)\Big)\\ &= Q''\big(s(t)\big)\big(s'(t)\big)^2 + Q'\big(s(t)\big)s''(t) = P''(t).\end{aligned}$$

Since Q has unit speed the Frenet–Serret equations imply

$$Q''\big(s(t)\big) = \kappa(t)N(t)$$

and

$$P''(t) = \big(s'(t)\big)^2\kappa(t)N(t) + s''(t)T(t).$$

Hence

$$\begin{aligned}P'(t)\times P''(t) &= s'(t)T(t)\times\Big(s'(t)^2\kappa(t)N(t) + s''(t)T(t)\Big)\\ &= s'(t)^3\kappa(t)B(t)\end{aligned}$$

since $T\times N = B$ and $T\times T = 0$. Since $\|B(t)\| = 1$ and $s'(t) = \|P'(t)\|$ this implies

$$\kappa(t) = \frac{\|P'(t)\times P''(t)\|}{\|P'(t)\|^3} \tag{8.9}$$

and

$$B(t) = \frac{P'(t)\times P''(t)}{\|P'(t)\times P''(t)\|}. \tag{8.10}$$

The simplest way to obtain N is to use the formula

$$N = B\times T. \tag{8.11}$$

The Frenet–Serret equation for N' implies

$$\begin{aligned}Q'''\big(s(t)\big) = (\kappa N)'(t) &= \kappa'(t)N(t) + \kappa(t)N'(t)\\ &= \kappa'(t)N(t) + \kappa(t)\big(-\kappa(t)T(t) + \tau(t)B(t)\big)\end{aligned}$$

and hence

$$\langle Q'''\big(s(t)\big), B(t)\rangle = \kappa(t)\tau(t).$$

On the other hand

$$\begin{aligned}\frac{d^3}{dt^3}\Big(Q\big(s(t)\big)\Big) &= P'''(t)\\ &= Q'''\big(s(t)\big)s'(t)^3 + 3Q''\big(s(t)\big)s'(t)s''(t) + Q'\big(s(t)\big)s'''(t)\\ &= Q'''\big(s(t)\big)s'(t)^3 + 3\kappa(t)s'(t)s''(t)N(t) + s'''(t)T(t).\end{aligned}$$

By orthogonality

$$\begin{aligned}\left\langle P'''(t),\ \frac{P'(t)\times P''(t)}{\|P'(t)\times P''(t)\|}\right\rangle &= \langle P'''(t), B(t)\rangle = s'(t)^3\langle Q'''\big(s(t)\big), B(t)\rangle\\ &= \|P'(t)\|^3\kappa(t)\tau(t).\end{aligned}$$

Finally

$$\tau(t) = \frac{\langle P'''(t), P'(t)\times P''(t)\rangle}{\|P'(t)\times P''(t)\|\cdot\|P'(t)\|^3\kappa(t)} = \frac{\langle P'''(t), P'(t)\times P''(t)\rangle}{\|P'(t)\times P''(t)\|^2}. \tag{8.12}$$

Equations (8.8–8.12) are the Frenet–Serret apparatus at $P(t)$ for Γ in terms of the parametrization P.

Example 8.4 Calculate the Frenet–Serret apparatus of the curve parametrized by

$$P(t) = (t - \cos t, \sin t, t).$$

We first calculate the required derivatives of P; P', P'' and P'''. We have $P'(t) = (1+\sin t, \cos t, 1)$, $P''(t) = (\cos t, -\sin t, 0)$ and $P'''(t) = (-\sin t, -\cos t, 0)$. Next, we obtain the cross product

$$P'(t)\times P''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1+\sin t & \cos t & 1\\ \cos t & -\sin t & 0\end{vmatrix} = (\sin t, \cos t, -\sin t - 1)$$

and finally the norms or lengths

$$\begin{aligned}\|P'(t)\| &= (1 + 2\sin t + \sin^2 t + \cos^2 t + 1)^{1/2} = (3 + 2\sin t)^{1/2}\\ \|P'(t)\times P''(t)\| &= (\sin^2 t + \cos^2 t + \sin^2 t + 2\sin t + 1)^{1/2}\\ &= (2 + 2\sin t + \sin^2 t)^{1/2}.\end{aligned}$$

Hence

$$T(t) = \frac{P'(t)}{\|P'(t)\|} = \frac{(1+\sin t, \cos t, 1)}{(3+2\sin t)^{1/2}}$$

$$\begin{aligned}
\kappa(t) &= \frac{\|P'(t)\times P''(t)\|}{\|P'(t)\|^3} = \frac{(2+2\sin t+\sin^2 t)^{1/2}}{(3+2\sin t)^{3/2}}\\
B(t) &= \frac{P'(t)\times P''(t)}{\|P'(t)\times P''(t)\|} = \frac{(\sin t, \cos t, -\sin t - 1)}{(2+2\sin t+\sin^2 t)^{1/2}}\\
\tau(t) &= \frac{\langle P'''(t), P'(t)\times P''(t)\rangle}{\|P'(t)\times P''(t)\|^2}\\
&= \frac{(-\sin t, -\cos t, 0)\cdot(\sin t, \cos t, -\sin t - 1)}{2+2\sin t+\sin^2 t}\\
&= \frac{-1}{2+2\sin t+\sin^2 t}\\
N(t) &= B(t)\times T(t)\\
&= \frac{1}{\sqrt{3+2\sin t}\sqrt{2+2\sin t+\sin^2 t}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin t & \cos t & -\sin t - 1\\ 1+\sin t & \cos t & 1\end{vmatrix}\\
&= \frac{(2\cos t+\sin t\cos t, -1-3\sin t-\sin^2 t, -\cos t)}{(6+10\sin t+7\sin^2 t+2\sin^3 t)^{1/2}}
\end{aligned}$$

Example 8.5 In Example 8.2 we showed that curvature and torsion together allowed us to deduce properties of spherical curves. In this example we show that curvature and torsion completely determine the shape of a curve in $\mathbb{R}^3$. Let Γ and Γ_1 denote two directed curves having the same length l in $\mathbb{R}^3$. We suppose that both have positive curvature. Now transfer Γ_1 so that its initial point coincides with the initial point of Γ and rotate it so that the tangents, normals and binormals of Γ and Γ_1 coincide at the initial point. These operations do not affect the shape of Γ_1. Let $P\colon [0,l]\to\Gamma$ and $P_1\colon [0,l]\to\Gamma_1$ denote unit speed parametrizations. We now suppose that the curvature and torsion of Γ and Γ_1 at $P(t)$ and $P_1(t)$ coincide for all t and thus we have the Frenet–Serret apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa, \tau\}$ for Γ and Γ_1 respectively. Using the dot product in $\mathbb{R}^3$ we define $g\colon [0,l]\to\mathbb{R}$ by

$$g(t) = T(t)\cdot T_1(t) + N(t)\cdot N_1(t) + B(t)\cdot B_1(t).$$

By our hypothesis $g(0)=3$ and by the Cauchy–Schwarz inequality (Example 3.4) $-3\le g(t)\le 3$ for all t and $g(t)=3$ if and only if $T(t)=T_1(t)$, $N(t)=N_1(t)$ and $B(t)=B_1(t)$. From the Frenet–Serret equations

$$g' = \kappa N\cdot T_1 + \kappa T\cdot N_1 + (-\kappa T+\tau B)\cdot N_1 + N\cdot(-\kappa T_1+\tau B_1) - \tau N\cdot B_1 - \tau B\cdot N_1 = 0.$$

Hence g is a constant mapping and, since $g(0)=3$, we have $g(t)=3$ for all t and the Frenet–Serret apparatus is the same for both curves. In particular

$$(P-P_1)'(t) = P'(t) - P_1'(t) = T(t) - T_1(t) = 0$$

and $P(t) = P_1(t) + C$ for all t. Since $P(0) = P_1(0)$ this implies $P(t) = P_1(t)$ for all t and one curve lies on top of the other. We conclude that Γ and Γ_1 have the same shape.

Exercises

8.1 Show that each of the directed curves in Exercise 7.5 is a generalised helix. In each case find a unit vector $\mathbf{u}$ such that $\langle T(t), \mathbf{u}\rangle$ is independent of t.

8.2 Let $P(t) = (t, 1 + t^{-1}, t^{-1} - t)$, $1 \leq t \leq 2$, denote a parametrization of the curve Γ in $\mathbb{R}^3$. Show that $B(t) = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ for all t and hence deduce that Γ lies in the plane $x - y + z = -1$. Find the Frenet–Serret apparatus for Γ.

8.3 If Γ is parametrized by

$$P(\theta) = (\log\cos\theta, \log\sin\theta, \sqrt{2}\theta), \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}$$

show that Γ has curvature $\sin 2\theta/\sqrt{2}$ at $P(\theta)$.

8.4 For the curve parametrized by $P(t) = (3t^2, 3t - t^3, 3t + t^3)$, $-1 \leq t \leq 1$, show that

$$\kappa(t) = -\tau(t) = \frac{1}{3(1+t^2)^2}.$$

Find a unit vector $\mathbf{u}$ such that $\langle T(t), \mathbf{u}\rangle$ is independent of t.

8.5 Find the curvature and torsion of the curve parametrized by

$$P(t) = (\mathrm{e}^t \cos t, \mathrm{e}^t \sin t, \mathrm{e}^t), \quad t \in \mathbb{R}.$$

8.6 The plane through a point on a curve perpendicular to the tangent line is called the *normal plane* to the curve at the point. Show that a curve lies on a sphere if the intersection of all normal planes is non-empty. Hence show that the curve parametrized by

$$P(\theta) = (-\cos 2\theta, -2\cos\theta, \sin 2\theta), \quad \theta \in [0, 2\pi]$$

lies in a sphere. Find the centre and radius of the sphere.

8.7 Show that the curve parametrized by $P(t) = (at, bt^2, t^3)$, $ab \neq 0$, is a generalised helix if and only if $4b^4 = 9a^2$.

Chapter 9
Double Integration

Summary *We define the double integral of a function over an open subset of* $\mathbb{R}^2$ *and use Fubini's theorem to evaluate such integrals. We discuss the fundamental theorem of calculus in* $\mathbb{R}^2$*—Green's theorem.*

We discuss *(double)* integration of a real-valued function of *two* variables $f(x, y)$ over an open set Ω in $\mathbb{R}^2$. Motivated by the one-dimensional theory we divide Ω into rectangles—the natural analogue of intervals—by first drawing horizontal and vertical lines and thus partitioning the x- and y-axes.

Let x_i denote a typical element of the partition of the the x-axis and let y_j be a typical element on the y-axis. The resulting grid of rectangles gives a partition of Ω (Fig. 9.1) and we form the Riemann sum

$$\sum_i \sum_j f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j, \quad \Delta x_i = x_{i+1} - x_i, \quad \Delta y_j = y_{j+1} - y_j$$

where we sum over all rectangles which are strictly contained in Ω. If this sum tends to a limit as we take finer and finer partitions we say that f is *integrable over* Ω and denote the limit by

$$\iint_\Omega f(x, y)\, dxdy.$$

We call this the *integral* (or *double integral*) of f over Ω. If Ω is the inside of a closed curve Γ and f is *continuous* on $\overline{\Omega}$ it can be shown that f is integrable over Ω. When $f(x, y) = 1$ for all $(x, y) \in \Omega$ the Riemann sum is the area of the rectangles in the partition inside Γ and on taking a limit we obtain

$$\iint_\Omega dxdy = Area\, of\, \Omega.$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_9,

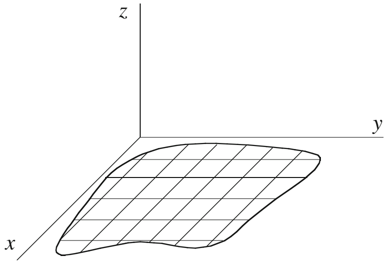


Fig. 9.1

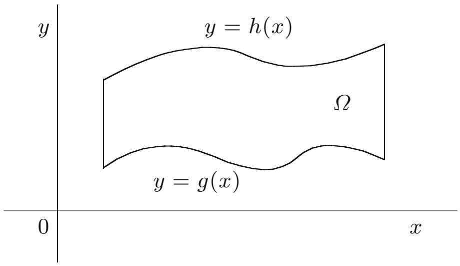


Fig. 9.2

If $f(x, y) \geq 0$ then the *volume* of the solid over Ω and beneath the graph of f is

$$\iint\limits_{\Omega} f(x, y)dxdy.$$

We only evaluate double integrals over rather simple open sets. An open set is said to be of *type I* if it is bounded above by the graph of a continuous function $y = h(x)$, bounded below by the graph of a continuous function $y = g(x)$ and on the left and right by vertical lines of finite length (see Fig. 9.2).

Take a fixed interval in the partition of the x-axis, say (x_i, x_{i+1}), and consider the terms in the Riemann sum

$$\sum_i \sum_j f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j$$

which only involve $\Delta x_i = x_{i+1} - x_i$. This gives the sum

$$\Big(\sum_j f(x_i, y_j) \cdot \Delta y_j\Big)\Delta x_i.$$

Taking limits—this can be justified for *continuous* functions—we get

$$\sum_j f(x_i, y_j) \cdot \Delta y_j \longrightarrow \int_{g(x_i)}^{h(x_i)} f(x_i, y)\, dy$$

as we take finer and finer partitions of the y-axis. Let

$$H(x) = \int_{g(x)}^{h(x)} f(x, y)\, dy.$$

Then

$$\sum_{i,j} f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j \approx \sum_i H(x_i) \cdot \Delta x_i \to \int_a^b H(x)\, dx$$

and taking the limit on both sides we get

$$\iint_\Omega f(x, y)\, dydx = \int_a^b H(x)\, dx = \int_a^b \left\{ \int_{g(x)}^{h(x)} f(x, y)\, dy \right\} dx.$$

This method of integration, together with the similar method obtained by reversing the roles of x and y, is known as *Fubini's theorem.* We define an open set to be of *type II* if it is bounded on the left and right by the graphs of continuous functions of y, k and l, which are defined on the interval $[c, d]$ and above and below by horizontal lines of finite length. For domains of type II Fubini's theorem is

$$\iint_\Omega f(x, y)\, dxdy = \int_c^d \left\{ \int_{k(y)}^{l(y)} f(x, y)\, dx \right\} dy.$$

If we are given an open set we recognise it is of type I if each *vertical* line cuts the boundary at two points except possibly at the end points and type II if each *horizontal* line cuts the boundary at two points except, perhaps, at the end points.

Example 9.1 Evaluate

$$\iint_\Omega \frac{x}{\sqrt{16 + y^7}}\, dxdy$$

over the set Ω bounded above by the line $y = 2$, below by the graph of $y = x^{1/3}$ and on the left by the y-axis (Fig. 9.3). By inspection the domain Ω is of type I and type II and we have a choice of method, i.e. we can integrate first with respect to

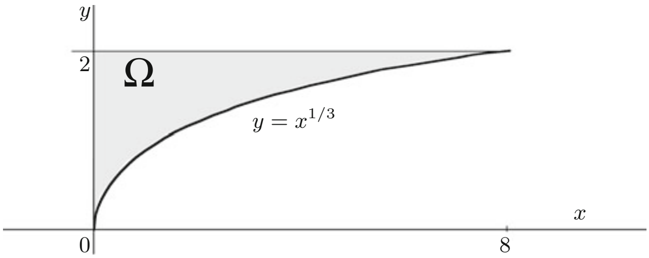


Fig. 9.3

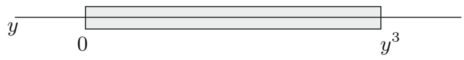


Fig. 9.4

either variable. Our choice may be important since one method may be very simple and the other quite difficult.

We have to evaluate two integrals of a single variable. In the first integral, the inner integral, one of the variables takes a fixed value and is really a constant. Thus we have to first evaluate either

$$\int x\,dx \qquad \text{or} \qquad \int \frac{dy}{\sqrt{16+y^7}}.$$

In these situations looks are usually not deceiving and we opt for the simpler looking integral. So, we choose to integrate first with respect to x. The limits of integration in the first integral will influence the degree of difficulty that arises in evaluating the second, or outer, integral. If you run into problems with the second integral you should consider starting again using a different order of integration. In our case we have decided to consider

$$\int \left\{ \int \frac{x}{\sqrt{16+y^7}}\,dx \right\} dy$$

and now need to determine the variation in x for fixed y. We draw a typical line through Ω on which y is constant—i.e. a horizontal line.

We must now express the end points in terms of y. Using once more Fig. 9.3 we see that the end points of the line of variation of x are $(0, y)$ and (x, y) where $y = x^{1/3}$. Hence $y^3 = x$ and we have the required variation of x (Fig. 9.4).

We see also that y varies from 0 to 2. Hence

$$\iint_{\Omega} \frac{x}{\sqrt{16+y^7}}\,dxdy = \int_0^2 \left\{ \int_0^{y^3} \frac{x\,dx}{\sqrt{16+y^7}} \right\} dy$$

$$
\begin{aligned}
&= \int_0^2 \frac{x^2\,dx}{2\sqrt{16+y^7}}\bigg|_0^{y^3}\,dy \\
&= \int_0^2 \frac{y^6}{2\sqrt{16+y^7}}\,dy \\
&= \frac{1}{14}\int \frac{dw}{\sqrt{w}} \qquad \begin{matrix} w = 16+y^7 \\ dw = 7y^6\,dy \end{matrix} \\
&= \frac{1}{14}\cdot\frac{w^{1/2}}{1/2} = \frac{1}{7}(16+y^7)^{1/2}\bigg|_0^2 \\
&= \frac{1}{7}\big((144)^{1/2} - (16)^{1/2}\big) = \frac{8}{7}.
\end{aligned}
$$

Example 9.2 In this example we reverse the order of integration and evaluate

$$\int_0^3 \left\{ \int_1^{\sqrt{4-y}} (x+y)\,dx \right\} dy.$$

From the limits of integration in the inner integral the left-hand side of the domain of integration, Ω, is bounded by the line $x = 1$ and the right-hand side by points satisfying $x = \sqrt{4-y}$, i.e. $x^2 = 4 - y$ or $y = 4 - x^2$. Hence the right-hand side is bounded by the graph of $y = 4 - x^2$ and we have the following diagram for our domain (Fig. 9.5).

Reversing the order of integration we get

$$\int_1^2 \left\{ \int_0^{4-x^2} (x+y)\,dy \right\} dx = \int_1^2 \left(xy + \frac{y^2}{2}\right)\bigg|_0^{4-x^2}\,dx$$

$$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2}\right) dx = \frac{241}{60} \text{ (eventually).}$$

The *fundamental theorem of one-variable calculus*

$$f(b) - f(a) = \int_a^b f'(t)dt \tag{9.1}$$

is used to evaluate integrals of certain functions over intervals from their boundary values. In this theorem integration on the right-hand side is over a *directed interval*

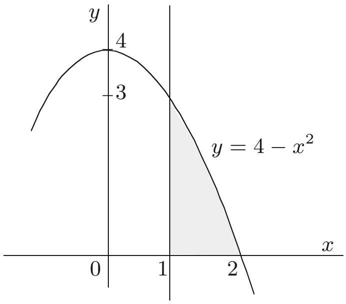


Fig. 9.5

while on the left positive and negative signs are assigned to the initial and final points of the interval respectively. Thus we see that a certain coherence has to be established between the orientations on the two sides of (9.1). The *fundamental theorem of two-variable calculus* is known as *Green's theorem*. To obtain this result by an immediate application of the one-variable theorem it is usual to begin with an open subset Ω in $\mathbb{R}^2$ which is of type I and type II. In applying (9.1) it is necessary to be careful with signs and this means that in Green's theorem the boundary of Ω, Γ, is oriented in an *anticlockwise* or *counterclockwise* direction.

Theorem 9.3 (Green's Theorem) *Let P and Q denote real-valued functions with continuous first-order partial derivatives on the open subset U of $\mathbb{R}^2$. If Γ is a closed curve directed in an anticlockwise direction such that the interior (inside) Ω of Γ is an open set of type I and type II and $\Gamma \cup \Omega \subset U$ then*

$$\int_{\Gamma} Pdx + Qdy = \iint_{\Omega} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy. \tag{9.2}$$

Proof We show

$$\int_{\Gamma} Qdy = \iint_{\Omega} \frac{\partial Q}{\partial y}dxdy$$

and for this we use the type II property of Ω. From the representation in Fig. 9.6 we have

$$\iint_{\Omega} \frac{\partial Q}{\partial x}dxdy = \int_{c}^{d} \Big\{ \int_{k(y)}^{l(y)} \frac{\partial Q}{\partial x}dx \Big\} dy.$$

By (9.1)

$$\int_{k(y)}^{l(y)} \frac{\partial Q}{\partial x}(x, y)dx = Q(x, y)\Big|_{k(y)}^{l(y)} = Q\big(l(y), y\big) - Q\big(k(y), y\big)$$

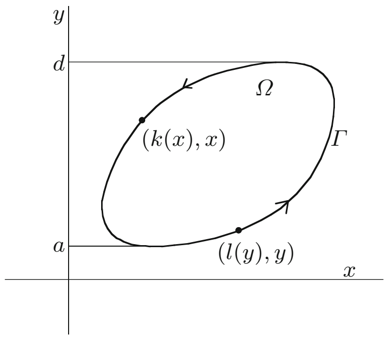


Fig. 9.6

and

$$\iint_{\Omega} \frac{\partial Q}{\partial x} dx dy = \int_{c}^{d} \Big(Q\big(l(y), y\big) - Q\big(k(y), y\big) \Big) dy.$$

On the other hand, using the parametrizations $y \to \big(l(y), y\big)$ and $y \to \big(k(y), y\big)$, we obtain

$$\begin{aligned} \int_{\Gamma} Q dy &= \int_{\text{(graph of } l)} Q dy - \int_{\text{(graph of } k)} Q dy \\ &= \int_{c}^{d} Q\big(l(y), y\big) dy - \int_{c}^{d} Q\big(k(y), y\big) dy. \end{aligned}$$

This proves

$$\iint_{\Omega} \frac{\partial Q}{\partial x} dx dy = \int_{\Gamma} Q dy.$$

The equality

$$\iint_{\Omega} (-\frac{\partial P}{\partial y}) dx dy = \int_{\Gamma} P dx$$

is obtained in the same way and this completes the proof. □

Green's theorem is true for many other sets Ω and the proof usually proceeds by partitioning Ω into sets $(\Omega_i)_i$ and applying the simple case above to each Ω_i (Fig. 9.7). Note that each new curve created in partitioning Ω appears as part of the boundary of *two* Ω_i's and each direction along these new curves appears precisely once. Hence, when we apply (9.2) to each Ω_i and add them together the integrals

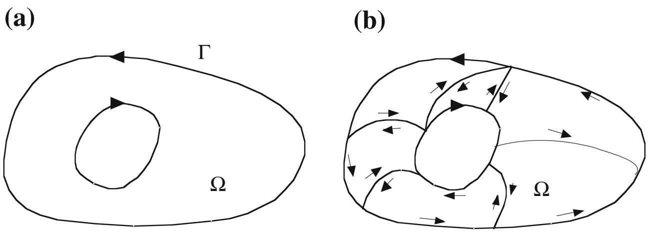


Fig. 9.7

along the newly created curves cancel and we are left with a line integral over the original curve Γ.

A glance at Fig. 9.7a shows that we no longer have an anticlockwise oriented boundary. In our new situation a *finite* number of piecewise smooth curves form the boundary of Ω and as we proceed along Γ in the given direction the open set Ω lies on the *left-hand side*. With these modifications Green's theorem is still true. We are, of course, always assuming that P and Q are nice smooth functions.

Example 9.4 We wish to calculate

$$I = \int_\Gamma (5 - xy - y^2)dx + (2xy - x^2)dy$$

where Γ is the boundary of the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. By Green's theorem

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \Big(\frac{\partial}{\partial x}(2xy - x^2) - \frac{\partial}{\partial y}(5 - xy - y^2)\Big)dxdy \\
&= \int_0^1 \int_0^1 (2y - 2x + x + 2y)dxdy \\
&= \int_0^1 \int_0^1 (4y - x)dxdy \\
&= \Big(\int_0^1 4ydy\Big)\Big(\int_0^1 dx\Big) - \Big(\int_0^1 dy\Big)\Big(\int_0^1 xdx\Big) \\
&= \frac{4y^2}{2}\Big|_0^1 \, x\Big|_0^1 - y\Big|_0^1 \, \frac{x^2}{2}\Big|_0^1 \\
&= 2 - \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

The change of variables rule for double integrals can be considered as a special case of the same rule for triple integrals and this is discussed in Chap. 14.

Exercises

9.1 Find $\iint\limits_U x^2 \sin^2 y dxdy$ where $A = \{(x, y) \in \mathbb{R}^2, \quad 0 < x < 1, \ 0 < y < \pi/4\}$.

9.2 Evaluate

(a) $\iint\limits_U x\cos(x + y)dxdy$ where U is the subset of $\mathbb{R}^2$ bounded by the triangle with vertices $(0, 0)$, $(\pi, 0)$ and (π, π).

(b) $\iint\limits_U (x^2 + y^2)dxdy$ where $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 2y\}$.

(c) $\iint\limits_U \frac{y^2}{x^2}dxdy$ where U is the region bounded by $y = x$, $y = 2$ and $xy = 1$.

9.3 Find the area bounded by the curves $x = y^2$ and $x = 4y - y^2$.

9.4 If Γ is a closed anticlockwise directed curve in $\mathbb{R}^2$ with interior Ω and $\mathbf{F}$ is a smooth vector field on an open set containing $\Omega \cup \Gamma$ show that Green's theorem is equivalent to

$$\iint\limits_\Omega \mathrm{div}(\mathbf{F}) = \int (\mathbf{F} \cdot \mathbf{n})\mathrm{ds}$$

where $\mathbf{n}$ is the outward normal to Γ.
If f is harmonic on Ω, i.e.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and continuous on $\Omega \cup \Gamma$ show that $\int\limits_\Gamma (\nabla f \cdot \mathbf{n})ds = 0$.

9.5 By reversing the order of integration find

(a) $\int\limits_0^6 \left\{ \int\limits_{x/3}^2 \mathrm{e}^{y^2} dy \right\} dx$

(b) $\int\limits_0^4 \left\{ \int\limits_{y/2}^{\sqrt{y}} \mathrm{e}^{y/x} dx \right\} dy.$

9.6 Find the volume of

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 \leq 1, \quad x \geq 0, \ y \geq 0, \ 0 \leq z \leq 1 - xy\}.$$

9.7 If Ω is an open set enclosed by the anticlockwise directed curve Γ show, using Green's theorem, that

$$\text{Area }(\Omega) = \int_\Gamma x\,dy = \int_\Gamma -y\,dx = \frac{1}{2}\int_\Gamma x\,dy - y\,dx\,.$$

Using one of these line integrals find the area of the interior of the ellipse $(x/4)^2 + (y/5)^2 = 1$.

9.8 Verify Green's theorem for the following integrals:

(a) $\int_\Gamma xy^2dx + 2x^2ydy$, Γ is the ellipse $4x^2 + 9y^2 = 36$

(b) $\int_\Gamma (x^2 + 2y^3)dy$, Γ is the circle $(x-2)^2 + y^2 = 4$

(c) $\int_\Gamma 2x^2y^2dx - 3yxdy$, Γ is the square bounded by the lines $x = 3$, $x = 5$, $y = 1$, $y = 4$

where Γ is always directed in an anticlockwise direction.

Chapter 10
Parametrized Surfaces in $\mathbb{R}^3$

Summary *We discuss theoretical and practical approaches to parametrizing a surface in* $\mathbb{R}^3$.

In this chapter we begin a systematic study of surfaces in $\mathbb{R}^3$, a topic which will occupy the remaining chapters of this book. We begin with an informal discussion of the background we bring to this investigation and reveal our general intentions.

Daily we encounter most of the classical Euclidean surfaces in $\mathbb{R}^3$ such as a *sphere* (football, globe), *cone* (ice-cream cone), *ellipsoid* (egg, American football), *cylinder* (jar, can), *plane* (floor, wall, ceiling) and the non-Euclidean *torus* (doughnut, tube). Many of our examples involve these surfaces and a solid geometric understanding of these should be deliberately cultivated. Initially these surfaces are merely subsets of $\mathbb{R}^3$ which have a certain recognisable shape. We have already established mathematical contact with them as level sets and graphs of functions. Our studies have been confined to the classical problem in differential calculus of finding the maximum and minimum of a scalar-valued function over a surface and for this we introduced the geometric concepts of *tangent plane* and *normal line*. This essentially summarises the formal knowledge we have acquired but we do have at our disposal a range of mathematical ideas and techniques which, if not directly applicable, will often hint at the way forward.

Our plan is to develop integration theory on surfaces and to investigate the geometry of surfaces. Keep in mind the following topics as particularly relevant to the theories we develop:

(a) integration over open subsets of $\mathbb{R}^2$ (Chap. 9)
(b) integration along directed curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Chap. 6)
(c) the geometry of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Chaps. 7 and 8).

For instance an open subset U in $\mathbb{R}^2$ can be considered, or identified with, a surface $\widetilde{U}$ in $\mathbb{R}^3$ by means of the mapping

$$(x, y) \in U \longrightarrow (x, y, 0) \in \widetilde{U}.$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_10,

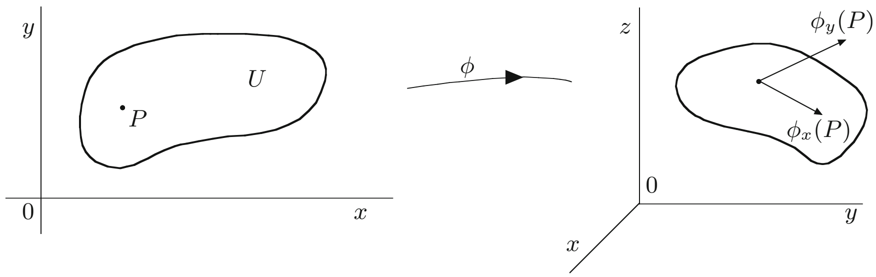


Fig. 10.1

Thus any theory of integration over a surface will extend integration theory over open subsets of $\mathbb{R}^2$. We can say more since parametrizations (Definition 10.2) are a means of identifying a simple surface with an open subset of $\mathbb{R}^2$ and this eventually allows us to reduce integration over a surface to integration over an open subset of $\mathbb{R}^2$ in much the same way that we reduced line integrals in $\mathbb{R}^n$ to integration over an interval in $\mathbb{R}$ (the key word in this sentence is unfortunately “eventually”). Thus our theory of integration over surfaces is obtained by combining and developing techniques already used in (a) and (b). We shall see later how (b) and (c) and, in particular, the methods used to derive the Frenet–Serret equations, can be extended to investigate the geometry of surfaces in $\mathbb{R}^3$.

We begin our formal study of surfaces by defining the concept of a parametrized surface in $\mathbb{R}^3$. Our definition is mathematically simple, a good starting point, and carries us a long way but does have certain inadequacies that we discuss as we proceed.

Definition 10.1 A parametrized surface in $\mathbb{R}^3$ consists of a pair (S, ϕ) where S is a subset of $\mathbb{R}^3$ and ϕ is a bijective mapping from an open subset of $\mathbb{R}^2$ onto S such that the following conditions hold:

(i) ϕ has derivatives of all orders (we say that ϕ is smooth or $\mathscr{C}^\infty$)
(ii) $\phi_x \times \phi_y \neq 0$ at all points.

Condition (ii) is the analogue of $P' \neq 0$ for a parametrized curve in $\mathbb{R}^3$ and is *equivalent* to the requirement that ϕ_x and ϕ_y are linearly independent vectors at all points (see Fig. 10.1).

Definition 10.2 A simple surface in $\mathbb{R}^3$ is a subset S of $\mathbb{R}^3$ for which there exists a mapping ϕ such that (S, ϕ) is a parametrized surface. We call ϕ a parametrization of S.

One notable difference between the above definitions and the corresponding definition of directed curve (Definition 5.1) is the change from a *closed* interval in $\mathbb{R}$ to an *open* subset of $\mathbb{R}^2$ for the domain of parametrization. For a directed curve the inclusion of end points in the domain of definition leads to a sense of *direction* along the curve and we will need an analogous concept, a sense of *orientation*, to develop vector-valued integration theory over a surface. However, a sense of direction along a curve can also be obtained by using tangent vectors at interior points of the curve and we develop the concept we require for surfaces by using the interior of the surface.

Unfortunately, many of the classical Euclidean surfaces, e.g. the sphere, are not simple surfaces but, fortunately, for many practical purposes, e.g. the calculation of surface area and the evaluation of surface integrals, they may be considered as simple surfaces. We will make this precise later and also define a general surface in $\mathbb{R}^3$.

We now examine three specific examples—*graphs*, *surfaces of revolution* and the classical *ellipsoid*. These, although apparently rather limited, appear in many different contexts and we allow them to divert us to essential ideas associated with any parametrized surface.

Example 10.3 (see Example 7.1) Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ denote a smooth function defined on the open subset U of $\mathbb{R}^2$ and let S denote the *graph* of f, i.e.

$$S = \big\{\big(x, y, f(x, y)\big) : (x, y) \in U\big\}.$$

The form of S immediately gives a parametrization ϕ defined by

$$\phi(x, y) = \big(x, y, f(x, y)\big)$$

with domain U. Since $\phi_x = (1, 0, f_x)$ and $\phi_y = (0, 1, f_y)$ we have

$$\phi_x \times \phi_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).$$

As the final coordinate of $\phi_x \times \phi_y$ is always 1 we have $\phi_x \times \phi_y \neq 0$. If $\phi(x_1, y_1) = \phi(x_2, y_2)$ then

$$\big(x_1, y_1, f(x_1, y_1)\big) = \big(x_2, y_2, f(x_2, y_2)\big).$$

Hence $x_1 = x_2$ and $y_1 = y_2$ and so ϕ is injective. Clearly $\phi(U) = S$.

As a particular example consider the unit sphere $S : x^2 + y^2 + z^2 = 1$. We have $z^2 = 1 - x^2 - y^2$ and $z = \pm(1 - x^2 - y^2)^{1/2}$. Hence the *upper hemisphere* of S is the graph of the function

$$g(x, y) = (1 - x^2 - y^2)^{1/2}$$

and the mapping $\widetilde{g}: (x, y) \to \big(x, y, (1 - x^2 - y^2)^{1/2}\big)$ is a parametrization of the upper hemisphere. The domain of $\widetilde{g}$ is the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. To visualise this physically imagine the hemisphere as a dome S and take U to be the floor of the dome. If you can now accept yourself as a point p on the floor and look directly upwards your eyes will focus on a unique point q in the dome S (Fig. 10.2). Moving around you will be *identifying* the points of U with points of the surface. The floor U is the domain of the parametrization while the parametrization itself is the mapping which takes p to q. The fact that you are always looking in the same direction means

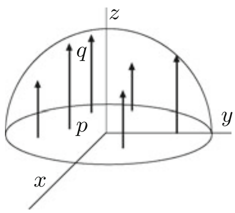


Fig. 10.2

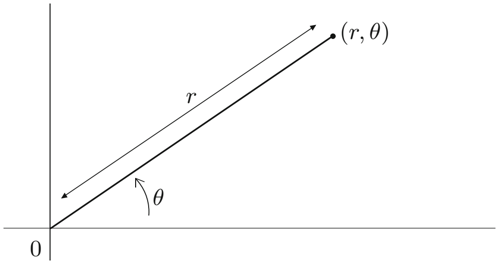


Fig. 10.3

that the identification or parametrization is given by a set of *parallel* lines (Fig. 10.2) and since parallel lines never meet the mapping is injective or one-to-one. Since each point on the dome is hit by an arrow we also have an onto or surjective mapping and thus a bijective mapping. In general any natural identification of a flat set with a surface by means of parallel lines will lead to a parametrization (see Example 11.2).

A point P in $\mathbb{R}^2$ can be identified by means of its distance r from the origin and the angle θ between the positive x-axis and the vector OP (Fig. 10.3).

We call (r, θ) the *polar coordinates* of P. Polar coordinates are particularly useful when dealing with circles and discs and with functions involving the expression $x^2 + y^2$ in Cartesian coordinates. Unfortunately, the correspondence

$$(r, \theta) \longrightarrow (x, y) = (r\cos\theta, r\sin\theta)$$

is not bijective but we get around this difficulty by removing a small portion of the domain. The domain of the parametrization of the hemisphere S given above is the disc $x^2 + y^2 < 1$ and suggests the use of polar coordinates. Since

$$(1 - x^2 - y^2)^{1/2} = (1 - r^2\cos^2\theta - r^2\sin^2\theta)^{1/2} = (1 - r^2)^{1/2}$$

the mapping

$$F\colon (r, \theta) \in (0, 1) \times (-\pi, \pi) \to \bigl(r\cos\theta, r\sin\theta, (1 - r^2)^{1/2}\bigr)$$

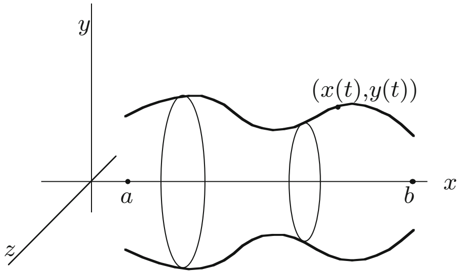


Fig. 10.4

is a bijective mapping onto $S \setminus \Gamma$ where Γ is the set

$$\left\{(-r, 0, (1-r^2)^{1/2}),\ 0 \leq r < 1\right\}.$$

On $(0, 1) \times (-\pi, \pi)$

$$F_r \times F_\theta = \left(\frac{r^2 \cos\theta}{(1-r^2)^{1/2}}, \frac{r^2 \sin\theta}{(1-r^2)^{1/2}}, r\right)$$

is non-zero and we have found a second parametrization of (almost) the whole hemisphere.

Example 10.4 Let $P(t) = \big(x(t), y(t)\big)$, $t \in [a, b]$ denote a directed curve Γ in $\mathbb{R}^2$. We suppose $y(t) > 0$ for all t. The surface obtained by revolving this curve about the x-axis is called the *surface of revolution* of P about the x-axis (Fig. 10.4).

Consider a typical point on the circle obtained by rotating the point $\big(x(t), y(t)\big)$. The first coordinate remains unchanged. The second coordinate will generate a circle in the (y, z)-plane with centre $(0, 0)$ and radius $y(t)$. Using the standard parametrization of the circle we see that a typical point on this circle has coordinates $\big(y(t)\cos\theta, y(t)\sin\theta\big)$. Putting these together we obtain a parametrization

$$(t, \theta) \longrightarrow \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$

where $t \in (a, b)$ and $\theta \in (0, 2\pi)$. Now

$$\phi_t = \big(x'(t), y'(t)\cos\theta, y'(t)\sin\theta\big)$$

and

$$\phi_\theta = \big(0, -y(t)\sin\theta, y(t)\cos\theta\big).$$

Hence

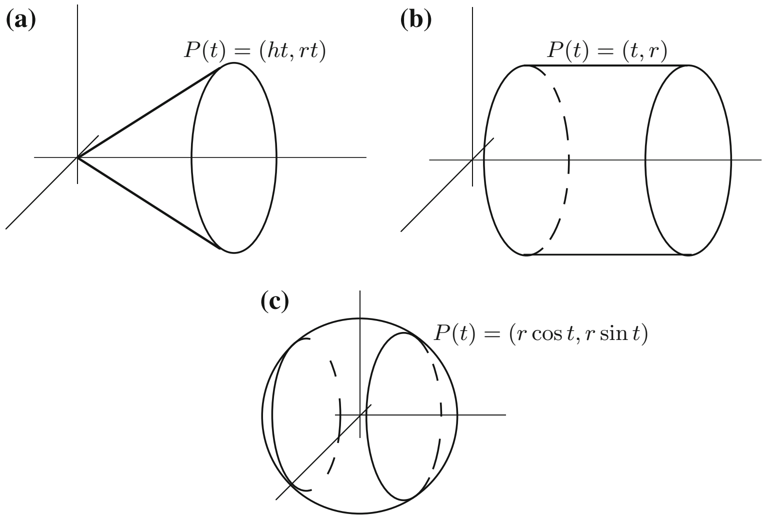


Fig. 10.5

$$\begin{aligned}\phi_t \times \phi_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= \big(y'(t)y(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta\big).\end{aligned}$$

Since

$$\begin{aligned}\|\phi_t \times \phi_\theta\| &= \big((y')^2y^2 + (x')^2y^2\cos^2\theta + (x')^2y^2\sin^2\theta\big)^{1/2} \\ &= y(t)\big(y'(t)^2 + x'(t)^2\big)^{1/2} = y(t)\|P'(t)\| \neq 0\end{aligned}$$

we have $\phi_t \times \phi_\theta \neq 0$. Since the mapping ϕ is bijective as long as we do not include θ and $\theta + 2\pi$ in the domain we let $U = (a, b) \times (0, 2\pi)$ and obtain a bijective mapping onto the surface of revolution with one curve, the original or *profile* curve, removed. Many classical surfaces may be realised as surfaces of revolution; e.g. the cone (Fig. 10.5a), cylinder (Fig. 10.5b) and sphere (Fig. 10.5c).

These induce the following parametrizations:

Cone	$(t, \theta) \longrightarrow$	$(ht, rt\cos\theta, rt\sin\theta)$,
Cylinder	$(t, \theta) \longrightarrow$	$(t, r\cos\theta, r\sin\theta)$,
Sphere	$(t, \theta) \longrightarrow$	$(r\cos t, r\sin t\cos\theta, r\sin t\sin\theta)$.

We parametrize, in Example 11.1 the *torus* as a surface of revolution.

We now examine in considerable detail a particular surface—the standard ellipsoid. This includes the sphere as a special case. We use this example as an excuse to explore and comment on many of the practical and theoretical considerations that arise in studying any surface and to shed some light on the necessity and significance of later developments. In studying this example the reader should keep in mind that parametrizations are nothing more than coordinate systems, that we are always trying to interpret mathematical facts geometrically and constantly attempting to articulate mathematically our geometric observations.

Example 10.5 We consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

This specialises to the sphere of radius a centred at the origin when $a=b=c$. Two of the most frequently used coordinate systems on the sphere are *spherical polar coordinates* and *geographical coordinates*. These, as we shall see, are closely related. Spherical polar coordinates are more popular in the mathematical literature but since geographical coordinates are in everyone's normal experience we devote more time to them here.

The representation of the ellipsoid as a level set of a sum of squares suggests we use the elementary identity $\sin^2\theta+\cos^2\theta=1$ to develop our parametrization. Rewriting the formula for the ellipsoid as a sum of *two* squares we get

$$\left(\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

Now let

$$\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}=\cos\theta \quad \text{and} \quad \frac{z}{c}=\sin\theta.$$

Hence $z=c\sin\theta$ and

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2\theta, \quad \text{i.e.} \quad \left(\frac{x}{a\cos\theta}\right)^2+\left(\frac{y}{b\cos\theta}\right)^2=1.$$

A further similar substitution gives

$$\frac{x}{a\cos\theta}=\cos\psi \quad \text{and} \quad \frac{y}{b\cos\theta}=\sin\psi.$$

This implies $x=a\cos\theta\cos\psi$ and $y=b\cos\theta\sin\psi$. Our parametrization, denoted by F, has the form

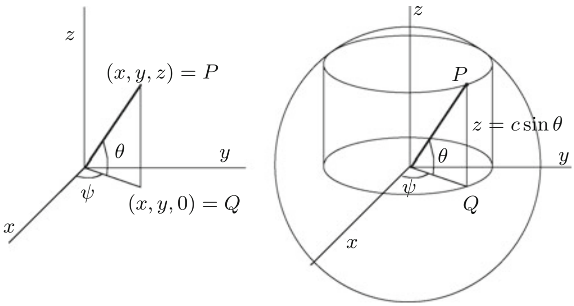


Fig. 10.6

$$\begin{aligned} F\colon U &\longrightarrow S \text{ (ellipsoid)} \\ (\theta, \psi) &\longrightarrow (a\cos\theta\cos\psi, b\cos\theta\sin\psi, c\sin\theta) \end{aligned}$$

but we must specify U, check that F is bijective and see that conditions (i) and (ii) are satisfied. Geometrically we have the following diagram (Fig. 10.6) in the case of the sphere $a = b = c$.

Thus we project the point $P = (x, y, z)$ onto the xy-plane to get the point $Q = (x, y, 0)$. The angle θ is the angle between the vectors OQ and OP and hence $\theta = \tan^{-1}\big(z \;/\; (x^2 + y^2)^{1/2}\big)$. The level set of the surface through P parallel to the xy-plane is a circle (remember we are just considering a sphere here) and this is also projected onto the xy-plane and the angle ψ is obtained by using *polar coordinates* on this circle. Hence $\psi = \tan^{-1}(y/x)$. From Fig. 10.6 it is clear that θ ranges over the interval $(-\pi/2, \pi/2)$ and ψ over the interval $(0, 2\pi)$. Clearly the smoothness condition (i) is satisfied no matter what domain U we choose for F. In many examples condition (ii) is easily checked and formal identification of the range of F is obtained using a diagram. In this example, however, we adopt a more analytic approach in verifying that F is a parametrization.

We have

$$F_\theta = (-a\sin\theta\cos\psi, -b\sin\theta\sin\psi, c\cos\theta)$$

and

$$F_\psi = (-a\cos\theta\sin\psi, b\cos\theta\cos\psi, 0).$$

Hence,

$$\begin{aligned} F_\theta \times F_\psi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\psi & -b\sin\theta\sin\psi & c\cos\theta \\ -a\cos\theta\sin\psi & b\cos\theta\cos\psi & 0 \end{vmatrix} \\ &= (-bc\cos^2\theta\cos\psi, -ac\cos^2\theta\sin\psi, -ab\cos\theta\sin\theta) \\ &= -abc\cos\theta\left(\frac{\cos\theta\cos\psi}{a}, \frac{\cos\theta\sin\psi}{b}, \frac{\sin\theta}{c}\right). \end{aligned}$$

If $a = b = c$ then

$$\begin{aligned}\|F_\theta \times F_\psi\| &= a^2 \cos\theta(\cos^2\theta\cos^2\psi + \cos^2\theta\sin^2\psi + \sin^2\theta)^{1/2}\\ &= a^2\cos\theta. \end{aligned} \tag{10.1}$$

Before proceeding we make a brief observation which we develop in more detail in Chaps. 16–18. If we fix ψ and let θ vary we obtain a mapping

$$\theta \longrightarrow F(\theta, \psi)$$

where θ ranges over an interval in $\mathbb{R}$. This defines a directed curve which lies in S with tangent F_θ. Similarly F_ψ is tangent to the curve $\psi \to F(\theta, \psi)$ in S. These curves are called *coordinate curves* (of the parametrization). The vectors F_θ and F_ψ lie in the tangent space of S at P and are called *tangent vectors*. Tangent vectors enable us to define directional derivatives of functions defined on the surface. If $G : S \to \mathbb{R}^n$ and $P \in S$ then the *directional derivative* of G at P in the direction of the tangent vector $\mathbf{v}$ at P is given by

$$D_{\mathbf{v}}G(p) = \frac{d}{dt}(G \circ \phi)(t)|_{t=0}$$

where ϕ is *any* differentiable mapping from $(-a, a)$ into S such that $\phi(0) = P$ and $\phi'(0) = \mathbf{v}$.

Condition (ii) says that F_ϕ and F_θ span the tangent space and the following relationship holds in *any* surface S.

$$\begin{aligned}\{\text{The tangent space of } S \text{ at } P\} &= \left\{\begin{array}{c}\text{set of tangents at } P \text{ of all curves}\\ \text{in } S \text{ which pass through } P\end{array}\right\}\\ &= \left\{\begin{array}{c}\text{all linear combinations spanned}\\ \text{by the partial derivatives at } P\\ \text{of any parametrization}\end{array}\right\}\\ &= \left\{\begin{array}{c}\text{set of directional derivatives}\\ \text{at } P \text{ which operate on}\\ \text{functions defined on } S.\end{array}\right\}\end{aligned}$$

Since $F_\theta \times F_\psi \neq 0$, the vector $F_\theta \times F_\psi$ is perpendicular to the tangent space and hence parallel to the normal. In our case the surface S is the level set $g^{-1}(0)$ where

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

We now confine ourselves to the special case of the unit sphere, i.e. the ellipsoid with $a = b = c = 1$. The adjustments necessary, when this case has been considered, in order to examine the general situation are fairly minor. We have seen in Chap. 2 that the normal is parallel to ∇g and by inspecting (10.1) we see that $\nabla g(P)$ is parallel to $F_\theta(P) \times F_\psi(P)$. This observation can be used to partially check calculations and, in some cases, may even be used to avoid calculating $F_\phi \times F_\psi$. We have thus related our earlier concepts of *tangent space* and *normal* with terms which can be calculated from any parametrization.

We next turn to find a domain U for F and to verify that all requirements necessary for a parametrization are satisfied by (F, U). Initially, it is usually better to choose the domain of parametrization to be as large and as simple as possible. These twin aims may not always be compatible with the requirements for a parametrization or, indeed, with one another so some judgement is necessary and this usually comes with experience.

By (10.1), $F_\theta \times F_\psi \neq 0$ if and only if $\cos\theta \neq 0$ and a natural choice for the domain of θ is the interval $-\pi/2 < \theta < \pi/2$. This is also suggested by Fig. 10.6. Since both the sine and cosine functions have period 2π the length of the interval of definition of ψ *cannot* be greater than 2π. The interval $(0, 2\pi)$ is a natural choice and leads to

$$U = \{(\theta, \psi) : -\pi/2 < \theta < \pi/2,\ 0 < \psi < 2\pi\}$$

as our domain for F. Clearly the set U is open. If $F(\theta_1, \psi_1) = F(\theta_2, \psi_2)$ then, comparing final coordinates in F, we get $\sin\theta_1 = \sin\theta_2$. Since the sine function is injective on $(-\pi/2, \pi/2)$ this implies $\theta_1 = \theta_2$. Moreover, $\cos\theta \neq 0$ on $(-\pi/2, \pi/2)$ and comparing the first two coordinates of F we see that $\cos\psi_1 = \cos\psi_2$ and $\sin\psi_1 = \sin\psi_2$. Since $(\cos(\psi), \sin(\psi)), 0 < \psi < 2\pi$, are the coordinates of a unique point on the unit circle we have $\psi_1 = \psi_2$ and F is injective on U.

We have thus shown that F is a parametrization of its range or image, $F(U)$. By our construction the image of F lies in the ellipsoid. Does it cover the full ellipsoid? If $F(\theta, \psi) = (0, 0, c)$ then $\sin\theta = 1$. However, $-\pi/2 < \theta < \pi/2$, and so $(0, 0, c) \notin F(U)$ and F does *not* cover the ellipsoid. To find the range of F we proceed as follows. The set U is an *open* rectangle in $\mathbb{R}^2$ and its boundary is the perimeter of this rectangle. The function F has a nice smooth extension from U to its boundary (Fig. 10.7) and the image of this boundary is the boundary of the image of F in the surface we are examining.

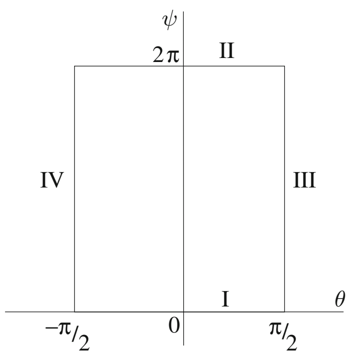


Fig. 10.7

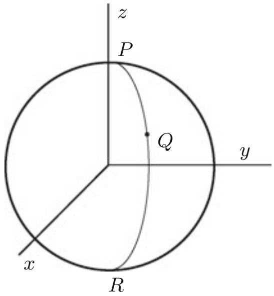


Fig. 10.8

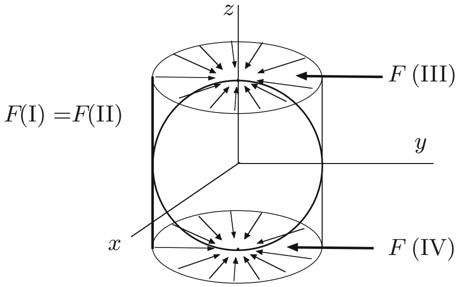


Fig. 10.9

Since $F(\theta, 0) = F(\theta, 2\pi) = (-a\cos\theta, 0, c\sin\theta)$, $F(\pi/2, \psi) = (0, 0, c)$, $F(-\pi/2, \psi) = (0, 0, -c)$, and $-\pi/2 < \theta < \pi/2$ the boundary of U is mapped onto one half of the ellipse $(x/a)^2 + (z/c)^2 = 1, y = 0$. Hence F is a parametrization of the ellipsoid with one curve removed, the half ellipse *PQR* in Fig. 10.8. As we develop geometric insight we will find that a good sketch often leads to a rapid identification of the domain of definition of the parametrization and it will not be necessary to go through the protracted investigation we have just completed.

A different choice of domain would have led to a parametrization which almost certainly would have covered a different part of the ellipsoid and perhaps covered the curve that we missed. However, *no parametrization* will cover the full ellipsoid. This, although intuitively clear, is a highly non-trivial mathematical result.

A parametrization identifies, with F as the mode of identification, a flat set of points, the open subset U of $\mathbb{R}^2$, with a subset of the surface, $F(U)$, in a one-to-one fashion. So essentially we are taking a sheet of paper U and using F to wrap it around a sphere. The first stage of wrapping turns the sheet of paper into a cylinder with the sphere inside it (Fig. 10.9). This identifies the boundary lines I and II with one another (Fig. 10.7) and is equivalent mathematically to $F(\theta, 0) = F(\theta, 2\pi)$ for all θ.

The final steps in the wrapping collapses III onto $(0, 0, c)$ and IV onto $(0, 0, -c)$. Mathematically this says $F(\pi/2, \psi) = (0, 0, c)$ and $F(-\pi/2, \psi) = (0, 0, -c)$ for

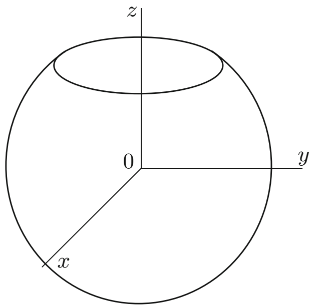


Fig. 10.10

all ψ. We thus see geometrically that if we *include* the boundary of U in the domain of F then the curve, which we previously missed, is covered twice and F is not one-to-one. If we do *not include* the boundary then this curve is not in the image of F.

Next we discuss parametrizations as *coordinate systems*. A useful initial approach to any such system is to sketch and examine the coordinate curves on the surface. In our case, since we are still looking at the unit sphere, we consider the curves obtained by fixing one of the variables of the function

$$F(\theta, \psi) = (\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta), \quad -\pi/2 < \theta < \pi/2,\ 0 < \psi < 2\pi.$$

Since F is bijective, a point on the surface corresponds to a unique pair (θ_1, ψ_1) and $F(\theta_1, \psi_1)$ is the point of intersection of the *coordinate curves* $\theta \to F(\theta, \psi_1)$, $\psi \to F(\theta_1, \psi)$. For this reason, the pair (θ, ψ) is often referred to as the *curvilinear coordinates* of the point $F(\theta, \psi)$. Fixing θ is clearly equivalent to taking a fixed value of z and amounts to taking a cross-section of the unit sphere parallel to the xy-plane. Geometrically we get a circle (Fig. 10.10).

This may also be seen analytically since the mapping $F(\theta, \psi)$ can be written as

$$(0, 0, \sin\theta) + \cos\theta(\cos\psi, \sin\psi, 0).$$

As ψ varies over $(0, 2\pi)$ we get a circle (with one point removed) in a plane parallel to the xy-plane, with centre $(0, 0, \sin\theta)$ on the z-axis, and radius $\cos\theta$. A number of level sets of θ are given in Fig. 10.11.

Now fix ψ and let θ range over the interval $-\pi/2 < \theta < \pi/2$. We have

$$F(\theta, \psi) = \cos\theta(\cos\psi, \sin\psi, 0) + \sin\theta(0, 0, 1).$$

Since $(\cos\psi, \sin\psi, 0)$ and $(0, 0, 1)$ are perpendicular unit vectors we get, by Pythagoras' theorem, a semicircle of radius 1. We do not get the full circle because of the restricted range of θ. This is easily seen, either geometrically or from the above formula, to be a semicircle running from $(-1, 0, 0)$ to $(+1, 0, 0)$, i.e.

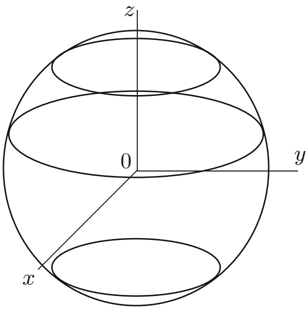


Fig. 10.11

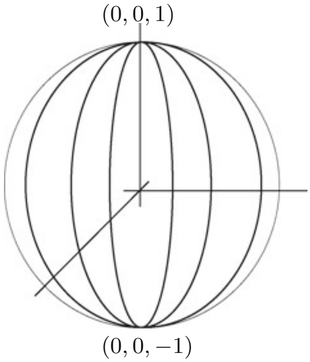


Fig. 10.12

from the bottom to the top of the sphere. In Fig. 10.12 we sketch a number of these level sets of ψ.

One can find a ready-made example of this by examining a *globe* representing the Earth. The level sets of θ are the *lines* or *parallels of latitude* with the equator corresponding to $\theta = 0$. Instead of $+$ and $-$ the terms North and South are used. The level sets of ψ are called *lines of longitude* or *meridians* with the level set $\psi = 0$ going through Greenwich, England. In place of $+$ and $-$ we use East and West (of Greenwich). The range here is from 0 to 180° while for θ we use 0 to 90°. If we represent the Earth on a map we are unwrapping the sphere, or mathematically, taking the inverse of a parametrization. The most common map (Fig. 10.13) is obtained by using *Mercator's projection*. This first unwraps the sphere onto a cylinder in such a way that when the cylinder is unrolled shapes are preserved (but area is distorted—see Exercise 11.8). What happens if we replace the cylinder by a cone?

Note that East meets West on the International Date Line in the middle of the Pacific Ocean. There is a certain ambiguity about this line—is it 180° East or 180° West of Greenwich? This, once more, reflects the fact that we do not obtain a bijective mapping if we include the boundary in the domain of definition of the parametrization.

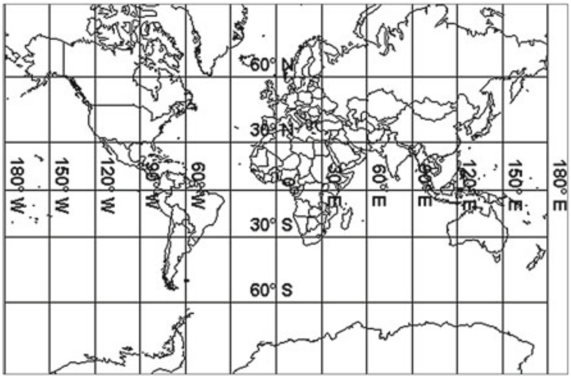


Fig. 10.13

Different maps result from using different parametrizations or coordinate systems. The mapping F that we have been discussing is called the *geographical coordinate system*. It is as useful and as intuitive a system as the more popular spherical polar coordinate system that we now discuss.

A small change in our method of introducing geographical coordinates leads to the definition of spherical polar coordinates. Recall that, for the ellipsoid, we have

$$\left(\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{1/2}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

Now we interchange the role of sine and cosine and let

$$\frac{z}{c}=\cos\theta \quad \text{and} \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{1/2}=\sin\theta.$$

This implies

$$\left(\frac{x}{a\sin\theta}\right)^2+\left(\frac{y}{b\sin\theta}\right)^2=1\,.$$

Let $x/a\sin\theta=\cos\psi$ and $y/b\sin\theta=\sin\psi$. This defines the mapping

$$G(\theta,\psi)=(a\sin\theta\cos\psi,\,b\sin\theta\sin\psi,\,c\cos\theta)$$

which are the *spherical polar coordinates* of the point (x,y,z).

From Fig. 10.14 we see that $0<\theta<\pi$ and $0<\psi<2\pi$. The angle θ is called the *colatitude* and ψ is sometimes called the *azimuth*. Note that we obtain the same two sets of coordinate curves for geographical and spherical polar coordinates—the labelling is, however, different.

For a sphere of radius r with centre at the origin this gives

$$G(\theta,\psi)=(r\sin\theta\cos\psi,\,r\sin\theta\sin\psi,\,r\cos\theta).$$

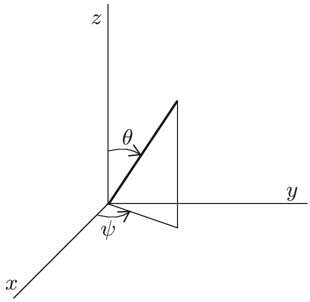


Fig. 10.14

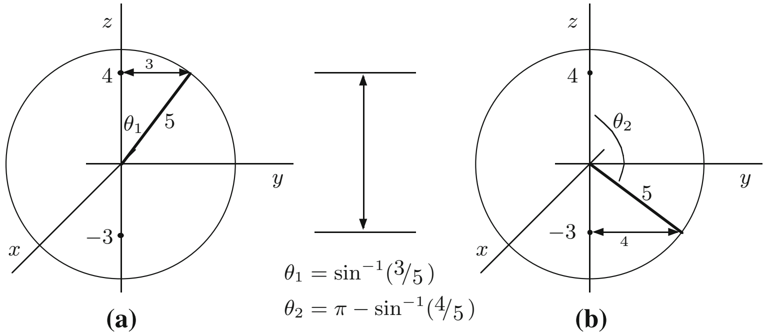


Fig. 10.15

Note that

$$G_\theta \times G_\psi = r^2 \sin\theta(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

and

$$\|G_\theta \times G_\psi\| = r^2 \sin\theta.$$

If we wish to parametrize a part of the sphere we can still use G—and hence all calculations involving G and its derivatives—but it is necessary to identify the restricted domain on which we are working. Fig. 10.14 is useful for this purpose. Suppose, for instance, we wish to parametrize that portion of the sphere of radius 5 with centre at the origin which lies between the planes $z = -3$ and $z = 4$.

Figure 10.15a and b gives the minimum and maximum values of θ, θ_1 and θ_2, and there is no restriction on ψ. Hence our parametrization G has domain

$$U = \{(\theta, \psi) : \sin^{-1}\Big(\frac{3}{5}\Big) < \theta < \pi - \sin^{-1}\Big(\frac{4}{5}\Big),\ 0 < \psi < 2\pi\}.$$

From Fig. 10.14 we get the spherical polar coordinates in terms of the Cartesian coordinates (x, y, z):

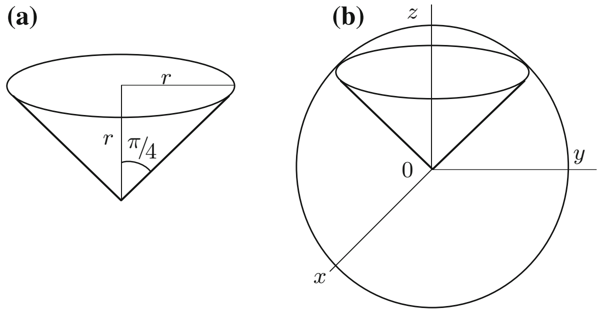


Fig. 10.16

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \theta = \tan^{-1}\left(\frac{(x^2 + y^2)^{1/2}}{z}\right) \quad \text{and} \quad \psi = \tan^{-1}\left(\frac{y}{x}\right).$$

The parametrization of the sphere can also be used to parametrize the *inverted vertical cone* of height a and base radius a (Fig. 10.16a).

Note that the angle between the z-axis and the curved surface of the cone is $\pi/4$. If we take the cross-section of the cone of height r, $0 < r < a$, then we obtain a circle of colatitude $\pi/4$ on the sphere of radius $r\sqrt{2}$ (Fig. 10.16b). The spherical polar coordinates of this circle are

$$\begin{aligned}\left(r\sqrt{2}\sin\frac{\pi}{4}\cos\psi,\ r\sqrt{2}\sin\frac{\pi}{4}\sin\psi,\ r\sqrt{2}\cos\frac{\pi}{4}\right), \quad 0 < \psi < 2\pi,\\ = (r\cos\psi, r\sin\psi, r)\end{aligned}$$

and we have the parametrization

$$(r, \psi) \longrightarrow (r\cos\psi, r\sin\psi, r), \quad 0 < \psi < 2\pi,\ 0 < r < a.$$

This completes our analysis of the ellipsoid and sphere. We remark that many of our comments apply to a wide variety of surfaces.

A simple surface may have *different parametrizations* and depending on the circumstances one may be more useful than another. In Examples 10.3 and 10.4 we derived other parametrizations for the “sphere”. An important technique that we discuss in Chap. 14 is how to go from one parametrization to another. Since parametrizations are coordinate systems involving variables this process is known as a *change of variable*.

In the above discussion we have seen that ellipsoids and spheres are not simple surfaces, although they may be written as a union of a simple surface and a parametrized curve and for practical purposes, such as integration, they may be regarded as simple surfaces. Nevertheless, it is desirable to have a definition of surface in $\mathbb{R}^3$ which implies, for instance, that a sphere is a surface. Such a definition, as we have just noted, requires more than one parametrization.

Definition 10.6 A subset of $\mathbb{R}^3$ is a surface if it can be covered by a collection of (generally overlapping) simple surfaces.

With the aid of the following important and highly non-trivial result we outline a relationship between surfaces, simple surfaces, level sets and graphs. This also shows that an ellipsoid is a surface.

Theorem 10.7 (Inverse Mapping Theorem.) *If $F : U(open) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable and $F'(p)$ is invertible at $p \in U$ then there exist V and W open in $\mathbb{R}^n$, $p \in V$, such that the restriction of F to V defines a bijective mapping from V onto W with continuously differentiable inverse.*

Theorem 10.8 *If S is a subset of $\mathbb{R}^3$ then the following are equivalent:*

1. *S is a surface,*
2. *for each $P \in S$ there exists $\varepsilon > 0$ such that $S \cap \{X : \|X - P\| < \varepsilon\}$ is a simple surface,*
3. *for each $P \in S$ there exists $\varepsilon > 0$ such that $S \cap \{X : \|X - P\| < \varepsilon\}$ is the graph of a smooth function,*
4. *for each $P \in S$ there exists $\varepsilon > 0$ and a smooth function $f : \{X : \|X - P\| < \varepsilon\} \longrightarrow \mathbb{R}$ such that $\nabla f(X) \neq 0$ for $\|X - P\| < \varepsilon$ and*

$$S \cap \{X : \|X - P\| < \varepsilon\} = \{X : \|X - P\| < \varepsilon, f(X) = 0\}.$$

Proof The first two conditions are equivalent by Definition 10.6. By Example 1.7, graphs are level sets and, by the Implicit Function Theorem (Theorem 2.1), level sets are locally graphs and the final two conditions are also equivalent. By Example 10.3 every graph is a simple surface and thus it suffices to show that every simple surface is locally a level set.

Suppose $\phi : U(\text{open in } \mathbb{R}^2) \longrightarrow S$ is a parameterized surface. Define $F : U \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by letting

$$F(x, y, z) = \phi(x, y) + zG(x, y)$$

where $G(x, y, z) = \phi_x(x, y) \times \phi_y(x, y)$. Then F is continuously differentiable and

$$F'(x, y, 0) = [\phi_x(x, y), \phi_y(x, y), G(x, y)]$$

is invertible. If $P \in S$ then, by the Inverse Mapping Theorem, we can find open subsets of $\mathbb{R}^3$, V and W, such that $P \in W$, $F(V) = W$ and F restricted to V has a continuously differentiable inverse. Let $g = \pi_3 \circ F^{-1}$ where $\pi_3(x, y, z) = z$. If $(x, y, z) \in W$, then for a unique $(x', y', z') \in V$ we have

$$(x, y, z) = F(x', y', z') = \phi(x', y') + G(x', y')z'.$$

This implies

$$\begin{aligned} g(x,y,z) = \pi_3 \circ F^{-1}(x,y,z) = 0 &\Longleftrightarrow \pi_3 \circ F^{-1} \circ F(x',y',z') = z' = 0 \\ &\Longleftrightarrow (x,y,z) = F(x',y',0) = \phi(x',y') \\ &\Longleftrightarrow (x,y,z) \in S \end{aligned}$$

and hence $g^{-1}(0) \cap W = S \cap W$. Since $\nabla g(X) \neq 0$ for all $X \in W$ this completes the proof. □

The above shows that graphs are much more than simple examples. This will become very evident when we examine curvature in the final three chapters.

Exercises

10.1 By using $\cosh^2 - \sinh^2 = 1$ and $\cos^2 + \sin^2 = 1$ parametrize the surfaces

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$

10.2 Show that the following are parametrizations of simple surfaces

(a) $P_1(r,\theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 < \theta < 2\pi,\ 0 < r < 1$
(b) $P_2(x,\theta) = (x\cos\theta, y\sin\theta, x+y), \quad 0 < x < 1,\ 0 < \theta < 2\pi, y \neq 0$
(c) $P_3(u,v) = (u+v, u-v, uv), \quad 0 < u < 1,\ 0 < v < 1.$

10.3 By using a parametrization of an ellipsoid parametrize an inverted cone of height h and base radius r.

10.4 Use geographical and spherical polar coordinates to parametrize that portion of the sphere $x^2+y^2+z^2 = r^2$ which lies in the first octant $\{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}$.

10.5 Let $S = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. If $(u,v) \in \mathbb{R}^2$ then the line determined by $(u,v,0)$ and $(0,0,1)$ intersects S in a point other than $(0,0,1)$. Let $\phi(u,v)$ denote this point. Find the coordinates of $\phi(u,v)$ in terms of u and v. Show that $\phi(\mathbb{R}^2)$ is a simple surface parametrized by ϕ. Identify $\phi(\mathbb{R}^2)$.

10.6 Find the length of the portion of the curve $u = v$ lying in the surface parametrized by

$$\phi(u,v) = (u\cos v, u\sin v, u\sqrt{3})$$

which lies between the planes $z = 0$ and $z = 2\sqrt{3}$.

10.7 Let $f(x,y,z) = x+xy+yz$ and let $S = f^{-1}(1)$. Find two parametrized surfaces which lie in S and cover S.

10.8 Consider a line L initially lying along the positive y-axis and attached orthogonally to the z-axis. If L moves along the z-axis with constant speed a and at the same time rotates about the z-axis with constant speed b show that it sweeps out a surface parametrized by

$$\phi(u,t) = (u\sin bt, u\cos bt, at), \quad u > 0,\ t > 0.$$

Chapter 11
Surface Area

Summary *We define and calculate surface area.*

We follow the method used in Chap. 5 to calculate the length of a curve in order to define the (surface) area of a simple surface S in $\mathbb{R}^3$. Let $\phi\colon U \to S$ denote a parametrization of S where U is an open subset of $\mathbb{R}^2$. We take a rectangular partition of U (Fig. 11.1), find the approximate area of the image of each rectangle in the partition, form a Riemann sum and obtain the surface area as the limit of the Riemann sums.

If R denotes a typical rectangle in U (Fig. 11.2) then

$$\phi(x + \Delta x, y) - \phi(x, y) \approx \phi_x(x, y)\Delta x$$

and

$$\phi(x, y + \Delta y) - \phi(x, y) \approx \phi_y(x, y)\Delta y.$$

If $\theta = \theta(x, y)$ is the *angle* between $\phi_x(x, y)$ and $\phi_y(x, y)$ then using the well-known formula for the *area* of a triangle, $\frac{1}{2}ab \sin C$, we get

$$\begin{aligned}\text{Area}\big(\phi(R)\big) &\approx 2 \cdot \frac{1}{2}\|\phi_x(x, y)\| \cdot \|\phi_y(x, y)\| \cdot |\sin\theta(x, y)|\Delta x\Delta y\\ &= \|\phi_x \times \phi_y(x, y)\|\Delta x\Delta y.\end{aligned}$$

If ϕ is integrable over U, and this will be the case if, for instance U is bounded, with smooth boundary, and ϕ has a continuous extension to $\overline{U}$, then

$$\begin{aligned}\sum_{i,j}\text{Area}\big(\phi(R_{ij})\big) &\cong \sum_{i,j}\|\phi_x \times \phi_y(x_i, y_j)\|\Delta x_i\Delta y_j\\ &\longrightarrow \iint_U \|\phi_x \times \phi_y\|\, dxdy\end{aligned}$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_11,

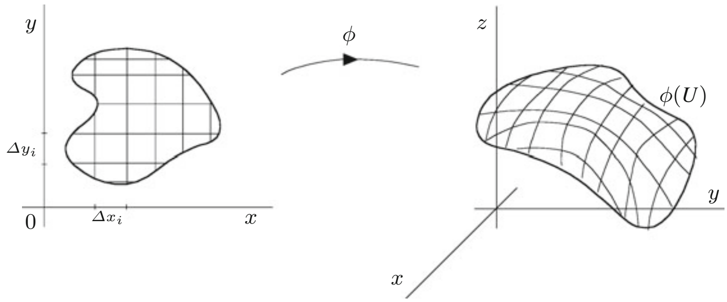


Fig. 11.1

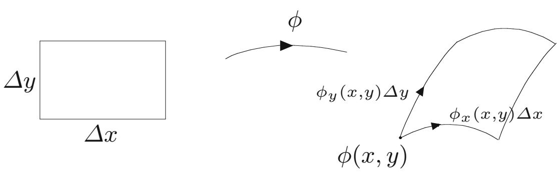


Fig. 11.2

as we take finer and finer partitions of U. Hence the *surface area* of S, $A(S)$, has the form

$$A(S) = \iint\limits_U \|\phi_x \times \phi_y\| \, dxdy$$

and is calculated using a parametrization. In general, a surface will admit many different parametrizations but, as we will see later, they all give the same value for surface area. In this chapter we are using the usual *physical* idea of area and angle. These *non-negative absolute quantities* do not require a sense of direction or orientation on the surface and lead, as we have just seen, to a relatively straightforward form of integration. In the next chapter we require more sophisticated concepts of area and angle to integrate *vector fields* over a surface.

We now obtain another formula for surface area which avoids the cross product. We maintain the notation $\phi\colon U \to S$ for our parametrization and introduce, in their traditional form, three quantities that make regular and important appearances in the remaining chapters of this book. Let

$$\begin{aligned} E \text{ (or } E(x,y)) &= \phi_x \cdot \phi_x = \|\phi_x\|^2 \\ F \text{ (or } F(x,y)) &= \phi_x \cdot \phi_y \\ G \text{ (or } G(x,y)) &= \phi_y \cdot \phi_y = \|\phi_y\|^2. \end{aligned}$$

We have

$$\|\phi_x \times \phi_y\|^2 = \|\phi_x\|^2 \|\phi_y\|^2 \sin^2 \delta$$

where δ is the angle between ϕ_x and ϕ_y. Hence

$$\cos\delta = \frac{\phi_x \cdot \phi_y}{\|\phi_x\| \cdot \|\phi_y\|}$$

and

$$\begin{aligned}\|\phi_x \times \phi_y\|^2 &= \|\phi_x\|^2\|\phi_y\|^2(1-\cos^2\delta) \\ &= \|\phi_x\|^2\|\phi_y\|^2\Big(1 - \frac{(\phi_x\cdot\phi_y)^2}{\|\phi_x\|^2\|\phi_y\|^2}\Big) \\ &= \|\phi_x\|^2\|\phi_y\|^2 - (\phi_x\cdot\phi_y)^2 = EG - F^2.\end{aligned}$$

This gives the following useful formula for surface area

$$A(S) = \iint_U \sqrt{EG - F^2}\, dxdy.$$

Figure 11.2 shows how E, F and G quantify the *distortion* of a rectangle by the parametrization. The stretching or contraction of the sides is measured by E and G while F measures the change in angle. Thus we see that *shape* is preserved if $E = G$ and $F = 0$ while (relative) *area* is preserved if $EG - F^2$ is constant. For many important parametrizations, including geographical and spherical polar coordinates, $F = 0$. This implies that angles between curves are preserved and, in particular, parallels (of latitude) and meridians (of longitude) cross one another at right angles. For geographical coordinates on a sphere of radius r, $E = r^2$ and $G = r^2\cos^2\theta$ and hence neither shape nor area are preserved. On the Equator, where $\theta = 0$, we have $E = G = r^2$ but as one moves towards the North and South Poles, $\theta \to \pm\pi/2$ and $G \to 0$ while $E = r^2$. Consequently, near the Equator shape is fairly well preserved but as one moves towards the polar regions it becomes more and more distorted. Mercator's projection is a *modification* of geographical coordinates—notice how the lines of latitude in Fig. 10.13 are not equally spaced in order to preserve shape. If the cylinder in Mercator's construction is replaced by a cone we obtain *Lambert's equal area projection* which preserves (relative) area but distorts shape and distance.

In Chap. 10 we noted that many well-known surfaces S could be written in the form $S = S_1 \cup \Gamma$ where S_1 is a simple surface and Γ is a curve parametrized by a *smooth* function on a closed interval. We see now that the surface area of Γ is zero and from this conclude that $A(S) = A(S_1)$ and thus we may use a parametrization of S_1 to calculate the surface area of S. To show $A(\Gamma) = 0$ enclose Γ in a finite union of rectangles of small width ε (Fig. 11.3).

The sum of the lengths of the rectangles is approximately the length of Γ, $l(\Gamma)$. Hence

$$A(\Gamma) \approx \varepsilon \times l(\Gamma).$$

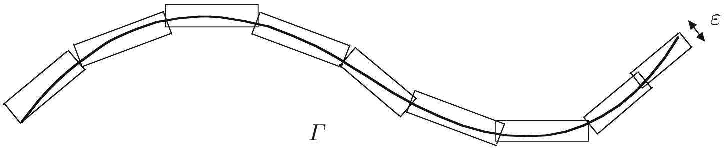


Fig. 11.3

Since ε is arbitrarily small this implies $A(\Gamma) = 0$. This intuitive "proof" can be developed into a rigorous proof using the *Mean Value Theorem* for differentiable functions of one variable. A differentiable parametrization of the curve is *necessary*, although our intuition might suggest otherwise, since there is a famous example of a *square filling curve* parametrized by a continuous function.

Example 11.1 In this example we consider the surface of revolution S parametrized as in Example 10.4. Let $P\colon [a, b] \to \mathbb{R}^2$ denote a parametrization of the curve to be revolved. Our parametrization ϕ of S is defined on $U = \{(t, \theta) : a < t < b,\ 0 < \theta < 2\pi\}$ and has the form

$$\phi(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$

where $P(t) = \big(x(t), y(t)\big)$. We have already shown in Example 10.4 that

$$\|\phi_t \times \phi_\theta\| = y(t)\|P'(t)\|$$

and so the surface area of S is

$$\int_0^{2\pi}\int_a^b y(t)\|P'(t)\|dtd\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_a^b y(t)\|P'(t)\|dt\right)$$

$$= 2\pi\int_a^b y(t)\|P'(t)\|dt.$$

For the *cone* of height h and (base) radius r, $P(t) = (ht, rt)$, $P'(t) = (h, r)$, $\|P'(t)\| = (h^2 + r^2)^{1/2}$ and the curved surface area is

$$2\pi\int_0^1 rt(r^2 + h^2)^{1/2}dt = 2\pi r(h^2 + r^2)^{1/2}\int_0^1 tdt = \pi r(h^2 + r^2)^{1/2}.$$

For the *cylinder* of radius r and height h we have $P(t) = (t, r)$, $P'(t) = (1, 0)$ and $\|P'(t)\| = 1$. Hence the curved surface area of the cylinder is

$$2\pi \int_0^h r \cdot 1 dt = 2\pi rh.$$

For the *sphere* of radius r, $P(t) = (r\cos t, r\sin t)$, $P'(t) = (-r\sin t, r\cos t)$ and $\|P'(t)\| = r$ and the surface area of the sphere of radius r is

$$2\pi \int_0^\pi r\sin t \cdot r dt = 2\pi r^2(-\cos t)\Big|_0^\pi = 4\pi r^2.$$

We now parametrize the *torus* by realising it as a surface of revolution. We adopt a slightly different approach to that in Example 10.4 in order to obtain what is regarded as the standard parametrization. Place a circle of radius r in the (y, z)-plane with centre on the y-axis at a distance b, $r > b$, from the origin and revolve this circle about the z-axis (Fig. 11.4).

The coordinates of a typical point on the original curve in the (x, z)-plane are $(b + r\cos\theta, r\sin\theta)$. When rotated about the z-axis the third coordinate remains unchanged while the first two coordinates describe a circle of radius $b + r\cos\theta$ about the origin. This gives a parametrization f with formula

$$f(\theta, \psi) = \big((b + r\cos\theta)\cos\psi, (b + r\cos\theta)\sin\psi, r\sin\theta\big)$$

and domain $(0, 2\pi) \times (0, 2\pi)$. Hence

$$\begin{aligned} f_\theta &= (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta) \\ f_\psi &= \Big(-(b + r\cos\theta)\sin\psi, (b + r\cos\theta)\cos\psi, 0\Big). \end{aligned}$$

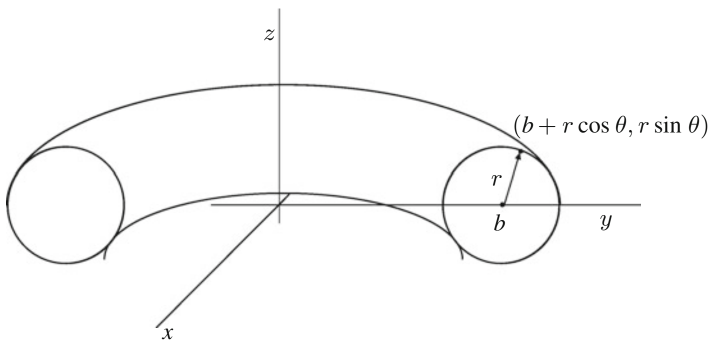


Fig. 11.4

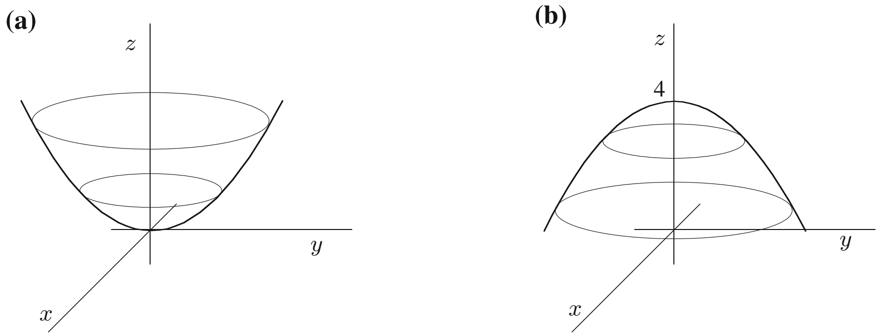


Fig. 11.5

By inspection we see that f_θ and f_ψ are perpendicular and hence $F = 0$. We have

$$\begin{aligned} E = \|f_\theta\|^2 &= r^2 \sin^2\theta \cos^2\psi + r^2 \sin^2\theta \sin^2\psi + r^2\cos^2\theta \\ &= r^2 \sin^2\theta(\cos^2\psi + \sin^2\psi) + r^2\cos^2\theta \\ &= r^2\sin^2\theta + r^2\cos^2\theta = r^2 \end{aligned}$$

and

$$\begin{aligned} G = \|f_\psi\|^2 &= (b + r\cos\theta)^2 \sin^2\psi + (b + r\cos\theta)^2\cos^2\psi \\ &= (b + r\cos\theta)^2. \end{aligned}$$

Hence, $\sqrt{EG - F^2} = r(b + r\cos\theta)$, and

$$\begin{aligned} A(\text{Torus}) &= \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\, d\theta d\psi \\ &= \int_0^{2\pi}\int_0^{2\pi} r(b + r\cos\theta)\, d\theta d\psi \\ &= \left(\int_0^{2\pi} d\psi\right)\left(\int_0^{2\pi} r(b + r\cos\theta)\, d\theta\right) \\ &= 4\pi^2 rb. \end{aligned}$$

Some geometric insight might have led you to this answer without *any* calculations or integration (see Example 14.5). The coordinates (θ, ψ) for the torus defined above are called *toroidal polar coordinates*.

Example 11.2 We wish to find the surface area of the portion of the *paraboloid* $z = 4 - x^2 - y^2$ that lies above the xy-plane. This is the graph of the function $f(x, y) = 4 - x^2 - y^2$ and we could find a parametrization using the method in Example 10.3. We prefer, however, to take a more geometric approach.

The standard paraboloid $z = x^2 + y^2$ can be sketched by noting that cross-sections parallel to the xy-plane are circles (Fig. 11.5a). This surface is turned upside down by taking $z = -x^2 - y^2$ and then moved up 4 units in the direction of the z-axis to give us the original surface $z = 4 - x^2 - y^2$ (Fig. 11.5b).

The surface cuts the xy-plane when $0 = z = 4 - x^2 - y^2$, i.e. on the circle $x^2 + y^2 = 4$. Clearly the geometry shows that we can project the surface in a one-to-one fashion onto the disc $x^2 + y^2 \leq 4$ and if we reverse this we obtain the parametrization

$$(x, y) \in \{x^2 + y^2 < 4\} \longrightarrow (x, y, 4 - x^2 - y^2) \in \text{ Paraboloid.}$$

We could proceed directly to use this parametrization to compute the surface area. This would require a change of variable in working out the double integral. It is just as easy to initially use a more appropriate parametrization which avoids a later change of variable.

We note that the domain of the above parametrization is a *disc* and the formula for the parametrization function involves $x^2 + y^2$. Either of these on their own suggest polar or spherical coordinates. Since, however, the disc shape and $x^2 + y^2$ only involve the first two coordinates this inclines us towards polar coordinates for the pair (x, y). We use the parametrization

$$f\colon (r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, 4 - r^2), \quad 0 < r < 2 \text{ and } 0 < \theta < 2\pi.$$

We have $f_r = (\cos\theta, \sin\theta, -2r)$, $f_\theta = (-r\sin\theta, r\cos\theta, 0)$. Hence $E = 1 + 4r^2$, $G = r^2$ and since $f_r \cdot f_\theta = 0$ we have $F = 0$. The surface area is

$$\begin{aligned}
\int_0^2 \int_0^{2\pi} \sqrt{EG - F^2}\, dr d\theta &= \int_0^2 \int_0^{2\pi} r\sqrt{1 + 4r^2}\, dr d\theta \\
&= 2\pi \int_0^2 r\sqrt{1 + 4r^2}\, dr \\
&= \frac{2\pi}{8} \int u^{1/2} du, \qquad 1 + 4r^2 = u,\ 8rdr = du \\
&= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} = \frac{\pi}{6}(1 + 4r^2)^{3/2} \Big|_0^2 \\
&= \frac{\pi}{6}(17^{3/2} - 1).
\end{aligned}$$

Example 11.3 We wish to find in this example the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 4z$ which lies outside the paraboloid $x^2 + y^2 = 3z$.

For this problem we first get some idea of the geometry and see, once this is reasonably clear, that the parametrization, the domain of the parametrization and even the integration are relatively straightforward.

We have two surfaces, a sphere and a paraboloid. Rewriting the equation for the sphere by completing the square we get

$$x^2 + y^2 + z^2 - 4z + 4 = 4$$

i.e.

$$x^2 + y^2 + (z - 2)^2 = 2^2$$

and this is the sphere of radius 2 with centre (0, 0, 2) (Fig. 11.6).

If we take the cross-section of the paraboloid corresponding to the plane $z = c$ we get a circle of radius $\sqrt{3c}$ and so we can sketch the paraboloid (Fig. 11.7). The points of intersection of the two surfaces are identified by letting

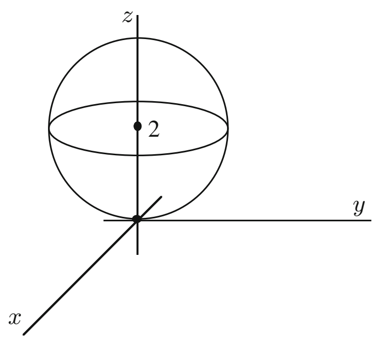


Fig. 11.6

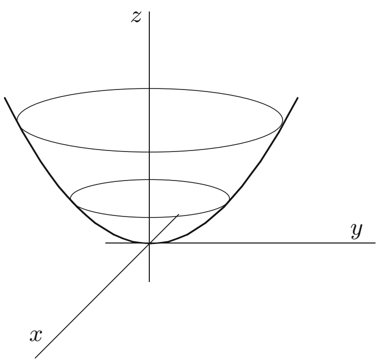


Fig. 11.7

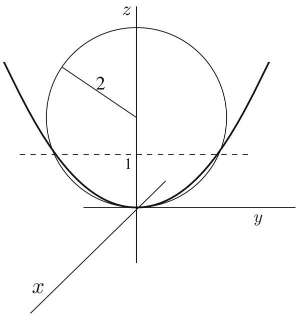


Fig. 11.8

$$x^2 + y^2 + z^2 - 4z = 0 = x^2 + y^2 - 3z.$$

This implies $z^2 - 4z = -3z$, i.e. $z^2 - z = 0$ and $z = 1$ or $z = 0$. When $z = 0$ we get the point $(0, 0, 0)$ on both surfaces and when $z = 1$ we get the curve $x^2 + y^2 = 3$, i.e. a circle of radius $\sqrt{3}$. This usually indicates where one surface crosses over from the inside to the outside of the other and gives us the working diagram (Fig. 11.8).

To find when the sphere is outside the paraboloid it suffices to compare the points (x, y, z_p) and (x, y, z_s) on the paraboloid and sphere, respectively, which project onto the *same* point $(x, y, 0)$ in the (x, y)-plane. From the defining equations of the surfaces we have

$$x^2 + y^2 = 4z_s - z_s^2 = 3z_p.$$

Hence

$$z_s - z_s^2 = 3z_p - 3z_s, \quad \text{i.e.} \quad z_s(1 - z_s) = 3(z_p - z_s).$$

Since $z_s \geq 0$ we have $z_p \geq z_s$ if and only if $1 - z_s \geq 0$, i.e. when $0 \leq z_s \leq 1$. Thus the part of the sphere which lies outside the paraboloid is precisely that portion which lies between the parallel planes $z = 0$ and $z = 1$.

We thus have to find the surface area of the sphere $x^2 + y^2 + (z - 2)^2 = 2^2$ between the planes $z = 0$ and $z = 1$. We use the formula for spherical polar coordinates obtained in Example 10.5 translated so that the origin is at the point $(0, 0, 2)$, i.e. let

$$x = 2\sin\theta\cos\psi, \quad y = 2\sin\theta\sin\psi, \quad z - 2 = 2\cos\theta$$

and it is now only necessary to find the domain of the parametrization. Since $0 \leq z \leq 1$ we consider $-2 \leq z - 2 \leq -1$, i.e. $-2 \leq 2\cos\theta \leq -1$. This implies (see Fig. 10.14) $2\pi/3 \leq \theta \leq \pi$ and, since there is no restriction on ψ, $0 < \psi < 2\pi$.

Denote this parametrization by f. Then

$$f_\theta = (2\cos\theta\cos\psi, 2\cos\theta\sin\psi, -2\sin\theta)$$

and

$$f_\psi = (-2\sin\theta\sin\psi, 2\sin\theta\cos\psi, 0).$$

Hence

$$\begin{aligned} E &= 4\cos^2\theta\cos^2\psi + 4\cos^2\theta\sin^2\psi + 4\sin^2\theta \\ &= 4(\cos^2\psi + \sin^2\psi)\cos^2\theta + 4\sin^2\theta \\ &= 4\cos^2\theta + 4\sin^2\theta = 4 \end{aligned}$$

and

$$G = 4\sin^2\theta\sin^2\psi + 4\sin^2\theta\cos^2\psi = 4\sin^2\theta.$$

Since $f_\theta \cdot f_\psi = 0$, $F = 0$ and the surface area is

$$\begin{aligned} \int_0^{2\pi}\int_{2\pi/3}^{\pi} 2\cdot 2\sin\theta d\theta d\psi &= \Big(4\int_0^{2\pi} d\psi\Big)\Big(\int_{2\pi/3}^{\pi} \sin\theta d\theta\Big) \\ &= 8\pi(-\cos\theta)\Big|_{2\pi/3}^{\pi} = 8\pi(1-1/2) = 4\pi. \end{aligned}$$

Example 11.4 We calculate the surface area of the level set $x^2 - 2y - 2z = 0$ which projects onto the region U of the xy-plane bounded by the lines $x = 2$, $y = 0$ and $y = 4x$. On the surface S we have $z = \frac{1}{2}x^2 - y$ and the surface is the graph of the function $g(x, y) = \frac{1}{2}x^2 - y$. Using Example 10.3 we obtain the parametrization

$$\phi\colon (x, y) \in U \longrightarrow (x, y, \frac{1}{2}x^2 - y)$$

and, moreover,

$$\|\phi_x \times \phi_y\| = \|(-g_x, -g_y, 1)\| = \|(x, -1, 1)\| = (x^2 + 2)^{1/2}.$$

Note that only part of the surface lies above the xy-plane. Our surface area is

$$\iint_U (x^2 + 2)^{1/2}\, dxdy.$$

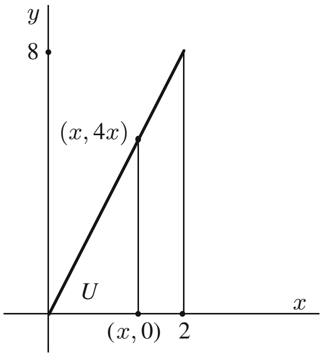


Fig. 11.9

We integrate first with respect to y and note, from Fig. 11.9, that y varies from 0 to $4x$ and afterwards integrate with respect to x which varies from 0 to 2. We have

$$\begin{aligned}\iint_U (x^2+2)^{1/2}\,dxdy &= \int_0^2 \left\{\int_0^{4x} (x^2+2)^{1/2} dy\right\} dx \\ &= \int_0^2 (x^2+2)^{1/2} y \Big|_0^{4x} dx = \int_0^2 4x(x^2+2)^{1/2} dx \\ &= 2\int u^{1/2} du = 2\frac{u^{3/2}}{3/2} = \frac{4}{3}(x^2+2)^{3/2}\Big|_0^2 \\ &= \frac{4}{3}\left(6^{3/2}-2^{3/2}\right) = \frac{8}{3}\left(3\sqrt{6}-\sqrt{2}\right)\end{aligned}$$

where $u = x^2+2$, $du = 2xdx$.

Exercises

11.1 Calculate the surface area of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 0$ and $z = 4$.

11.2 Find the area of the surface generated by revolving about the x-axis the curves

(a) $y = x^3, 0 \le x \le 1$,
(b) $y = x^2, 0 \le x \le 1$.

11.3 Show that the area of the *helicoid* defined by the parametrization

$$P(r,\theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 < \theta < 2\pi,\ 0 < r < 1$$

is $\pi\big(\sqrt{2} + \log(1+\sqrt{2})\big)$.

11.4 If U is open in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a smooth function show

$$\text{Area}\,\big(\text{graph}(f)\big) = \iint_U \big(1 + \|\nabla f\|^2\big)^{1/2}\, dxdy.$$

11.5 Find the surface area of the paraboloid in Example 11.3 that lies inside the sphere.

11.6 Find the area of the portion of the surface $z = xy$ which lies inside the cylinder $x^2 + y^2 = a^2$.

11.7 Use the parametrization $(r,\theta) \to (r^2/16, r\cos\theta, r\sin\theta)$ to find the area of the part of $y^2 + z^2 = 16x$ which lies in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ between the planes $x = 0$ and $x = 12$ and inside the cylinder $y^2 = 4x$.

11.8 Let $\phi : U \to S$ denote a parametrization of the simple surface S in $\mathbb{R}^3$. Give some intuitive reasons why ϕ preserves

(a) angles if $F = 0$
(b) relative area if $EF - G^2$ is constant
(c) shape if $E = G$ and $F = 0$.

11.9 Use the Cauchy–Schwarz inequality (Example 3.4) to show that $EG - F^2 \geq 0$ where $E,\ F$ and G are calculated from any parametrization of a surface.

11.10 Let Γ denote a directed curve in a surface parametrized by a mapping ϕ. Show that Γ has a parametrization of the form $P : [a,b] \longrightarrow \Gamma$, $P(t) = \phi(\alpha(t), \beta(t))$ where α and β are real valued functions. Show that

$$l(\Gamma) = \int_a^b \sqrt{E(P(t))\alpha'(t)^2 + 2F(P(t))\alpha'(t)\beta'(t) + G(P(t))\beta'(t)^2}\, dt.$$

(see Table 11.1)

Table 11.1 Useful parametrizations

1. Sphere $x^2 + y^2 + z^2 = r^2$; **geographical coordinates**

$$f(\theta, \psi) = (r\cos\theta\cos\psi, r\cos\theta\sin\psi, r\sin\theta), \ -\pi/2 < \theta < \pi/2, \ 0 < \psi < 2\pi$$

$$f_\theta = (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta), \quad f_\psi = (-r\cos\theta\sin\psi, r\cos\theta\cos\psi, 0)$$

$$f_\theta \times f_\psi = -r^2\cos\theta(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

$$E = r^2, \ F = 0, \ G = r^2\cos^2\theta, \ \|f_\theta \times f_\psi\| = \sqrt{EG - F^2} = r^2\cos\theta$$

$$\mathbf{n} = -(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

2. Sphere $x^2 + y^2 + z^2 = r^2$; **spherical polar coordinates**

$$f(\theta, \psi) = (r\sin\theta\cos\psi, r\sin\theta\sin\psi, r\cos\theta), \quad 0 < \theta < \pi, \ 0 < \psi < 2\pi$$

$$f_\theta = (r\cos\theta\cos\psi, r\cos\theta\sin\psi, -r\sin\theta), \quad f_\psi = (-r\sin\theta\sin\psi, r\sin\theta\cos\psi, 0)$$

$$f_\theta \times f_\psi = r^2\sin\theta(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

$$E = r^2, \ F = 0, \ G = r^2\sin^2\theta, \ \|f_\theta \times f_\psi\| = \sqrt{EG - F^2} = r^2\sin\theta$$

$$\mathbf{n} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

3. Cylinder $x^2 + y^2 = r^2, \ 0 < z < h$; **cylindrical polar coordinates**

$$f(\theta, z) = (r\cos\theta, r\sin\theta, z), \quad 0 < \theta < 2\pi, \ 0 < z < h$$

$$f_\theta = (-r\sin\theta, r\cos\theta, 0), \quad f_z = (0, 0, 1)$$

$$f_\theta \times f_z = (r\cos\theta, r\sin\theta, 0)$$

$$E = r^2, \ F = 0, \ G = 1, \ \|f_\theta \times f_z\| = \sqrt{EG - F^2} = r$$

$$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$

4. Inverted cone $x^2 + y^2 = z^2, \ 0 < z < a$

$$f(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 < r < a, \ 0 < \theta < 2\pi$$

$$f_r = (\cos\theta, \sin\theta, 1), \quad f_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$f_r \times f_\theta = (-r\cos\theta, -r\sin\theta, r)$$

$$E = 2, \ F = 0, \ G = r^2, \ \|f_r \times f_\theta\| = \sqrt{EG - F^2} = \sqrt{2}r$$

$$\mathbf{n} = -\tfrac{1}{\sqrt{2}}(\cos\theta, \sin\theta, -1)$$

(Continued)

Table 11.1 (Continued)

5. Torus—rotate circle of radius r in yz-plane, centre $(b, 0)$, $b > r$, about z-axis; **toroidal polar coordinates**

$$f(\theta, \psi) = \big((b + r\cos\theta)\cos\psi, (b + r\cos\theta)\sin\psi, r\sin\theta\big), \quad 0 < \theta, \psi < 2\pi$$

$$f_\theta = (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta),$$

$$f_\psi = \big(-(b + r\cos\theta)\sin\psi, (b + r\cos\theta)\cos\psi, 0\big)$$

$$f_\theta \times f_\psi = -r(b + r\cos\theta)(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

$$E = r^2,\ F = 0,\ G = (b + r\cos\theta)^2, \|f_\theta \times f_\psi\| = \sqrt{EG - F^2} = r(b + r\cos\theta)$$

$$\mathbf{n} = -(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

6. Graph of $f : U \subset \mathbb{R}^2 \to \mathbb{R}$

$$F(x, y) = (x, y, f(x, y)),\ (x, y) \in U$$

$$F_x = (1, 0, f_x),\ F_y = (0, 1, f_y),\ F_x \times F_y = (-f_x, -f_y, 1)$$

$$E = 1 + f_x^2,\ F = f_x f_y,\ G = 1 + f_y^2,\ \|F_x \times F_y\| = \sqrt{EG - F^2} = \sqrt{1 + f_x^2 + f_y^2}$$

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

7. Surface of revolution of Γ parametrized by $P(t) = ((x(t), y(t)),\ a \le t \le b,$ $y(t) > 0$ for all t, $\|P'(t)\| \neq 0$

$$f(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big), \quad a < t < b,\ 0 < \theta < 2\pi$$

$$f_t = \big(x'(t), y'(t)\cos\theta, y'(t)\sin\theta\big), \quad f_\theta = \big(0, -y(t)\sin\theta, y(t)\cos\theta\big)$$

$$f_t \times f_\theta = \big(y'(t)y(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta\big)$$

$$E = \|P'(t)\|^2,\ F = 0,\ G = y(t)^2,\ \|f_t \times f_\theta\| = \sqrt{EG - F^2} = y(t)\|P'(t)\|$$

$$\mathbf{n} = \tfrac{1}{\|P'(t)\|}\big(y'(t), -x'(t)\cos\theta, -x'(t)\sin\theta\big)$$

Chapter 12
Surface Integrals

Summary *We define the integral of a vector field over an oriented surface. Geometrical interpretations are discussed.*

Integrals are used to measure quantities such as length, area, expected value, etc., and as with all measurements, there has to be a unit of measurement. Our basic unit of measurement in integration theory over $\mathbb{R}$ is obtained by assigning the value 1 to the rectangle of height 1 over an interval of length 1 measured from *left to right*. From this we are able to define Riemann sums and afterwards the Riemann integral of a continuous function over a closed interval. The inclusion of "left to right" is crucial for without it we would have an ambiguous definition—and to a mathematician an ambiguous definition is no definition. To emphasis this point we call an interval $[a, b]$ directed from left to right a *positive* interval in $\mathbb{R}$.

Now suppose Γ is a curve in $\mathbb{R}^2$ and, for the sake of simplicity, we suppose that Γ is not closed and that we are interested in defining the integral of a function over Γ. We have seen in Chap. 7 that we can define *two* integrals over Γ since Γ can be directed in two different ways and we have to decide, before doing any calculations, which interests us. We may think of a curve as having *two sides*—one for each direction of motion—like two escalators side by side, one going up and the other coming down (Fig. 12.1). We call one side positive and the other negative. If we are interested in evaluating integrals over $\{\Gamma, A, B\}$ we call Γ directed from A to B the *positive side*.

We use a parametrization to transfer the integral over $\{\Gamma, A, B\}$ to an integral over an interval in $\mathbb{R}$ which we subsequently evaluate. However, the parametrization must be directed correctly. This means that it must map a *positive* interval in $\mathbb{R}$ onto the *positive* side of Γ. Surface integrals are defined in the same way—just step up a dimension.

Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. It has two *sides*—an inside and an outside. More generally, the surface S which is the level set $\{X \in U : f(X) = 0 \text{ and } \nabla f(X) \neq 0\}$ has $\pm\nabla f(X)/\|\nabla f(X)\|$ as unit normals at each point and we may consider $\nabla f(X)/\|\nabla f(X)\|$ and $-\nabla f(X)/\|\nabla f(X)\|$ as lying on different sides. We *distinguish* between the sides of a surface by *using normals*.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_12,

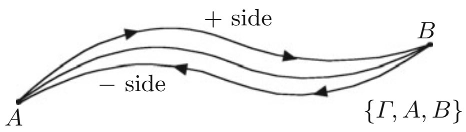


Fig. 12.1

Definition 12.1 An oriented surface in $\mathbb{R}^3$ is a pair $(S, \mathbf{n})$ where S is a surface and $\mathbf{n}$ is a continuous mapping from S into $\mathbb{R}^3$ such that $\mathbf{n}(P)$ is a unit normal to the surface S at P.

We will also use $\mathbf{S}$ to denote an oriented surface. The appearance of $\mathbf{S}$ signals that we have fixed an orientation on the underlying surface S. The notation $\mathbf{S}$ is simpler but if we need to specify the normal or if there is any possibility of confusion we write $(S, \mathbf{n})$.

Clearly if $(S, \mathbf{n})$ is an oriented surface, then $(S, -\mathbf{n})$ is also an oriented surface and the continuity requirement in the choice of normal implies that a *connected surface* can have at most *two orientations*. There exist, however, surfaces which do not admit *any* orientation and integration theory *cannot* be defined over such surfaces. Fortunately, every simple surface admits an orientation. If $\phi\colon U \subset \mathbb{R}^2 \to S$ is a parametrization of S then $\big(S, \phi_x \times \phi_y \big/ \|\phi_x \times \phi_y\|\big)$ and $\big(S, -\phi_x \times \phi_y \big/ \|\phi_x \times \phi_y\|\big)$ are two oriented surfaces associated with S and if S is connected these are the only two oriented surfaces associated with S. Given an oriented surface $(S, \mathbf{n})$ we call the side of S containing $\mathbf{n}$ the *positive side* and the side which contains $-\mathbf{n}$ the negative side. In the case of a simple oriented surface $(S, \mathbf{n})$ a parametrization ϕ is said to be *consistent* with the orientation if

$$\frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|} = \mathbf{n}.$$

If $\mathbf{F}$ is a continuous vector field on a simple oriented surface $\mathbf{S}$ we define

$$\iint_{\mathbf{S}} \mathbf{F}$$

by using a parametrization (ϕ, U) consistent with the orientation and a process involving Riemann sums similar to that used to define surface area in the previous chapter (see Figs. 11.2 and 11.3 for reference). We find it convenient to use (u, v) for the variables in the domain of ϕ and (x, y, z) for the range, i.e. $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$. We begin by partitioning the domain U into small rectangles and consider a typical rectangle R in this partition. Now, however, instead of taking the *absolute area* of $\phi(R)$ we take the *(signed) volume* of the *parallelepiped* with base $\phi(R)$ and height determined by $\mathbf{F}$ (see Fig. 12.2).

We use $\mathbf{n}$ as our *positive unit of measurement* perpendicular to the surface. This is reasonable since we are just considering a small portion of the surface $\phi(R)$, which

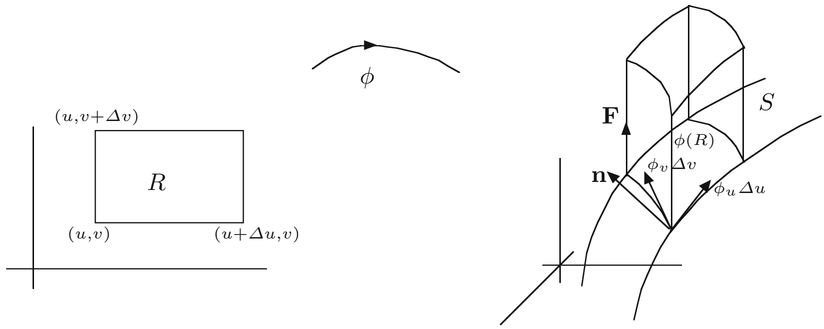


Fig. 12.2

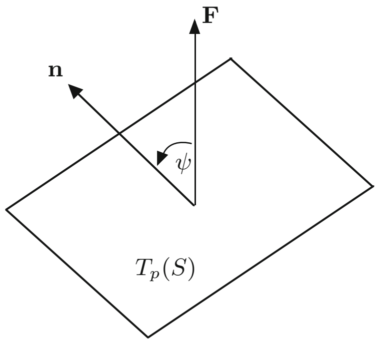


Fig. 12.3

lies approximately in the tangent plane, and $\mathbf{n}\big(\phi(u, v)\big)$ is perpendicular to the tangent plane at $\phi(u, v)$. The height of the parallelepiped is $\|\mathbf{F}\| \cos \psi = \mathbf{F} \cdot \mathbf{n}$ where ψ is the angle between $\mathbf{F}$ and $\mathbf{n}$ at $\phi(u, v)$ (see Fig. 12.3).

We have already seen, in calculating surface area, that the area of $\phi(R)$, the base of the parallelepiped, is $\|\phi_x \times \phi_y\| \Delta x \Delta y$ and hence the (signed) volume is

$$(\mathbf{F} \cdot \mathbf{n})\|\phi_u \times \phi_v\| \Delta u \Delta v = (\mathbf{F} \cdot \phi_u \times \phi_v) \Delta u \Delta v.$$

Now taking the limit of Riemann sums in the usual way we define

$$\begin{aligned} \iint\limits_{\mathbf{S}} \mathbf{F} &= \iint\limits_{U} (\mathbf{F} \cdot \mathbf{n})\|\phi_u \times \phi_v\| du dv \\ &= \iint\limits_{U} (\mathbf{F} \cdot \phi_u \times \phi_v) du dv. \end{aligned}$$

From this analysis we expect the volume of the solid one unit high on the positive side of the surface to equal the total surface area and to be obtained by integrating the vector field **n** over **S**. Indeed, we have

$$\iint_{(S,\mathbf{n})} \mathbf{n} = \iint_{U} (\mathbf{n}\cdot\mathbf{n})\|\phi_u \times \phi_v\| du dv = \text{Surface area } (S).$$

It is now natural to define the surface integral of a *scalar-valued function* f on $(S, \mathbf{n})$ by identifying f with $f\mathbf{n}$, i.e.

$$\iint_{(S,\mathbf{n})} f \overset{\text{def}}{=} \iint_{(S,\mathbf{n})} f\mathbf{n} = \iint_{S} f\big(\phi(u,v)\big)\|\phi_u \times \phi_v\| du dv$$

where ϕ is a parametrization consistent with the orientation. This is analogous to the way we defined, in Chap. 7, the integral of a scalar-valued function f along a curve by identifying f with fT. Although we needed an orientation to define this integral it is easily seen that

$$\iint_{(S,\mathbf{n})} f = \iint_{(S,-\mathbf{n})} f$$

and the value of the integral is independent of the orientation. This might suggest that it is possible to integrate a scalar-valued function over an *arbitrary* surface (see Definition 10.6). This is not true. What is required is an *orientable surface*, i.e. a surface which can be oriented. Simple surfaces are orientable. This means that we can use *any* parametrization to integrate a scalar-valued function over a simple surface.

If the vector field **F** is represented by vectors emanating from the surface and if these all lie on the *same* side (of the surface) as **n** then the integral of **F** over $(S, \mathbf{n})$ is non-negative. On the right-hand diagram in Fig. 12.2 we see that $\{\phi_u, \phi_v, \mathbf{n}\}$ follows the *right-hand rule* and note that **F** and **n** lie on the same side of $(S, \mathbf{n})$ if and only if the angle between them lies in $[-\pi/2, \pi/2]$.

Example 12.2 We evaluate $\iint_{(S,\mathbf{n})} \mathbf{F}$ where **S** is the sphere of radius r with centre at the origin oriented outwards and $\mathbf{F}(x, y, z) = (x, y, z)$. Although a sphere is not a simple surface we can treat it as one for integration theory (see our remarks in the previous chapter). Here, however, we do not need any parametrization. The unit normals at a point P on S are $\pm P/\|P\|$ and since S is oriented outwards

$$\mathbf{n}(P) = \frac{P}{\|P\|} = \frac{P}{r}.$$

Hence

$$\mathbf{F}(P)\cdot\mathbf{n}(P) = P\cdot\frac{P}{r} = \frac{\|P\|^2}{r} = \frac{r^2}{r} = r$$

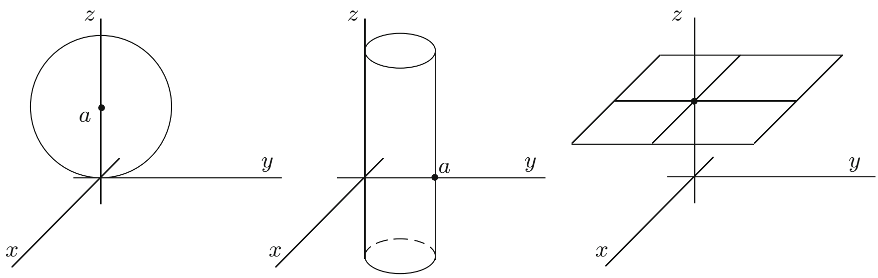


Fig. 12.4

and

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{S} r dA = r(\text{Surface Area of } S) = r \cdot 4\pi r^2 = 4\pi r^3.$$

Example 12.3 We compute

$$\iint_{\mathbf{S}} \frac{(x, y, z-a)}{\sqrt{2az - z^2}}$$

where **S** is the part of the surface $x^2 + y^2 + (z-a)^2 = a^2$ which lies inside the cylinder $x^2 + y^2 = ay$ and underneath the plane $z = a$ oriented with outward pointing normal.

We first sketch the surface over which we are integrating. Initially this appears as a rather formidable task but by sketching each part separately (Fig. 12.4) and then combining (Fig. 12.5) them it becomes relatively simple. The surface $x^2 + y^2 + (z-a)^2 = a^2$ is a sphere of radius a with centre $(0, 0, a)$. The equation $x^2 + y^2 = ay$ can be rewritten as

$$x^2 + y^2 - ay + \frac{a^2}{4} = \frac{a^2}{4}$$

i.e.

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

and this surface is the cylinder parallel to the z-axis above the circle with centre $(0, a/2)$ and radius $a/2$ in the xy-plane. The plane $z = a$ is parallel to the xy-plane and a units above it. The outward normal to the sphere of radius a with centre $(0, 0, a)$ at (x, y, z) is $(x, y, z-a)/a$. Hence

$$\iint_{\mathbf{S}} \frac{(x, y, z-a)}{\sqrt{2az - z^2}} = \iint_{S} \frac{(x, y, z-a)}{\sqrt{2az - z^2}} \cdot \frac{(x, y, z-a)}{a} dA$$

$$= \iint_{S} \frac{a}{\sqrt{2az - z^2}} dA.$$

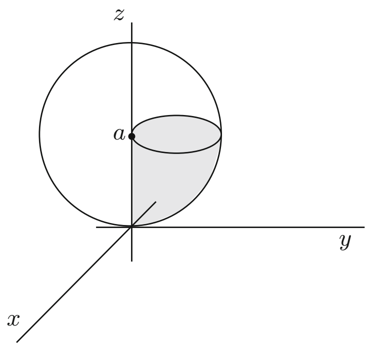


Fig. 12.5

From Fig. 12.5 we observe that our integral is over part of the lower right-hand quarter of the sphere. We use geographical coordinates about the point $(0, 0, a)$

$$F\colon (\theta, \psi) \longrightarrow (a\cos\theta\cos\psi, a\cos\theta\sin\psi, a + a\sin\theta).$$

Since θ refers to latitude and we are considering the lower portion we have $-\pi/2 < \theta < 0$. From the sketch $y > 0$ and hence $\sin\psi > 0$, i.e. $0 < \psi < \pi$, and since the surface lies inside the cylinder we also have $x^2 + y^2 < ay$, i.e. $a^2\cos^2\theta < a^2\cos\theta\sin\psi$. Hence $\cos\theta < \sin\psi = \cos(\psi - \frac{\pi}{2})$. By considering separately the cases $0 < \psi < \frac{\pi}{2}$ and $\frac{\pi}{2} < \psi < \pi$ we obtain $\frac{\pi}{2} + \theta < \psi < \frac{\pi}{2} - \theta$. Now

$$2az - z^2 = -a^2 + 2az - z^2 + a^2 = a^2 - (a-z)^2$$

and changing this into our new coordinates we get

$$\sqrt{2az - z^2} = (a^2 - a^2\sin^2\theta)^{1/2} = (a^2\cos^2\theta)^{1/2} = a\cos\theta.$$

From Example 10.5 we recall that

$$dA = \sqrt{EG - F^2}d\theta d\psi = a^2\cos\theta d\theta d\psi.$$

Hence

$$\iint_{\mathbf{S}} \frac{(x, y, z-a)}{\sqrt{2az - z^2}} = \int_{-\pi/2}^{0}\left\{\int_{\pi/2+\theta}^{\pi/2-\theta} \frac{a^2\cos\theta\cdot a}{a\cos\theta}d\psi\right\}d\theta = \pi^2a^2/4.$$

An alternative approach to vector-valued integration is possible by using oriented (coordinate) planes in $\mathbb{R}^3$. In oriented planes we can define *positive* and *negative area* and an anticlockwise sense of direction. The idea is to use consistent

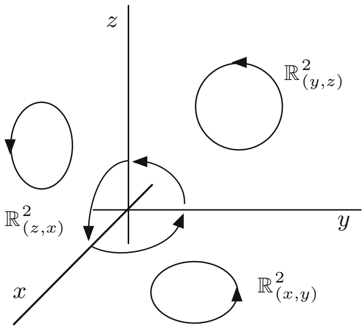


Fig. 12.6

parameterizations to transfer integration from an oriented surface in $\mathbb{R}^3$ to integration over subsets of the *coordinate planes* in $\mathbb{R}^3$, i.e. the xy-plane, the yz-plane and the xz-plane. Each of these planes is a surface in $\mathbb{R}^3$ and to define our integral we must assign a positive side to each of these planes for the same reason that we required positive intervals in $\mathbb{R}$.

We first define the *positive unit vectors* in the x, y and z directions in $\mathbb{R}^3$ as $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. These choices are natural in view of the way we sketch graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$. Consider now the xy-plane in $\mathbb{R}^3$ as the surface defined by $f^{-1}(0)$ where $f(x, y, z) = z$. The unit normals to this surface are $\pm(0, 0, 1)$ and we must choose between them to define a positive side to the xy-plane. We use the positive unit vectors in the x and y directions *in that order* and the *right-hand rule* or *cross product* to define the *positive side* of the xy-plane as that oriented by the normal

$$(1, 0, 0) \times (0, 1, 0) = (0, 0, 1).$$

We denote the positively oriented xy-plane in $\mathbb{R}^3$ by $\mathbb{R}^2_{(x,y)}$. Similarly the positive side of the yz-plane is defined by the normal $(0, 1, 0) \times (0, 0, 1) = (1, 0, 0)$ and denoted by $\mathbb{R}^2_{(y,z)}$ and the positive side of the zx-plane is defined by $(0, 0, 1) \times (1, 0, 0) = (0, 1, 0)$ and denoted by $\mathbb{R}^2_{(z,x)}$.

In Fig. 12.6 we see the direction of rotation (i.e. rotation to the *left*) used to measure angles in each of the coordinate planes and also the direction of a closed anticlockwise oriented curve.

If we look at each individual coordinate plane we get, using Fig. 12.6, the diagrams in Fig. 12.7 showing the anticlockwise directions.

In Chap. 9 we discussed Green's theorem

$$\oint_{\Gamma} Pdx + Qdy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{12.1}$$

where the boundary of Ω, Γ is oriented in an *anticlockwise* direction. From (12.1) it does not appear that Ω has been assigned any orientation but, in fact, if we identify

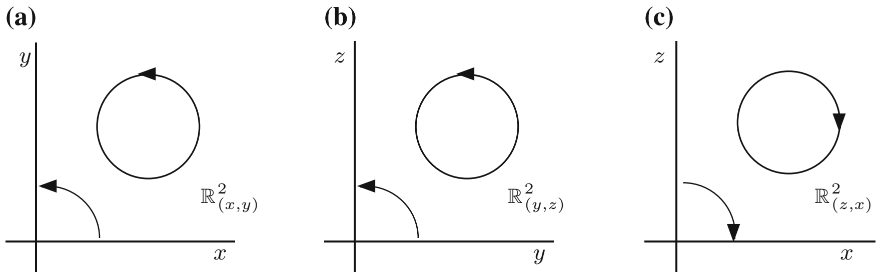


Fig. 12.7

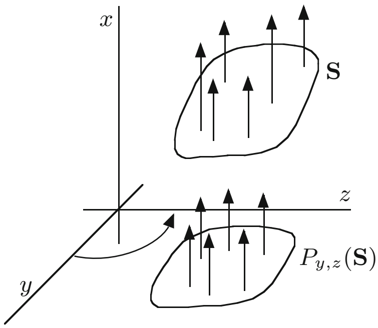


Fig. 12.8

$\mathbb{R}^2$ with $\mathbb{R}^2_{(x,y)} \subset \mathbb{R}^3$ then the integration on $\mathbb{R}^2$ is really over the *positive side* of $\mathbb{R}^2_{(x,y)}$. This is the matching up of orientations in Green's theorem.

Let $\mathbf{F} = (f, g, h)$ denote a continuous vector field on the simple oriented surface $\mathbf{S}$. Let

$$\phi\colon (u, v) \in U \longrightarrow \big(x(u, v), y(u, v), z(u, v)\big)$$

denote a parametrization consistent with the orientation. We now take an independent approach to defining separately $\iint_{\mathbf{S}} (f, 0, 0)$, $\iint_{\mathbf{S}} (0, g, 0)$ and $\iint_{\mathbf{S}} (0, 0, h)$.

First observe that $(f, 0, 0)$ is parallel to the x-axis and may be represented as in Fig. 12.8, where we have rotated the axes but preserved the orientation.

Since f points in the direction of the normal $(1, 0, 0)$ to $\mathbb{R}^2_{(y,z)}$ it is natural to project onto $\mathbb{R}^2_{(y,z)}$ and to define $\iint_{\mathbf{S}} (f, 0, 0)$ as $\iint_{P_{y,z}(\mathbf{S}) \subset \mathbb{R}^2_{(y,z)}} f$ where $P_{y,z}$ is the projection $(x, y, z) \rightarrow (y, z)$ and $P_{y,z}(\mathbf{S})$ is given the induced orientation from $\mathbb{R}^2_{(y,z)}$. Although the set $P_{y,z}(\mathbf{S})$ may not be open and the mapping

$$P_{y,z}\colon (u, v) \longrightarrow \big(y(u, v), z(u, v)\big)$$

is not necessarily a parametrization of $P_{y,z}(\mathbf{S})$ we *may* proceed as if they were. Partition U into rectangles, $(R_{ij})_{ij}$. We consider the "boundary" of $P_{y,z}(R_{ij})$ as a directed curve in $\mathbb{R}^2_{(y,z)}$ "parameterized" by $P_{y,z}$ restricted to the boundary of the anticlockwise oriented rectangle R_{ij} in the uv-plane. Hence $P_{y,z}(R_{ij})$ will have positive (or at least non-negative) area if it is oriented in an anticlockwise fashion in $\mathbb{R}^2_{(y,z)}$. This gives us the approximation

$$\text{Area}\big(P_{y,z}(R_{ij})\big) \text{ in } \mathbb{R}^2_{(y,z)} \approx \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}(u_i, v_j)\Delta u_i \Delta v_j.$$

Next we transfer the partition of U to $P_{y,z}(\mathbf{S})$ by using $P_{y,z}$ and ϕ and form the Riemann sum of f with respect to this partition. This analysis shows that the Riemann sum

$$\sum_i \sum_j f\big(\phi(u_i, v_j)\big) \times \Big[\text{ Area }\big(P_{y,z}(R_{ij})\big) \text{ in } \mathbb{R}^2_{(y,z)}\Big]$$

is approximately equal to

$$\sum_i \sum_j f\big(\phi(u_i, v_j)\big) \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}(u_i, v_j)\Delta u_i \Delta v_j$$

and on taking a limit we get

$$\iint_{\mathbf{S}} (f, 0, 0) = \iint_{U} f\big(\phi(u, v)\big) \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} dudv.$$

We proceed in the same way with $(0, g, 0)$ and obtain the diagram shown in Fig. 12.9. This implies that z precedes x (see also Fig. 12.7c) and so the correctly oriented "parametrization" in this case is

$$(u, v) \longrightarrow \big(z(u, v), x(u, v)\big)$$

and we obtain

$$\iint_{\mathbf{S}} (0, g, 0) = \iint_{U} g\big(\phi(u, v)\big) \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} dudv.$$

Similarly we are led to

$$\iint_{\mathbf{S}} (0, 0, h) = \iint_{U} h\big(\phi(u, v)\big) \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} dudv.$$

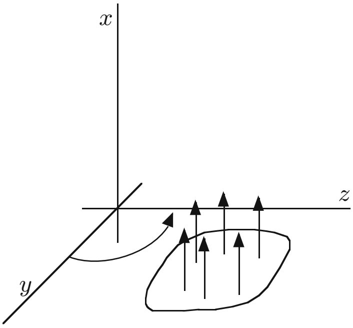


Fig. 12.9

Although each of these integrals looks rather complicated when we add them together we obtain a familiar expression. Since

$$\phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$$

we have $\phi_u = (x_u, y_u, z_u)$ and $\phi_v = (x_v, y_v, z_v)$. Hence

$$\phi_u \times \phi_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, -\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right)$$

$$\mathbf{F} \cdot \phi_u \times \phi_v = f \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} - g \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + h \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \det \begin{pmatrix} \mathbf{F} \\ \phi_u \\ \phi_v \end{pmatrix}$$

and

$$\iint\limits_{\mathbf{S}} (f, 0, 0) + \iint\limits_{\mathbf{S}} (0, g, 0) + \iint\limits_{\mathbf{S}} (0, 0, h)$$

$$= \iint\limits_{U} \left(f \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} + g \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} + h \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right) dudv \tag{12.2}$$

$$= \iint\limits_{U} \mathbf{F}\big(\phi(u, v)\big) \cdot \phi_u \times \phi_v(u, v) dudv \tag{12.3}$$

$$= \iint\limits_{U} \det \begin{pmatrix} \mathbf{F} \\ \phi_u \\ \phi_v \end{pmatrix} dudv. \tag{12.4}$$

Formula (12.3) is our original definition of the integral of **F** over **S** and thus (12.2) and (12.4) are new expressions for $\iint_{\mathbf{S}} \mathbf{F}$. We have also shown, provided we use the correct orientations on $P_{y,z}(\mathbf{S})$, $P_{z,x}(\mathbf{S})$ and $P_{x,y}(\mathbf{S})$, that

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{P_{y,z}(\mathbf{S})} f + \iint_{P_{z,x}(\mathbf{S})} g + \iint_{P_{x,y}(\mathbf{S})} h.$$

In certain cases these projections map **S** onto geometrically nice domains in the coordinate planes and this may lead to simpler calculations -see for instance our next example, Example 13.2 and Exercises 12.1, 12.2 and 12.8.

From (12.3) and the results of the previous chapter we also have

$$\begin{aligned}\iint_{\mathbf{S}} \mathbf{F} &= \iint_{U} (\mathbf{F}\cdot\mathbf{n})\|\phi_u \times \phi_v\| du dv \\ &= \iint_{U} (\mathbf{F}\cdot\mathbf{n})\sqrt{EG - F^2} du dv. \end{aligned} \tag{12.5}$$

Using the notation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}, \quad \text{etc.}$$

we obtain yet another formula for the integral:

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{U} (\mathbf{F}\cdot\mathbf{n}) \left[\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 \right]^{1/2} du dv. \tag{12.6}$$

In view of formula (12.5) the notation $\iint_{S} \langle \mathbf{F}, \mathbf{n}\rangle dA$ is also used in place of $\iint_{(S,\mathbf{n})} \mathbf{F}$ where *dA* denotes surface area.

We have established five formulae for calculating surface integrals. Furthermore, our excursion into oriented planes has led to geometric insights on the construction of the integral and to a method of evaluating integrals by projecting onto the coordinate planes. There are still a number of topics to be sorted out, e.g. independence of the parametrization, and we discuss these in examples as we proceed.

Example 12.4 In this simple example we use projections to calculate the area of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The fact that we can easily find the answer independently allows us to check our solution.

From Fig. 12.10, $P = (1/2, 1/2, 0)$ and

$$\|(1/2, 1/2, 0) - (0, 0, 1)\| = \left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2} = \sqrt{\frac{3}{2}}.$$

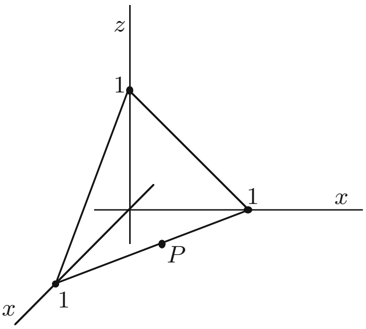


Fig. 12.10

Since $\|(1,0,0)-(0,1,0)\| = \sqrt{2}$ the (surface) area is $1/2 \cdot \sqrt{2} \cdot \sqrt{3/2} = \sqrt{3}/2$. We now calculate the area using $\iint\limits_{(S,\mathbf{n})} \mathbf{n}$. Since the triangle is part of a *plane* the normal $\mathbf{n}$ is constant on $\mathbf{S}$. By symmetry it is easily seen that the triangle lies in the plane $x+y+z=1$ and hence the unit normals are $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Let us take $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ as our unit normal. By symmetry we only have to calculate $\iint\limits_{(S,\mathbf{n})} \big(1/\sqrt{3}, 0, 0\big)$. Clearly $P_{y,z}$ projects onto the anticlockwise oriented triangle $A_1 \subset \mathbb{R}^2_{(y,z)}$ with vertices $(0,0)$, $(0,1)$ and $(1,0)$ and

$$\iint\limits_{(S,\mathbf{n})} \big(\frac{1}{\sqrt{3}}, 0, 0\big) = \frac{1}{\sqrt{3}} \text{ Area } (A_1) = \frac{1}{\sqrt{3}} \cdot \frac{1}{2}.$$

Hence

$$\iint\limits_{(S,\mathbf{n})} \mathbf{n} = 3 \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

and this agrees with our earlier result.

Exercises

12.1 Find $\iint\limits_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x,y,z) = (1,2,3)$ and $\mathbf{S}$ is the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ oriented so that the origin is on the negative side.

12.2 By using projections and the portion of the sphere that lies in the first octant calculate the area of a sphere of radius r.

12.3 Express as an integral over a region in $\mathbb{R}^2$ the integral $\iint\limits_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x,y,z) = \big(-2/y^3, -6xy^2, 2z^3/x^3\big)$, $\mathbf{S}$ is the graph of $f(u,v) = uv^3$, $(u,v) \in [1,2] \times [1,2]$ and the parametrization $\phi(u,v) = (u,v,uv^3)$ is consistent with the orientation. Evaluate the integral.

12.4 Let $0 < a < b$ and let Γ denote the circle of centre $(b, 0)$ and radius a in the xz-plane. Let S denote the surface obtained by rotating Γ about the z-axis. If S is oriented with outward normal and

$$\mathbf{F}(x, y, z) = \left(x - \frac{bx}{\sqrt{x^2 + y^2}}, y - \frac{by}{\sqrt{x^2 + y^2}}, z\right)$$

show that

$$\iint_{\mathbf{S}} \mathbf{F} = 4\pi^2 a^2 b.$$

12.5 Evaluate

(a) $\displaystyle\iint_{\mathbf{S}} y^2 + z^2$

(b) $\displaystyle\iint_{\mathbf{S}} \frac{1}{\left(x^2 + y^2 + (z + a)^2\right)^{1/2}}$

where $\mathbf{S}$ is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane oriented by the outward normal.

12.6 Describe the surface $S = \{(x, y, z) : z^2 = x^2 + y^2,\ 1 \le z \le 3\}$. If $\mathbf{n}$ is the outward pointing normal to S find

$$\iint_{(S,\mathbf{n})} \frac{(-xz, -yz, x^2 + y^2)}{x^2 + y^2}.$$

12.7 Let S denote the portion of the level set $z = \tan^{-1}(y/x)$ which lies between the planes $z = 0$ and $z = 2\pi$, inside the cone $x^2 + y^2 = z^2$ and outside the cylinder $x^2 + y^2 = \pi^2$. Let

$$\mathbf{G}(x, y, z) = \frac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that

$$F(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad \pi < r < \theta,\ \pi < \theta < 2\pi$$

is a parametrization of S. If $\mathbf{n} = \dfrac{F_r \times F_\theta}{\|F_r \times F_\theta\|}$ evaluate $\displaystyle\iint_{(S,\mathbf{n})} \mathbf{G}$.

12.8 Evaluate $\displaystyle\iint_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x, y, z) = (y^2 + z^2, \tan^{-1}(x/z), z\exp(x^2 + y^2))$ and $\mathbf{S}$ is the part of the sphere of radius a centered at the origin in the first octant oriented outwards by (a) using a parametrization and (b) projecting onto the coordinate planes.

Chapter 13
Stokes' Theorem

Summary *We discuss Stokes' theorem for oriented surfaces in* $\mathbb{R}^3$.

Stokes' theorem, the *fundamental theorem of calculus for surfaces*, generalises Green's theorem to oriented surfaces $\mathbf{S} = (S, \mathbf{n})$ with *edge* or *boundary* Γ (the term edge avoids confusion with our other use of the word boundary) consisting of a finite number of piecewise smooth directed curves. We suppose that the *positive side* of $\mathbf{S}$ lies on the *left-hand side* as we move along Γ in the *positive direction*. In practice this consistency between the orientations of the surface and its edge may be verified by sketching. In certain cases the normal $\mathbf{n}$ admits a *continuous extension* to the boundary and a parametrization P of the boundary has the correct orientation if at one point, say $P(t_0)$, we have $P'(t_0) \cdot \mathbf{n}\big(P(t_0)\big) > 0$ where $\mathbf{n}\big(P(t_0)\big)$ is the value of the extension of $\mathbf{n}$ at $P(t_0)$. If a consistent parametrization of the surface extends to give a parametrization of the boundary then the boundary is also correctly directed (see Example 13.5).

Theorem 13.1 (Stokes' Theorem) *Let* $\mathbf{S} = (S, \mathbf{n})$ *denote an oriented surface in* $\mathbb{R}^3$ *with boundary* Γ *consisting of a finite number of piecewise smooth directed curves. We suppose that the positive side of* $\mathbf{S}$ *lies on the left of the positive side of* Γ. *If* $\mathbf{F}$ *is a smooth vector field on* $S \cup \Gamma$ *then*

$$\int_\Gamma \mathbf{F} = \iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F}) \tag{13.1}$$

i.e.

$$\int_\Gamma \langle \mathbf{F}, T\rangle ds = \iint_S \langle \mathrm{curl}(\mathbf{F}), \mathbf{n}\rangle dA$$

where T *is the unit tangent to the directed curves* Γ.

The proof, which we omit, is obtained by applying Green's theorem to the projections onto the coordinate planes. In Chaps. 6 and 12 we developed techniques to evaluate the left- and right-hand sides of (13.1), respectively. Thus the only new

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_13,

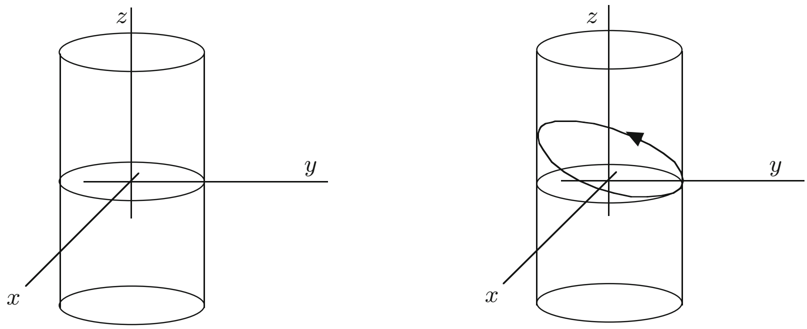


Fig. 13.1

factor in Stokes' theorem is the correlation between the orientations of the surface and its boundary.

Example 13.2 We use Stokes' theorem to evaluate the line integral

$$\int_C -y^3 dx + x^3 dy - z^3 dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 3$ and the plane $x + y + z = 1$ and the orientation on C is anticlockwise when viewed from a point sufficiently high up on the z-axis. Let $\mathbf{S}$ denote the portion of the plane inside the cylinder oriented so that the normal lies above the surface (Fig. 13.1).

Let $\mathbf{F} = (-y^3, x^3, -z^3)$ then

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2 + 3y^2).$$

By Stokes' theorem

$$\int_C \mathbf{F} = \iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F}) = \iint_{\mathbf{S}} (0, 0, 3x^2 + 3y^2).$$

Since only the final coordinate of curl($\mathbf{F}$) is non-zero our analysis in the previous chapter implies that we only need consider the projection of $\mathbf{S}$ onto $\mathbb{R}^2_{(x,y)}$. As C projects onto an anticlockwise oriented curve Γ_1 (Fig. 13.1) our projection is onto the positive side of $\mathbb{R}^2_{(x,y)}$. Hence

$$\int_C \mathbf{F} = \iint_{x^2+y^2 \leq 3} (3x^2 + 3y^2) dx dy.$$

We parametrize the surface $x^2 + y^2 < 3$ in $\mathbb{R}^3$ by

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, 0), \quad 0 < r < \sqrt{3},\ 0 < \theta < 2\pi.$$

We have $\phi_r = (\cos\theta, \sin\theta, 0)$ and $\phi_\theta = (-r\sin\theta, r\cos\theta, 0)$. Hence

$$\phi_r \times \phi_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r).$$

Since

$$\frac{\phi_r \times \phi_\theta}{\|\phi_r \times \phi_\theta\|} = (0, 0, 1)$$

our parametrization is consistent with the positive orientation of $\mathbb{R}^2_{(x,y)}$. We have

$$\int_C \mathbf{F} = \int_0^{\sqrt{3}} \int_0^{2\pi} 3r^2 \cdot \|\phi_r \times \phi_\theta\| dr d\theta = 6\pi \int_0^{\sqrt{3}} r^3 dr = \frac{27\pi}{2}.$$

Moral A reasonable sketch is not just optional but necessary. The form of $\mathbf{F}$, i.e. the fact that the first two coordinates were zero, combined with information on how surface integrals can be projected onto the coordinate planes greatly simplified the calculations required.

Example 13.3 We evaluate $\iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F})$ where

$$\mathbf{F}(x, y, z) = (y^2 \cos xz, x^3 e^{yz}, -e^{-xyz})$$

and $\mathbf{S}$ is the portion of the sphere $x^2 + y^2 + (z-2)^2 = 8$ which lies above the xy-plane oriented outwards. The edge or boundary of $\mathbf{S}$, Γ, is where the sphere cuts the xy-plane, i.e. where $z = 0$. We have $x^2 + y^2 + 4 = 8$, i.e. $x^2 + y^2 = 4$ (Fig. 13.2).

Since the positive side of the sphere is the outside we see from Fig. 13.2 that the surface $\mathbf{S}$ is on the left as we move along Γ in an anticlockwise direction in the xy-plane. This gives us our direction along Γ. By Stokes' theorem

$$\int_\Gamma \mathbf{F} = \iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F}).$$

But Γ with anticlockwise direction is also the boundary or edge of the surface

$$\mathbf{S}_1 = \{(x, y, z) : x^2 + y^2 < 4, z = 0\}$$

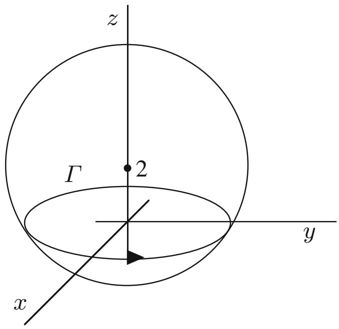


Fig. 13.2

oriented by the normal $(0, 0, 1)$. Hence a further application of Stokes' theorem implies

$$\int_\Gamma \mathbf{F} = \iint_{\mathbf{S}_1} \text{curl}(\mathbf{F}).$$

Now

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos xz & x^3 e^{yz} & -e^{-xyz} \end{vmatrix}.$$

Since, however, $\mathbf{S}_1$ projects onto smooth curves, which have zero surface area in $\mathbb{R}^2_{(y,z)}$ and $\mathbb{R}^2_{(x,z)}$, it suffices to consider the final coordinate of curl $(\mathbf{F})$. This is

$$\frac{\partial}{\partial x}\left(x^3 e^{yz}\right) - \frac{\partial}{\partial y}\left(y^2 \cos xz\right) = 3x^2 e^{yz} - 2y \cos xz$$

and, since $z = 0$ on $\mathbf{S}_1$, we need only evaluate

$$\iint_{x^2+y^2 \leq 4} (3x^2 - 2y)dxdy.$$

By symmetry

$$\iint_{x^2+y^2 \leq 4} (-2y)dxdy = 0.$$

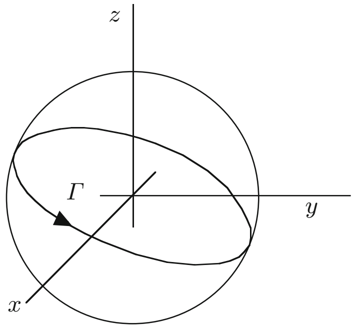


Fig. 13.3

If we use the parametrization

$$(r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, 0), \quad 0 < r < 2,\ 0 < \theta < 2\pi$$

then

$$\iint_{\mathbf{S}_1} \mathbf{F} = \int_0^2 \int_0^{2\pi} 3r^2\cos^2\theta r dr d\theta = \int_0^{2\pi} \cos^2\theta d\theta \int_0^2 3r^3 dr = 12\pi.$$

Moral A closed curve may be the edge or boundary of more than one surface and a suitable choice (of surface) may simplify calculations. Projections and symmetry are helpful.

Example 13.4 We wish to use Stokes' theorem to find a suitable orientation of the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$, Γ, so that

$$\int_\Gamma y dx + z dy + x dz = \sqrt{3}\pi a^2.$$

The curve Γ is the intersection of a sphere and a plane through the centre of the sphere and hence is a "great circle" or "equator" on a sphere (see Fig. 13.3).

The curve Γ is the edge or boundary of the two hemispheres on either side of it and also of a portion of the plane $x + y + z = 0$. Which we use will depend on the function being integrated. Let $\mathbf{F}(x, y, z) = (y, z, x)$. We have

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1).$$

The plane $x + y + z = 0$ has unit normals $\pm(1, 1, 1)/\sqrt{3} = \mathbf{n}_1(x, y, z)$ while the sphere has unit normals $\pm(x, y, z)/a = \mathbf{n}_2(x, y, z)$. By Stokes' theorem

$$\int_\Gamma ydx + zdy + xdz = \iint_{(S,\mathbf{n})} (-1, -1, -1)$$

where S and $\mathbf{n}$ have to be chosen and Γ directed. If we take $\mathbf{S}$ as part of the plane with normal $(1, 1, 1)/\sqrt{3}$ then

$$\iint_{\mathbf{S}} (-1, -1, -1) = \iint_S (-1, -1, -1) \cdot \frac{(1, 1, 1)}{\sqrt{3}} dA = -\frac{3}{\sqrt{3}} \iint_S dA$$
$$= -\sqrt{3}\pi a^2$$

since S is a disc of radius a.

Since we obtained a negative answer we have been using the incorrect orientation on S and so take $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ as the normal which describes the orientation. Hence Γ is oriented as in Fig. 13.3, i.e. it appears clockwise when looked at from, say, the point $(1, 1, 1)$ or from any point sufficiently far out in the first octant.

Moral The curl of a vector field is a form of derivative. If the entries are linear, as in this example, the curl is constant. If the entries are of degree 2 then the curl has linear entries.

Example 13.5 In this example we verify Stokes' theorem for the portion S of the surface $z = \tan^{-1}(y/x)$ which lies inside the cone $x^2 + y^2 = z^2$ and between the planes $z = 0$ and $z = 2\pi$ by using the vector field

$$\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} - (x^2 + y^2)\mathbf{k}.$$

We first examine the surface $z = \tan^{-1}(y/x)$. This can be parametrized as a graph using Cartesian coordinates (see Example 10.3) but it is preferable to use polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$ then

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

and we obtain the parametrization

$$(r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \theta) \tag{13.2}$$

where $r > 0$ and $0 < \theta < 2\pi$.

What sort of surface is this? Well, if we fix different values of r and let θ vary we obtain a *helix* (see Example 5.2). In Fig. 13.4 we have sketched the surface for $1/2 \leq r \leq 1$. We are considering the portion of the surface in Fig. 13.4, extended

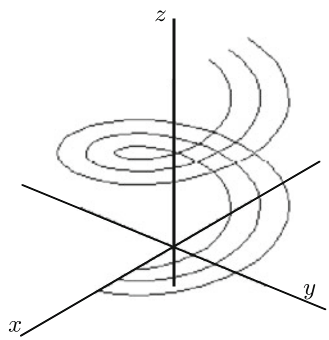


Fig. 13.4

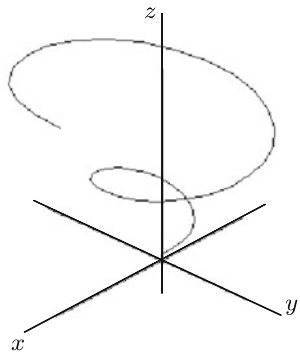


Fig. 13.5

over all r, which lies inside the cone and between the two planes. It is thus a *screw-shaped* surface or *spiral staircase* with the spirals or steps getting wider as we rise. The fact that the surface lies between the planes $z = 0$ and $z = \pi$ means that we have just one full twist of the screw or one turn of the staircase (Fig. 13.5).

We can use the parametrization (13.2) but the restriction caused by lying inside the other surfaces means that we must restrict the range. Translating the boundaries into polar coordinates, we get

$$x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2 = z^2$$

and since $0 \le z \le 2\pi$ this implies $r = z$, $0 \le r \le 2\pi$. Hence our parametrization of the surface is

$$f : (r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \theta), \quad 0 < r < \theta,\ 0 < \theta < 2\pi.$$

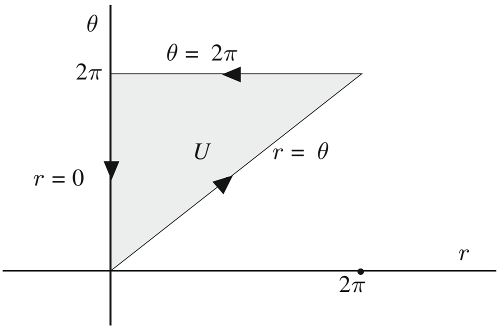


Fig. 13.6

We have $\mathbf{F}(f(r,\theta)) = (r\theta\cos\theta, r\theta\sin\theta, -r^2)$ and

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \text{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & yz & -(x^2+y^2) \end{vmatrix} = -3y\mathbf{i} + 3x\mathbf{j} = (-3r\sin\theta, 3r\cos\theta, 0).$$

We define our orientation on **S** by

$$f_r \times f_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r).$$

Hence

$$\begin{aligned} \iint_{\mathbf{S}} \text{curl}(\mathbf{F}) &= \int_{\substack{0 \\ (\theta)}}^{2\pi} \int_{\substack{0 \\ (r)}}^{\theta} \langle \text{curl}(\mathbf{F}), f_r \times f_\theta \rangle dr d\theta \\ &= \int_0^{2\pi} \Big(\int_0^{\theta} -3r dr \Big) d\theta = \int_0^{2\pi} -\frac{3r^2}{2} \Big|_0^{\theta} d\theta \\ &= -\frac{3}{2} \int_0^{2\pi} \theta^2 d\theta = -\frac{3}{2}\frac{\theta^3}{3} \Big|_0^{2\pi} \\ &= -\frac{(2\pi)^3}{2} = -4\pi^3. \end{aligned}$$

The parametrization of S is over the set U in $\mathbb{R}^2$ given in Fig. 13.6 and the boundary or edge of the surface can be found by examining

$$f(r,\theta) = (r\cos\theta, r\sin\theta, \theta) \tag{13.3}$$

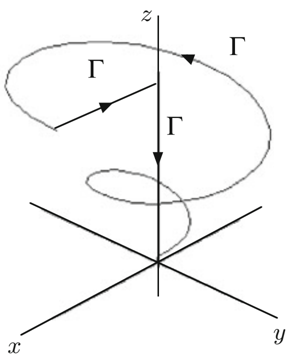


Fig. 13.7

on the boundary of U. We can also look at it geometrically by examining where the boundary curves intersect S. We first look at the curve of intersection of the cone and the screw. The cone is parametrized (Examples 10.4 and 10.5) by

$$(r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \theta)$$

and comparing this with (13.3) we get a curve of intersection Γ_1 when $r = \theta$. This curve is parametrized by

$$\theta \longrightarrow (\theta\cos\theta, \theta\sin\theta, \theta), \quad 0 \le \theta \le 2\pi.$$

The curve Γ_1 joins the origin to $P_1 = (2\pi, 0, 2\pi)$. The second curve Γ_2 is obtained by putting $\theta = 2\pi$ and from (13.3) this is parametrized by

$$r \longrightarrow (r\cos 2\pi, r\sin 2\pi, 2\pi) = (r, 0, 2\pi)$$

where $0 < r < 2\pi$. This is the straight line joining P_1 to $(0, 0, 2\pi)$. From Fig. 13.6 it runs in the *negative direction* and so we must reverse the orientation.

The third curve Γ_3 is obtained by letting $r = 0$ in (13.3) and we have a parametrization

$$\theta \longrightarrow (0, 0, \theta), \quad 0 \le \theta \le 2\pi.$$

Γ_3 joins $(0, 0, 2\pi)$ to the origin and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a closed piecewise smooth directed curve (Fig. 13.7).

We have oriented the surface and directed its boundary. Are these consistent in order to apply Stokes' theorem? Yes, because the parametrization f of Γ is obtained from a *continuous extension* of a consistent parametrization of the surface. You have already seen two other ways in which it is possible to check this consistency. We now have to evaluate

$$\int_{\Gamma} \mathbf{F} = \int_{\Gamma_1} \mathbf{F} + \int_{\Gamma_2} \mathbf{F} + \int_{\Gamma_3} \mathbf{F}.$$

We have

$$\begin{aligned}
\int_{\Gamma_1} \mathbf{F} &= \int_0^{2\pi} \Big(\frac{d}{d\theta}(\theta\cos\theta, \theta\sin\theta, \theta)\Big)\cdot(\theta^2\cos\theta, \theta^2\sin\theta, -\theta^2)\,d\theta \\
&= \int_0^{2\pi} (\theta^2\cos^2\theta - \theta^3\sin\theta\cos\theta + \theta^2\sin^2\theta + \theta^3\sin\theta\cos\theta - \theta^2)\,d\theta \\
&= \int_0^{2\pi} 0\,d\theta = 0 \\
\int_{\Gamma_2} \mathbf{F} &= \int_{2\pi}^0 \Big(\frac{d}{dr}(r, 0, 2\pi)\Big)\cdot(2\pi r, 0, -r^2)dr \\
&= \int_{2\pi}^0 2\pi r\,dr = \frac{2\pi r^2}{2}\Big|_{2\pi}^0 = -\pi(2\pi)^2 = -4\pi^3 \\
\int_{\Gamma_3} \mathbf{F} &= \int_{2\pi}^0 \Big(\frac{d}{d\theta}(0, 0, \theta)\Big)\cdot(0, 0, 0)d\theta = 0.
\end{aligned}$$

Hence

$$\int_\Gamma \mathbf{F} = -4\pi^3 = \iint_{\mathbf{S}} \text{curl}(\mathbf{F})$$

and we have verified Stokes' theorem.

Moral It is possible to verify Stokes' theorem.

Exercises

13.1 Let **S** denote the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ in the first octant which lies inside the cylinder $x^2 + y^2 = 2ax$ oriented outwards. Let Γ denote the boundary or edge of **S** directed in accordance with Stokes' theorem. Sketch **S** and Γ. Using Stokes' theorem evaluate

(a) $\displaystyle\int_\Gamma zdx - xdz,$

(b) $\displaystyle\int_\Gamma xdy - ydx,$

(c) $\displaystyle\int_\Gamma ydz - zdy.$

13.2 Sketch the surfaces $az = xy$ and $x^2 + y^2 = b^2$ in $\mathbb{R}^3$. Show that

$$\theta \longrightarrow (b\cos\theta, b\sin\theta, \frac{b^2\sin 2\theta}{2a}), \quad 0 \le \theta \le 2\pi$$

is a parametrization of the intersection Γ of the two surfaces oriented clockwise when viewed from a high point on the z-axis. Show that the surface parametrized by

$$P: (r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \frac{r^2\sin 2\theta}{2a}), \quad 0 < r < b,\ 0 < \theta < 2\pi$$

has Γ as its edge or boundary. Using Stokes' theorem find

$$\int_\Gamma ydx + zdy + xdz.$$

13.3 Let Γ denote the curve of intersection of $x + y = 2b$ and $x^2 + y^2 + z^2 = 2b(x + y)$ oriented in a clockwise sense when viewed from the origin. Sketch the appropriate diagram. Using Stokes' theorem evaluate

$$\int_\Gamma ydx + zdy + xdz.$$

13.4 Let $0 < a < b$ and let S denote the torus obtained by rotating the circle $(x - b)^2 + z^2 = a^2$ in the xz-plane about the z-axis. Sketch S. Let Γ denote the directed curve parametrized by

$$t \longrightarrow \big((b + a\cos nt)\cos t, (b + a\cos nt)\sin t, a\sin nt\big).$$

Show that Γ is a closed curve in S. Describe and sketch S. Let $\mathbf{S}$ denote the surface parametrized and oriented by

$$P: (r, t) \longrightarrow \big(r(b + a\cos nt)\cos t, r(b + a\cos nt)\sin t, a\sin nt\big)$$

where $0 \leq r < 1,\ 0 < t < 2\pi$. Show that Γ is the boundary of $\mathbf{S}$. Let $\mathbf{F}(x, y, z) = (-y, x, 0)$. By computing both $\int_\Gamma \mathbf{F}$ and $\int_{\mathbf{S}} \mathrm{curl}(\mathbf{F})$ verify Stokes' theorem. Show that the area of the projection of $\mathbf{S}$ onto $\mathbb{R}^2_{(x,y)}$ is $\pi(a^2+2b^2)/2$.

13.5 Let $\mathbf{S}$ denote the unit sphere oriented outwards. For $0 < b < c < 1$ let $\mathbf{S}_{b,c}$ denote the part of the sphere between the planes $z = b$ and $z = c$. Let $\mathbf{F}(X) = X\|X\|^{-3}$ for all $X \neq 0$ in $\mathbb{R}^3$. By using the result in Exercise 6.3 and Stokes Theorem find $\iint_{\mathbf{S}_{b,c}} \mathbf{F}$.

13.6 Let Γ denote the curve of intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, $a > 0$, $b > 0$. Use Stokes' theorem to find a direction along Γ so that $\int_\Gamma (y - z, z - x, x - y)$ is positive. Find the value of this positive number.

13.7 Use Stokes' theorem to find a suitable orientation of the curve of intersection Γ of the hemisphere $x^2 + y^2 + z^2 = 2ax$, $z > 0$, and the cylinder $x^2 + y^2 = 2bx$, $0 < b < a$, so that

$$\int_\Gamma (y^2 + z^2)dx + (x^2 + z^2)dy + (x^2 + y^2)dz = 2\pi ab^2.$$

Chapter 14
Triple Integrals

Summary *We define triple integrals of scalar-valued functions over open subsets of* $\mathbb{R}^3$, *discuss coordinate systems in* $\mathbb{R}^3$, *justify a change of variable formula and use Fubini's theorem to evaluate integrals.*

Let f be a real-valued function defined on an open subset U of $\mathbb{R}^3$. By using partitions of the coordinate axes to draw planes parallel to the coordinate planes (Fig. 14.1) we obtain a *grid* which partitions $\mathbb{R}^3$ into cubes. Let $(\overline{x}_i, \overline{y}_j, \overline{z}_k)$ denote a typical point in the cube $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$. The Riemann sum of f with respect to this grid

$$\sum_i \sum_j \sum_k f(\overline{x}_i, \overline{y}_j, \overline{z}_k)(x_{i+1} - x_i)\,(y_{j+1} - y_j)\,(z_{k+1} - z_k)$$

is formed by summing over all cubes that lie in U. If the Riemann sums converge, as we take finer and finer partitions and grids, to a limit then f is said to be *Riemann integrable* and the limit

$$\iiint_U f(x, y, z)\,dxdydz$$

is called the (Riemann) integral of f over U.

If U is a bounded open subset of $\mathbb{R}^3$ with smooth boundary and f is the restriction to U of a continuous function $\overline{f}$ on $\overline{U}$ then f is integrable over U. This result, proved using *uniform continuity* of $\overline{f}$ on the *compact* subset $\overline{U}$ of $\mathbb{R}^3$, implies the existence of an abundance of integrable functions.

If $f(x, y, z) \equiv 1$ the Riemann sum is the volume of all cubes inside U and in the limit equals the volume of U, Vol(U). Thus

$$\text{Vol}(U) = \iiint_U dxdydz.$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_14,

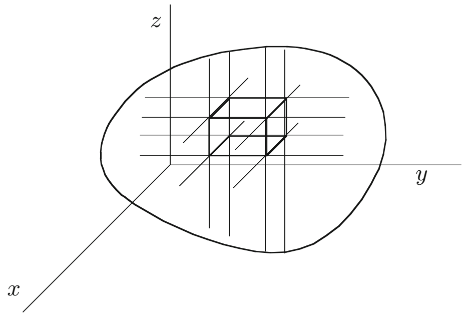


Fig. 14.1

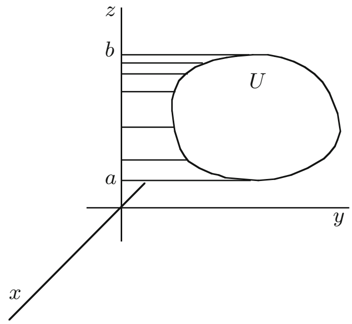


Fig. 14.2

In evaluating triple integrals we use an extension of *Fubini's theorem*. This is obtained from the Riemann sum by first summing over i, taking a limit, then summing over j and taking a limit and finally summing over k and taking a limit. To justify this process it is usual to assume that the domain of integration has a "box-like" appearance, i.e. it is bounded above and below by surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, front and rear by surfaces $x = g_1(y, z)$ and $x = g_2(y, z)$ and on the left and right by surfaces $y = h_1(x, z)$ and $y = h_2(x, z)$. The situations we discuss are of this type but by no means reflect the full range of examples to which Fubini's theorem applies. Many open sets can be partitioned into a finite union of sets and Fubini's theorem applies to each of these. We refer to our remarks on Green's theorem in Chap. 9 for further details.

To apply Fubini's theorem we must examine various cross-sections of the domain of integration U. First suppose that the set of *non-empty cross-sections* of U parallel to the xy-plane, i.e. those obtained by fixing the z-coordinate, determine an interval (a, b) on the z-axis (Fig. 14.2).

Let $A(z)$ denote the cross-section defined by fixing z in (a, b), i.e.

$$A(z) = \{(x, y) \in \mathbb{R}^2_{(x,y)} : (x, y, z) \in U\}.$$

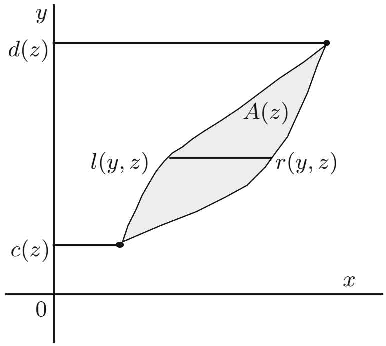


Fig. 14.3

This means

$$\iiint_U f(x, y, z)\, dxdydz = \int_a^b \left\{ \iint_{A(z)} f(x, y, z)\, dxdy \right\} dz$$

and we now have to evaluate the inner integral of *two* variables. In some cases it is possible to do this directly, see for instance Example 14.1, but usually we apply the two-variable Fubini's theorem to each $A(z)$. For this we assume that the region $A(z) \subset \mathbb{R}^2_{(x,y)}$ is bounded on the left and right by the graphs of functions of y over some interval (Fig. 14.3). Both the functions and the interval will depend on z and we denote the interval by $\big(c(z), d(z)\big)$ and the functions on the left and right by $y \to l(y, z)$ and $y \to r(y, z)$ respectively.

This implies

$$\iint_{A(z)} f(x, y, z)\, dxdy = \int_{c(z)}^{d(z)} \left\{ \int_{l(y,z)}^{r(y,z)} f(x, y, z) dx \right\} dy$$

and

$$\iiint_U f(x, y, z)\, dxdydz = \int_a^b \left\{ \int_{c(z)}^{d(z)} \left\{ \int_{l(y,z)}^{r(y,z)} f(x, y, z)\, dx \right\} dy \right\} dz.$$

Thus to evaluate triple integrals it is necessary to identify, by sketching, cross-sections of the open set U. Once this has been achieved and the result compared with the abstract figures above it is a matter of writing down the iterated integrals and evaluating them using one-variable integration theory. We have, as in the two-dimensional case a choice in the order of integration—in fact a total of $3! = 6$ choices, some may be easy, others difficult and some impossible. There are no definite rules.

A particularly simple situation occurs when $U = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ and $f(x, y, z) = g(x)h(y)k(z)$ where $g,\ h$ and k are functions of a single variable. In this case

$$\iiint\limits_U f(x, y, z)\, dxdydz = \Big(\int_{a_1}^{b_1} f(x)\, dx\Big)\Big(\int_{a_2}^{b_2} g(y)\, dy\Big)\Big(\int_{a_3}^{b_3} k(z)\, dz\Big).$$

For example, if $U = (0, a) \times (0, b) \times (0, c)$ then

$$\iiint\limits_U xy^2z^3\, dxdydz = \int_0^a xdx \cdot \int_0^b y^2dy \cdot \int_0^c z^3dz = \frac{a^2b^3c^4}{24}.$$

Example 14.1 Let B denote the solid ball of radius r with centre at the origin, i.e. $B = \{(x, y, z) : x^2 + y^2 + z^2 < r^2\}$. We calculate the volume of B by integrating the function $f \equiv 1$ over B. Figure 14.4a shows clearly that the values of z which give non-zero cross-sections lie in the interval $(-r, r)$ and the cross-section for fixed z is the disc $x^2 + y^2 \le r^2 - z^2$ (Fig. 14.4b).

In this special case a direct computation is possible since

$$\iint\limits_{A(z)} 1\, dxdy = \text{Area}\,(A(z)) = \pi(r^2 - z^2)$$

and

$$\text{Vol}(B) = \int_{-r}^{r} \Big\{ \int \int_{A(z)} 1\, dxdy \Big\}\, dz \tag{14.1}$$

$$= \int_{-r}^{r} \pi(r^2 - z^2)dz = \pi\left(r^2z - \frac{z^3}{3}\right)\Big|_{-r}^{r} = \frac{4}{3}\pi r^3. \tag{14.2}$$

We now consider the more typical approach to evaluating the inner integral over $A(z)$ by applying Fubini's theorem in two variables. In Fig. 14.4b we have sketched the cross-section $A(z)$ in the xy-plane. The equation $x^2 + y^2 = r^2 - z^2$ (z fixed) has two solutions

$$x = \pm\sqrt{r^2 - z^2 - y^2}.$$

These give the total variation of x and the boundary functions, l and r. We thus have

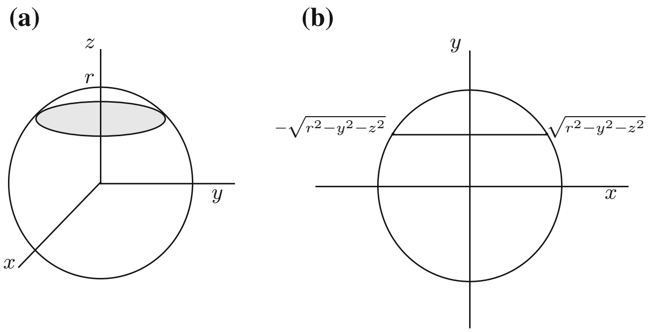


Fig. 14.4

$$
\begin{aligned}
\mathrm{Vol}(B) &= \int_{-r}^{r}\left\{\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}}\left\{\int_{-\sqrt{r^2-y^2-z^2}}^{+\sqrt{r^2-y^2-z^2}} dx\right\} dy\right\} dz \\
&= \int_{-r}^{r}\left\{\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} 2\sqrt{r^2-y^2-z^2}\, dy\right\} dz \\
&= 2\int_{-r}^{r}\left\{\int_{-\pi/2}^{+\pi/2}(r^2-z^2)\cos^2\theta d\theta\right\} dz \\
&= \pi\int_{-r}^{r}(r^2-z^2)dz = \pi\left(r^2z-\frac{z^3}{3}\right)\Big|_{-r}^{r} = \frac{4\pi r^3}{3}
\end{aligned}
$$

where we let $y = (r^2 - z^2)^{1/2}\sin\theta$, $dy = (r^2 - z^2)^{1/2}\cos\theta$.

The geometry of the previous example was reasonably straightforward. Many examples appear initially to involve a rather complicated geometric shape. However, we are usually dealing with a limited number of objects, mainly conic sections, mixed together and once familiarity with these has been established and sufficiently many cross-sections sketched—with the help of the defining inequalities—the correct approach often presents itself.

Example 14.2 We wish to find the volume of the region V lying below the plane $z = 3 - 2y$ and above the paraboloid $z = x^2 + y^2$, i.e. the set $\{(x, y, z) : x^2 + y^2 < z < 3 - 2y\}$. We begin by considering a full sketch (Fig. 14.5).

The coordinates of P and Q are found by solving between the equations $z = x^2 + y^2$ and $z = 3 - 2y$. These imply $x^2 + y^2 = 3 - 2y$, i.e. $x^2 + y^2 + 2y + 1 = 4$. Hence $x^2 + (y + 1)^2 = 2^2$. The extreme values of y are obtained by letting $x = 0$. This gives $(y + 1)^2 = 2^2$, i.e. $y + 1 = \pm 2$. Hence $y = -3$ or $+1$. The coordinates of P and Q are $(0, -3, 9)$ and $(0, 1, 1)$ respectively. Hence $-3 \le y \le +1$. From

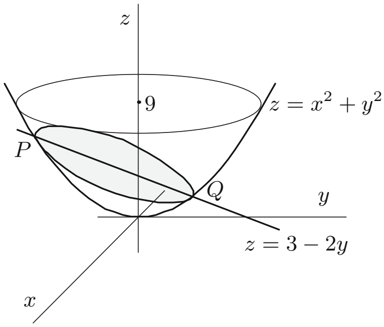


Fig. 14.5

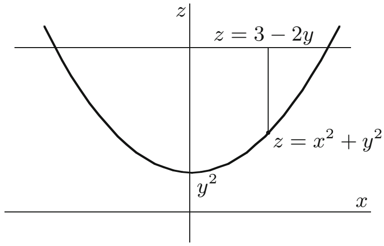


Fig. 14.6

Fig. 14.5 we see $0 \leq z \leq 9$ and from the equation $x^2 + (y+1)^2 = 2$ we deduce that $-2 \leq x \leq 2$. Having found the extremal values for non-empty sections we sketch the corresponding cross-sections.

First fix y, $-3 \leq y \leq 1$. In the zx-plane, $z = y^2 + x^2$ is a parabola and $z = 3 - 2y$ is a straight line. From Fig. 14.6 it follows that for fixed y,

$$-(3 - 2y - y^2)^{1/2} \leq x \leq (3 - 2y - y^2)^{1/2}$$

and for fixed x and y

$$x^2 + y^2 \leq z \leq 3 - 2y.$$

We have

$$\begin{aligned}\text{Vol}(V) &= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} \left\{ \int_{x^2+y^2}^{3-2y} dz \right\} dx \right\} dy \\ &= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} \Big[z \Big]_{x^2+y^2}^{3-2y} dx \right\} dy\end{aligned}$$

$$= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} (3 - 2y - y^2 - x^2)\, dx \right\} dy$$

$$= \int_{-3}^{1} \left[(3 - 2y - y^2)x - \frac{x^3}{3} \right]_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} dy$$

$$= \int_{-3}^{1} \frac{4}{3}(3 - 2y - y^2)^{3/2} dy = \frac{4}{3} \int_{-3}^{1} \left(4 - (y+1)^2\right)^{3/2} dy$$

Let $y + 1 = 2\sin\theta$, then $dy = 2\cos\theta d\theta$, $4 - (y+1)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$, so

$$\text{Vol}(V) = \frac{4}{3} \int_{-\pi/2}^{\pi/2} (4\cos^2\theta)^{3/2} \cdot 2\cos\theta d\theta$$

$$= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta$$

$$= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \frac{1}{4} \left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right) d\theta$$

$$= \frac{16}{3} \cdot \frac{3\pi}{2} = 8\pi.$$

We choose to take y as the final variable in our order of integration since it is clear from Fig. 14.5 that all cross-sections parallel to the xz-plane are of the same type whereas cross-sections parallel to the xy-plane, i.e. fixing z, are different for $z < 1$ and $z \geq 1$. A different order of integration can be used as a second opinion.

From the two previous examples we see that the evaluation of triple integrals proceeded in three stages. First we chose an order of integration and next examined the geometry of the domain of integration in order to determine the limits of integration in the inner integrals. Finally we evaluated a sequence of one variable integrals. The alternatives at each stage in the process point towards a useful technique commonly called *change of variable*. In place of a detailed motivation we briefly mention three pertinent ideas.

(a) The domain of integration U was partitioned into cubes, Fig. 14.7a, with element of volume ΔV easily calculated using $\Delta x \times \Delta y \times \Delta z$. We could use instead a grid based on spheres centred at the origin and planes through the origin to obtain a different element of volume, Fig. 14.7b, or a grid based on cylinders parallel to the z-axis and planes perpendicular and parallel to the z-axis (Fig. 14.7c).

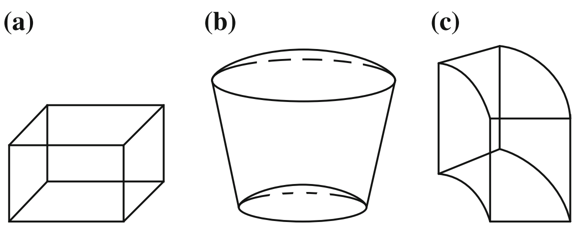


Fig. 14.7

These alternatives lead to more complicated formulae for ΔV but hopefully the new limits of integration and the resulting one-variable integrals are less complicated.

(b) We used Cartesian coordinates (x, y, z) to denote a typical point in the domain of integration U but we may consider other methods of identifying points in U. For instance, if U is the solid sphere of radius 1 then each point in U lies in a sphere of radius r, $0 \leq r \leq 1$, and using the parametrization of the sphere of radius r given in Example 10.5 we can identify points of U by means of r (the distance to the origin), θ the angle of latitude and ψ (the angle of longitude). In terms of the coordinates (r, θ, ψ) the domain U becomes the parallelepiped $(0, 1) \times (-\pi/2, \pi/2) \times (0, 2\pi)$ and, as previously noted, integration in this case is much more pleasant. The new set of coordinates gives a correspondence F between a domain in $\mathbb{R}^3_{(r,\theta,\psi)}$ and the original domain U in $\mathbb{R}^3_{(x,y,z)}$ (Fig. 14.8). By Example 10.5, $F(r, \theta, \psi) = (r \cos\theta \cos\psi, r \cos\theta \sin\psi, r \sin\theta)$. The idea now is to transfer the cubical grid on U, using F, to a grid, which is usually not cubical, on $F(U)$ and hence to evaluate the integral. In carrying out this operation it will be necessary to calculate

$$\text{Vol}\,(F(\Delta V)) = \text{Vol}\,(F(\Delta r \times \Delta\theta \times \Delta\psi))$$

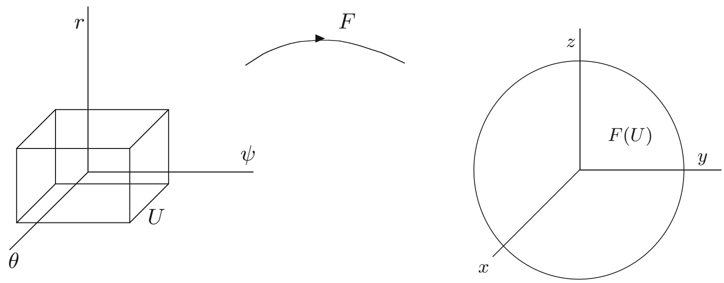


Fig. 14.8

and this is the problem that also arises in (a). The mapping F has many of the features that we have previously associated with a parametrization and, by now, the following definition should appear natural.

Definition 14.3 A parametrization of an open set V in $\mathbb{R}^n$ is a bijective differentiable mapping F from an open subset U of $\mathbb{R}^n$ onto V such that $F'(X)$ is an invertible linear operator for all X in U.

This definition contains the essential properties that we used to parametrize curves and surfaces and may also be regarded as a method of providing V with a *new coordinate system*. The requirement that $F'(X)$ be invertible (or equivalently that $\det\big(F'(X)\big) \neq 0$) is the three-dimensional analogue of the condition $P'(t) \neq 0$ for curves and $\phi_x \times \phi_y \neq 0$ for surfaces.

(c) A third, perhaps more obscure, approach is motivated by the substitutions that may arise in the one-variable integrals in the final stage. In Example 14.2 we needed one such change. Working backwards it may be possible to choose initially a coordinate system which does not require a change of variable in the iterated integrals.

We return now to (b) above to work out the formula for the change of variable. Let U denote an open subset of $\mathbb{R}^3_{(r,s,t)}$ and let $g\colon U \to g(U) = V$ denote a parametrization of the open subset V of $\mathbb{R}^3_{(u,v,w)}$. Note that g^{-1} is a parametrization of U. To avoid confusion we think of U and V as lying in different copies of $\mathbb{R}^3$ each with their own set of coordinates, (r, s, t) and (u, v, w) respectively. This explains the terminology "change of variables".

Let f denote an integrable function on $g(U)$ (Fig. 14.9). Take a cubical grid on U, transfer it by g to a grid on $g(U)$ and then form a Riemann sum of f. A typical term in this Riemann sum is

$$f\left(g(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right) \mathrm{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right).$$

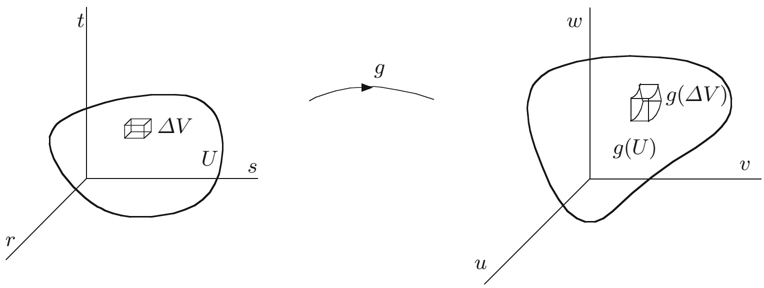


Fig. 14.9

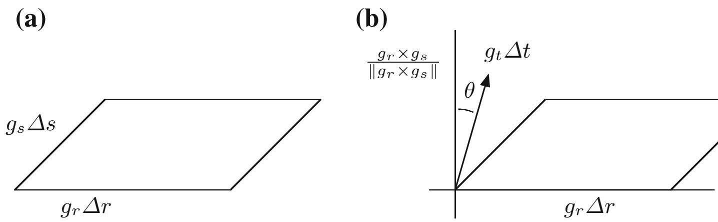


Fig. 14.10

Now $g(\Delta r_i \times \Delta s_j \times \Delta t_k)$ is approximately a parallelepiped with sides $g_r\Delta r$, $g_s\Delta s$ and $g_t\Delta t$. We have already discussed the volumes of parallelepipeds while introducing Stokes' theorem in Chap. 12. The area of the base of the parallelepiped is $\|g_r \times g_s\Delta r_i \cdot \Delta s_j\|$ (Fig. 14.10a) where the partial derivatives of g are evaluated at $(\overline{r}_i, \overline{s}_j, \overline{t}_k)$. The height of the parallelepiped is the length of the projection of $g_t\Delta t_k$ onto a direction perpendicular to the base. Since $g_r \times g_s$ is perpendicular to the base the required height is $\left(g_r \times g_s \cdot g_t \;\big/\; \|g_r \times g_s\|\right)\Delta t_k = \|g_t\|\cos\theta\,\Delta t_k$ (Fig. 14.10b).

Hence

$$\mathrm{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right) \approx \|g_r \times g_s \cdot g_t\|\Delta r_i\Delta s_j\Delta t_k.$$

By Exercise 6.5(a)

$$\|g_r \times g_s \cdot g_t\| = \left|\det\begin{pmatrix} g_r \\ g_s \\ g_t \end{pmatrix}\right| = \left|\det(g')\right|$$

and

$$\mathrm{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right) \approx \left|\det(g')\right|\Delta r_i\Delta s_j\Delta t_k.$$

Hence the Riemann sum of f over $g(U)$ is approximately

$$\sum_i\sum_j\sum_k f\left(g(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right)\left|\det\left(g'(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right)\right|\Delta r_i\Delta s_j\Delta t_k$$

where we sum over the cubes in the partition of U. In the limit we get the *change of variables formula*

$$\iiint\limits_{g(U)} f(u, v, w)dudvdw = \iiint\limits_{U} f\left(g(r, s, t)\right)\left|\det\left(g'(r, s, t)\right)\right| drdsdt.$$

The notations $J(g)$ and $\dfrac{\partial(u,v,w)}{\partial(r,s,t)}$ are also used in place of $\det(g')$ and this determinant is called the *Jacobian* of g.

If V is an open subset of $\mathbb{R}^2$ then we may identify V with the open subset $\widetilde{V} = V \times (0,1)$ in $\mathbb{R}^3$. A function $f\colon V \to \mathbb{R}$ is integrable over V if and only if $\widetilde{f}$, defined by $\widetilde{f}(x,y,z) = f(x,y)$, is integrable over $\widetilde{V}$ and, moreover,

$$\iiint\limits_{\widetilde{V}} \widetilde{f}(x,y,z)dxdydz = \iint\limits_{V} f(x,y)dxdy.$$

If $g\colon U \subset \mathbb{R}^2_{(r,s)} \to V \subset \mathbb{R}^2_{(u,v)}$ is a mapping between the open sets U and V then it is easily seen that g is a parametrization of V if and only if $\widetilde{g} : \widetilde{U} \subset \mathbb{R}^3_{(r,s,t)} \longrightarrow \widetilde{V} \subset \mathbb{R}^3_{(u,v.w)}$, defined by $\widetilde{g}(r,s,t) = (g(r,s),t)$, is a parametrization of $\widetilde{V}$. Since

$$\widetilde{g}' = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{g'}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\det(g') = \det(\widetilde{g}')$ and

$$\begin{aligned} \iint\limits_{V} f = \iiint\limits_{\widetilde{V}} \widetilde{f} &= \iiint\limits_{\widetilde{U}} \widetilde{f} \circ \widetilde{g} |\det(\widetilde{g}')| \\ &= \iint\limits_{U} f \circ g |\det(g')|. \end{aligned}$$

This justifies the change of variables formula for double integrals (Chap. 9) and yields the following familiar formula

$$\iint\limits_{V=g(U)} f(u,v)dudv = \iint\limits_{U} f\,(g(r,s)) \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} drds$$

where $g(r,s) = (u(r,s), v(r,s))$. In particular, for polar coordinates in the plane, $g\colon (r,\theta) \to (x,y) = (r\cos\theta, r\sin\theta)$, we have

$$|\det(g')| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

and

$$\iint\limits_{x^2+y^2<1} f(x,y)dxdy = \iint\limits_{\substack{0<r<1 \\ 0<\theta<2\pi}} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

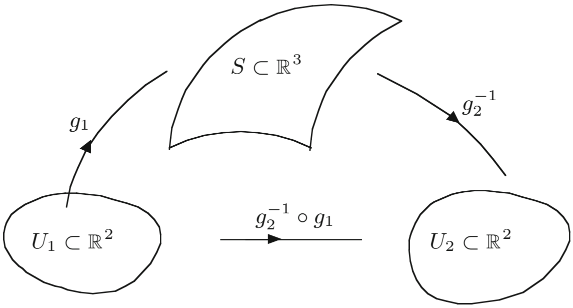


Fig. 14.11

If **S** is a simple oriented surface in $\mathbb{R}^3$ and (g_1, U_1) and (g_2, U_2) are parametrizations of S consistent with the orientation then the bijectivity and smoothness of g_1 and g_2 imply that the mapping $g_2^{-1} \circ g_1$ is a parametrization of U_2 (Fig. 14.11).

The change of variables formula for double integrals may now be used to show that integrals over **S** are *independent* of the parametrization and justifies the notation used in earlier chapters (we remark that g_1 and g_2 map onto the *same* side of **S** if and only if $\det\left((g_2^{-1} \circ g_1)'\right)$ is always *strictly positive*).

We have noted above how a solid sphere is a union of surfaces, i.e.

$$\{(x, y, z) : x^2 + y^2 + z^2 < r^2\} = \bigcup_{0 \leq s < r} \{(x, y, z) : x^2 + y^2 + z^2 = s^2\}$$

and using parametrization of surfaces we were able to fill in to obtain a parametrization of the solid sphere. In fact, we do not quite get a parametrization of the full solid, since to obtain a bijective mapping, we are forced to miss a small portion of the sphere. This is the same problem that we encountered and discussed fully in parametrizing the classical surfaces (Chap. 10). We do not enter into a full discussion here but remark that after a similar discussion we would arrive at an analogous conclusion for solids. The portion of the solid omitted has volume zero and so for integration purposes we may treat the mappings in the following example as parametrizations of the full solid.

Example 14.4 The procedure outlined above for the sphere can be applied to a number of solids and using Table 11.1 we obtain the following parametrizations.

(a) *Solid sphere of radius a* (spherical polar coordinates)

$$(0, a) \times (0, \pi) \times (0, 2\pi) \in \mathbb{R}^3 \longrightarrow \left\{(x, y, z) : x^2 + y^2 + z^2 < a^2\right\}$$
$$(r, \theta, \psi) \longrightarrow (r \sin\theta \cos\psi, r \sin\theta \sin\psi, r \cos\theta)$$

(b) *Solid ellipsoid* (elliptical polar coordinates)

$$(0,1)\times(0,\pi)\times(0,2\pi)\in\mathbb{R}^3 \longrightarrow \left\{(x,y,z): \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1\right\}$$
$$(r,\theta,\psi) \longrightarrow (ra\sin\theta\cos\psi, rb\sin\theta\sin\psi, rc\cos\theta)$$

(c) *Solid of revolution* generated by revolving the area beneath the plane curve $P(t)=(x(t),y(t))$, $t\in[a,b]$ and $y(t)>0$, about the x–axis (Example 10.4),

$$(0,1)\times(a,b)\times(0,2\pi)\in\mathbb{R}^3 \longrightarrow \text{Solid of Revolution}$$
$$(r,t,\theta) \longrightarrow (x(t), ry(t)\cos\theta, ry(t)\sin\theta)$$

(d) The *solid cylinder* of radius r and height h parallel to the z-axis is defined by $\{(x,y,z): x^2+y^2<r^2,\ 0<z<h\}$ and parametrized by

$$(s,\theta,z) \longrightarrow (s\cos\theta, s\sin\theta, z)$$

with domain $(0,r)\times(0,2\pi)\times(0,h)$ (cylindrical coordinates)

(e) The inverted *solid cone* $\{(x,y,z): x^2+y^2<z^2,\ 0<z<1\}$ is parametrized by

$$(r,\theta,z) \longrightarrow (r\cos\theta, r\sin\theta, z)$$

where $0<r<z, 0<\theta<2\pi, 0<z<1$.

(f) The inverted *solid paraboloid* defined by $\{(x,y,z): x^2+y^2<z,\ 0<z<1\}$ is parametrized by

$$(r,\theta,z) \longrightarrow (r\cos\theta, r\sin\theta, z)$$

where $0<r<\sqrt{z}, 0<\theta<2\pi, 0<z<1$.

Example 14.5 In this example we calculate the volume of a solid *torus*. We discussed the boundary surface of a torus in Example 11.1 and obtained the parametrization

$$f(\theta,\psi)=((b+r\cos\theta)\cos\psi, (b+r\cos\theta)\sin\psi, r\sin\theta)$$

where $0<\theta<2\pi, 0<\psi<2\pi$ (Fig. 11.4). We generate the *solid torus* by rotating the disc and obtain the parametrization

$$F: (0,2\pi)\times(0,2\pi)\times(0,r) \longrightarrow \text{Solid Torus}$$
$$(\theta,\psi,s) \longrightarrow ((b+s\cos\theta)\cos\psi, (b+s\cos\theta)\sin\psi, s\sin\theta)\,.$$

We will use the change of variable formula to calculate the volume but first note how the following geometric observation immediately gives the answer. The solid torus is obtained by rotating a disc of radius r with centre on the y-axis at a distance b

from the origin about the z-axis. Thus the disc is rotated through a distance $2\pi b$ and generates a solid whose volume is

$$\pi r^2 \cdot 2\pi b = 2\pi^2 r^2 b.$$

By the change of variable formula

$$\text{Volume (Torus)} = \iiint\limits_{[0,2\pi]\times[0,2\pi]\times[0,r]} \left|\det\left(F'(\theta,\psi,s)\right)\right| d\theta d\psi ds.$$

From Table 11.1 we obtain, on replacing r by s, the first two columns of F'; $F_\theta = f_\theta$ and $F_\psi = f_\psi$. Hence

$$F'(\theta,\psi,s) = \begin{pmatrix} -s\sin\theta\cos\psi & -(b+s\cos\theta)\sin\psi & \cos\theta\cos\psi \\ -s\sin\theta\sin\psi & (b+s\cos\theta)\cos\psi & \cos\theta\sin\psi \\ s\cos\theta & 0 & \sin\theta \end{pmatrix}.$$

From the matrix representation it is easily seen that F_θ, F_ψ and F_s, i.e. the columns of F', are mutually perpendicular vectors and hence generate a parallelepiped shaped like a rectangular box. In this case the volume is the product of the lengths of the sides. A further application of Table 11.1 gives us $\|F_\theta\| = \|f_\theta\| = \sqrt{E} = s$ and $\|F_\psi\| = \|f_\psi\| = \sqrt{G} = b + s\cos\theta$. Since $\|F_s\| = (\cos^2\theta\cos^2\psi + \cos^2\theta\sin^2\psi + \sin^2\theta)^{1/2} = 1$ we have

$$\left|\det\left(F'(\theta,\psi,s)\right)\right| = \|F_\theta\|\cdot\|F_\psi\|\cdot\|F_s\| = s(b+s\cos\theta).$$

and

$$\begin{aligned}\text{Volume (Torus)} &= \int\limits_{[0,2\pi]}\int\limits_{[0,2\pi]}\int\limits_0^r s(b+s\cos\theta)d\theta d\psi ds \\ &= \int\limits_0^{2\pi} d\psi \cdot \int\limits_0^r \left\{\int\limits_0^{2\pi}(sb+s^2\cos\theta)d\theta\right\}ds \\ &= 2\pi\int\limits_0^r (sb\theta + s^2\sin\theta)\Big|_0^{2\pi} ds \\ &= 2\pi\int\limits_0^r sb2\pi ds = 4\pi^2 b\frac{s^2}{2}\Big|_0^r = 2\pi^2 br^2.\end{aligned}$$

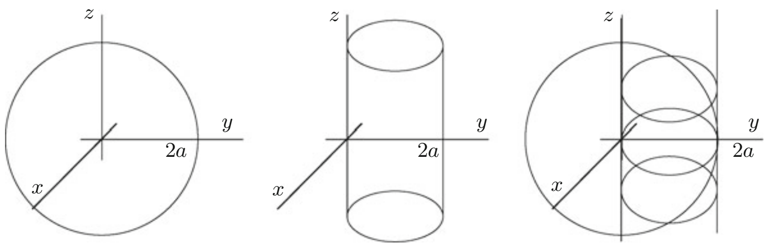


Fig. 14.12

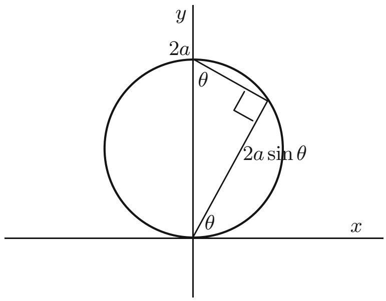


Fig. 14.13

Example 14.6 To find the volume of the solid contained within the sphere $x^2+y^2+z^2=4a^2$ and the cylinder $x^2+(y-a)^2=a^2$. In Example 12.3 we considered a geometric situation similar to the present one. By not moving the origin we adopt here a slightly different approach. From Fig. 14.12 we see that the solid lies above and below the plane disc $x^2+(y-a)^2 \leq a^2$ and this immediately suggests polar coordinates.

The volume V is equal to

$$\iiint\limits_{\substack{x^2+y^2+z^2<4a^2\\ x^2+(y-a)^2<a^2}} 1\, dxdydz.$$

We parametrize the solid using *polar coordinates* in the xy-plane and the usual Cartesian z coordinate, i.e. we use the *cylindrical coordinates*

$$F:\,(r,\theta,z)\longrightarrow(r\cos\theta, r\sin\theta, z).$$

Since

$$F'(r,\theta,z)=\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

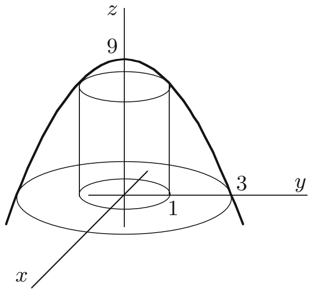


Fig. 14.14

we have $\det\big(F'(r,\theta,z)\big) = r$. To find the limits of integration consider Fig. 14.13 in the xy-plane.

We see that $0 < \theta < \pi$, $0 < r < 2a\sin\theta$ and, from Fig. 14.12, $z^2 < 4a^2 - x^2 - y^2 = 4a^2 - r^2$, i.e. $-\sqrt{4a^2-r^2} < z < \sqrt{4a^2-r^2}$. Hence

$$\text{Volume} = \int_0^{\pi}\left\{\int_0^{2a\sin\theta}\left\{\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r\,dz\right\}dr\right\}d\theta$$
$$= 2\int_0^{\pi}\left\{\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\,dr\right\}d\theta.$$

Let $s = 4a^2 - r^2$. Then $ds = -2r\,dr$ and

$$\text{Volume} = 2\int_0^{\pi}\left\{\int_{4a^2}^{4a^2\cos^2\theta}(-\frac{1}{2}s^{1/2})ds\right\}d\theta$$
$$= -\int_0^{\pi}\left(\frac{2s^{3/2}}{3}\Big|_{4a^2}^{4a^2\cos^2\theta}\right)d\theta$$
$$= \frac{16a^3}{3}\int_0^{\pi}(1-|\cos^3\theta|)d\theta = \frac{16a^3}{9}(3\pi-4).$$

Example 14.7 In this example we calculate the volume of the region U bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane and which lies outside the cylinder $x^2 + y^2 = 1$ (Fig. 14.14).

The presence of x^2+y^2 in the defining inequalities suggests cylindrical coordinates (r,θ,z) (the presence of $x^2+y^2+z^2$ would suggest geographical or spherical polar coordinates). From Fig. 14.14 we see that U projects onto $\{(x, y) : 1 \leq x^2 + y^2 \leq 9\}$ in the xy-plane. Hence $1 \leq r \leq 3$ and z varies over $0 \leq z \leq 9 - x^2 - y^2 = 9 - r^2$. From the previous example we know that the Jacobian is equal to r. Hence the required volume is

$$\int_0^{2\pi} \left\{ \int_1^3 \left\{ \int_0^{9-r^2} r\,dz \right\} dr \right\} d\theta = \int_0^{2\pi} \left\{ \int_1^3 (9r - r^3)\,dr \right\} d\theta$$

$$= 2\pi \cdot \left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Big|_1^3 = 32\pi.$$

Exercises

14.1 Show that the volume of the solid inside the cylinder $x^2 + y^2 - 2ay = 0$ and between the plane $z = 0$ and the paraboloid $4az = x^2 + y^2$ equals $3\pi a^3/8$.

14.2 Find the volume of the solid inside the cylinder $x^2 + y^2 = 2ay$ which lies between the plane $z = 0$ and the cone $x^2 + y^2 = z^2$.

14.3 Show that the volume of the solid defined by the inequalities $x^2 + y^2 \leq 1$ and $\tan^{-1}(y/x) \leq z \leq 2\pi$ equals π^2.

14.4 Let U denote the region above the plane $z = 0$ between the cone $z^2 = x^2 + y^2$ and the paraboloid $z = 2 - x^2 - y^2$. Show that this region projects onto the unit disc in the xy-plane. Using cylindrical coordinates or otherwise show that the volume of U equals $5\pi/6$.

14.5 Evaluate the following integrals:

(a) $$\int_{-3}^{1} \left\{ \int_{y^2}^{3-2y} \left\{ \int_{-(z-y^2)^{1/2}}^{(z-y^2)^{1/2}} dx \right\} dz \right\} dy$$

(b) $$\int_{-2}^{2} \left\{ \int_{-1-(4-x^2)^{1/2}}^{-1+(4-x^2)^{1/2}} \left\{ \int_{x^2+y^2}^{3-2y} dz \right\} dy \right\} dx$$

(c) $$\int_{0}^{9} \left\{ \int_{-\sqrt{z}}^{\sqrt{z}} \left\{ \int_{-(z-x^2)^{1/2}}^{\frac{3-z}{2}} dy \right\} dx \right\} dz.$$

14.6 Use the change of variables formula to calculate the volumes of the solids parametrized in Example 14.4.

14.7 Find the volume of the wedge of the cylinder $\{(x, y, z) : x^2 + y^2 \leq 1\}$ which lies above the xy-plane and between the planes $z = -y$ and $z = 0$.

14.8 Find the volume of the open set which lies above the square $0 < x < 1$, $0 < y < 1$ in the xy-plane and below the surface $z = (1 + x + y)^{1/2}$. Hence write down the volume of the set enclosed by the surfaces $z = (1+|x|+|y|)^{1/2}$, $z = -(1+|x|+|y|)^{1/2}$ and by the planes $x = 1, x = -1, y = 1$ and $y = -1$.

14.9 Find the volume (of the ice-cream cone) that lies above the xy-plane, inside the cone $3(x^2 + y^2) = z^2$ and the sphere $x^2 + y^2 + z^2 = 4a^2$.

14.10 Evaluate

$$\iiint_V z dx dy dz$$

where $V = \{(x, y, z) : 0 < z < x\sqrt{y},\ 0 < y < 1,\ 1 < x < 2\}$.

14.11 Use the change of variables

$$F : (u, v, w) \longrightarrow \big(u(1 - v), uv(1 - w), uvw\big)$$

to calculate

$$\iiint_V x dx dy dz \quad \text{and} \quad \iiint_V \frac{dx dy dz}{y + z}$$

where V is the tetrahedron cut from the first octant by the plane $x + y + z = 1$.

Chapter 15
The Divergence Theorem

Summary *We state, discuss and give examples of the divergence theorem of Gauss.*

The *divergence theorem of Gauss* is an extension to $\mathbb{R}^3$ of the fundamental theorem of calculus and of Green's theorem and is a close relative, but not a direct descendent, of Stokes' theorem. This theorem allows us to evaluate the integral of a scalar-valued function over an open subset of $\mathbb{R}^3$ by calculating the surface integral of a certain vector field over its boundary.

In Chap. 6 we defined the *divergence* of the vector field $\mathbf{F} = (f_1, f_2, f_3)$ as

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_1, f_2, f_3) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

and we have previously written symbolically

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}.$$

Carrying this symbolism a step further we now write

$$\mathrm{div}(\mathrm{curl}\ \mathbf{F}) = \nabla \cdot \nabla \times \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

This suggests, since the determinant of a matrix with two identical rows is zero, that $\mathrm{div}(\mathrm{curl}\ \mathbf{F}) = 0$ for any smooth vector field $\mathbf{F}$. This is indeed true. Symbolism, however, does *not* prove anything and it is necessary to verify this formally. At the same time it is a good example of the role of symbolism (and notation) in mathematics—it can be suggestive—and sometimes leads to true results that might otherwise be overlooked.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_15,

If **F** is the velocity of a gas then div(**F**) represents the rate of expansion (or compression) per unit volume. The divergence theorem states that the total expansion (or contraction) over a region U equals the total inflow (or outflow) across the boundary. Very little physical intuition is required in order to accept this as reasonable. This physical interpretation is responsible for the terminology "divergence".

We now formally state the divergence theorem.

Theorem 15.1 (Gauss' Divergence Theorem) *Suppose* **S** *is an oriented surface in* $\mathbb{R}^3$*, with outward pointing normal, enclosing an open set* U *and* **F** *is a smooth vector field on* $\overline{U}$ $(= U \cup S)$ *then*

$$\iint\limits_{S} \mathbf{F} = \iiint\limits_{U} \mathrm{div}(\mathbf{F}).$$

An initial proof uses Green's theorem and Fubini's theorem on cross-sections of U and, as noted in earlier chapters, this places certain geometrical restrictions on U and S. From the previous chapter we see that it suffices to have U bounded, top and bottom, above and below, front and back by graphs of functions which form the boundary S. This covers many examples involving classical Euclidean shapes.

Further examples, for instance the annulus $0 < r_1 < x^2 + y^2 + z^2 < r_2 < \infty$, are obtained by dividing the open set into a finite number of sets on each of which the initial proof applies (we refer to our discussion on Green's theorem in Chap. 9 for further details). In general, the boundary will be composed of a finite number of distinct simple surfaces.

The collection of sets to which the divergence theorem applies is quite large but difficult to formulate precisely without recourse to further concepts from differential geometry. For this reason we have carefully avoided proving the theorem and giving detailed hypotheses on U and S in the statement of the theorem. The theorem, as stated, is sufficient for the examples we consider.

Example 15.2 In this example we use the divergence theorem to evaluate

$$\iiint\limits_{V} \mathrm{div}(\mathbf{F})dxdydz$$

where V is the solid cylinder $\{(x, y, z) : x^2 + y^2 < 1,\ 0 < z < 1\}$ and

$$\mathbf{F}(x, y, z) = \big(1 - (x^2 + y^2)^3, 1 - (x^2 + y^2)^3, x^2z^2\big).$$

The boundary of V is composed of the cylinder $\{(x, y, z) : x^2+y^2 = 1,\ 0 < z < 1\}$ and the flat discs $\{(x, y, z) : x^2+y^2 \leq 1,\ z = 0\}$ and $\{(x, y, z) : x^2+y^2 \leq 1,\ z = 1\}$ (Fig. 15.1).

The outward normal on the curved surface at the point (x, y, z) is $(x, y, 0)$. On the upper and lower discs the outward normals are $(0, 0, 1)$ and $(0, 0, -1)$ respectively. On the curved surface $\mathbf{F} \cdot \mathbf{n} = 0$ since $x^2 + y^2 = 1$ and the third coordinate of $\mathbf{n}$

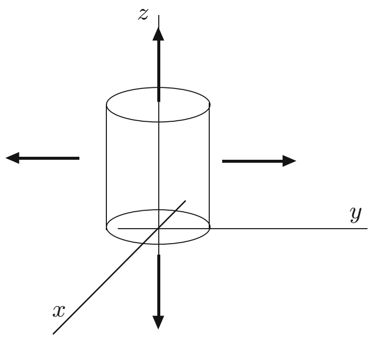


Fig. 15.1

is zero, on the bottom $\mathbf{F} \cdot \mathbf{n} = 0$ since $z = 0$ and on the top $\mathbf{F} \cdot \mathbf{n} = x^2$. By the divergence theorem

$$\iiint_V \operatorname{div}(\mathbf{F}) dxdydz = \iint_{x^2+y^2 \leq 1} x^2 dxdy.$$

The double integral is evaluated by using the polar coordinates

$$(r, \theta) \longrightarrow (r \cos\theta, r \sin\theta), \quad 0 < r < 1,\ 0 < \theta < 2\pi$$

on the unit disc and equals $\pi/4$.

Example 15.3 We use the divergence theorem to evaluate

$$\iint_{\partial \mathbf{W}} (x^2 + y + z)$$

where W is the solid sphere $x^2 + y^2 + z^2 < 1$ and $\partial \mathbf{W}$ is its boundary oriented outwards. The unit sphere has outward normal $\mathbf{n}(x, y, z) = (x, y, z)$. If $\mathbf{F} = (f_1, f_2, f_3)$ is a vector field on $\overline{W}$ then the divergence theorem implies that

$$\iint_{\partial \mathbf{W}} (f_1, f_2, f_3) = \iint_{\partial W} (xf_1 + yf_2 + zf_3) dA \tag{15.1}$$

$$= \iiint_W \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dxdydz. \tag{15.2}$$

Hence we can use any smooth vector field on the closed unit sphere for which $xf_1 + yf_2 + zf_3 = x^2 + y + z$. This allows a wide choice but in such cases the simplest

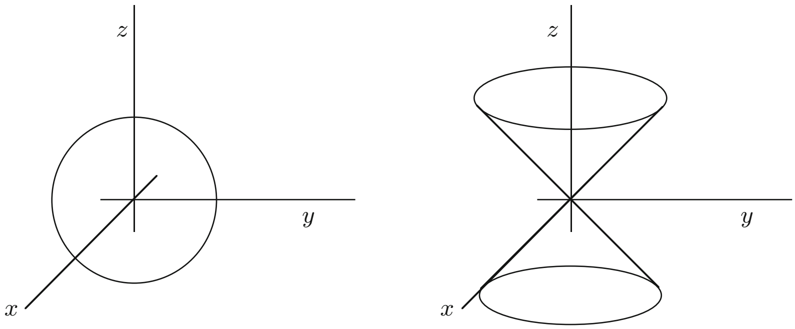


Fig. 15.2

and most obvious option is usually the best. Let $f_1(x, y, z) = x$, $f_2(x, y, z) = 1$ and $f_3(x, y, z) = 1$. Then

$$\iint\limits_{\partial W} (x^2 + y + z)dA = \iiint\limits_{W} 1dxdydz = \text{Vol}\,(W) = \frac{4}{3}\pi.$$

Example 15.4 In this example we verify a particular case of the divergence theorem. Let U denote the set defined by the inequalities $z \geq 0$, $x^2 + y^2 + z^2 \leq 4$ and $x^2 + y^2 \leq z^2$. These define the set of points, above the xy-plane, contained in the sphere of radius 2 centred at the origin, which lie within the cone $x^2 + y^2 \leq z^2$ (Fig. 15.2).

The boundaries of the solid sphere and cone are given by the equalities $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = z^2$ respectively. These intersect in a curve which satisfies both equations, i.e. $x^2 + y^2 + z^2 = 4 = z^2 + z^2$. Hence $z = \pm\sqrt{2}$ and $x^2 + y^2 = 2$. This is a circle of radius $\sqrt{2}$ parallel to the xy-plane, $\sqrt{2}$ units above it with centre at the point $(0, 0, \sqrt{2})$. The open set U is that portion of the cone which lies between the planes $z = 0$ and $z = \sqrt{2}$ capped by the top of the sphere (Fig. 15.3). The boundary of U consists of the portion S_1 of the cone defined by $x^2+y^2 = z^2, 0 \leq z \leq \sqrt{2}$, and the portion of the sphere, S_2, defined by $x^2 + y^2 + z^2 = 4$, $\sqrt{2} \leq z \leq 2$. Consider the vector field

$$\mathbf{F}(x, y, z) = (xz, yz, x^2 + y^2).$$

We have $\text{div}(\mathbf{F}) = \dfrac{\partial}{\partial x}(xz) + \dfrac{\partial}{\partial y}(yz) + \dfrac{\partial}{\partial z}(x^2 + y^2) = 2z$. To evaluate

$$\iiint\limits_{U} 2zdxdydz$$

we use spherical polar coordinates (Example 10.5):

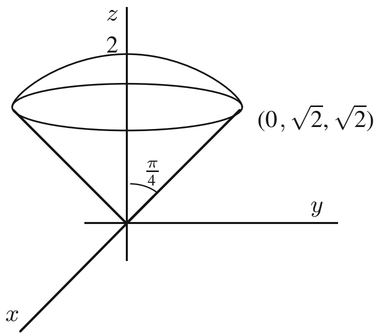


Fig. 15.3

$$w\colon (r, \theta, \psi) \longrightarrow (r \sin\theta \cos\psi, r \sin\theta \sin\psi, r \cos\theta).$$

From Fig. 15.3 the domain of w is

$$0 < \theta < \frac{\pi}{4}, \quad 0 < r < 2, \quad 0 < \psi < 2\pi.$$

Since the rows of

$$w' = \begin{pmatrix} w_r \\ w_\theta \\ w_\psi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \\ r\cos\theta\cos\psi & r\cos\theta\sin\psi & -r\sin\theta \\ -r\sin\theta\sin\psi & r\sin\theta\cos\psi & 0 \end{pmatrix}$$

are easily seen to be mutually orthogonal

$$\det(w') = 1 \cdot r \cdot r \sin\theta = r^2 \sin\theta.$$

Hence

$$\iiint_U 2z dx dy dz = 2\int_0^{\pi/4}\int_0^2\int_0^{2\pi} r^2 \sin\theta \cdot r\cos\theta dr d\theta d\psi$$
$$= \int_0^{\pi/4} \sin 2\theta d\theta \cdot \int_0^2 r^3 dr \cdot \int_0^{2\pi} d\psi$$
$$= 2\pi \cdot \frac{r^4}{4}\bigg|_0^2 \cdot \left(\frac{-\cos 2\theta}{2}\right)\bigg|_0^{\pi/4} = 2\pi \cdot \frac{16}{4} \cdot \frac{1}{2} = 4\pi.$$

To verify the divergence theorem we must show

$$\iint_{\mathbf{S_1}} \mathbf{F} + \iint_{\mathbf{S_2}} \mathbf{F} = 4\pi$$

where $\mathbf{S_1}$ and $\mathbf{S_2}$ both are oriented outwards. To ensure that we do not take an incorrect sign we use the formula

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{S} (\mathbf{F}\cdot\mathbf{n})\sqrt{EG-F^2}$$

to evaluate the surface integrals where $\mathbf{n}$ is the outer normal. Since S_1 is part of the level set $x^2+y^2-z^2=0$ its unit normals are

$$\pm\frac{(2x,2y,-2z)}{\Big((2x)^2+(2y)^2+(2z)^2\Big)^{1/2}} = \pm\frac{(x,y,-z)}{\sqrt{2}z}.$$

On S_1, $z>0$ and from Fig. 15.3 the outer normal has a negative z-component and hence

$$\mathbf{n_1}(x,y,z) = \frac{(x,y,-z)}{\sqrt{2}z}.$$

Hence

$$\mathbf{F}\cdot\mathbf{n_1} = (xz, yz, x^2+y^2)\cdot\frac{(x,y,-z)}{\sqrt{2}z} = \frac{x^2z+y^2z-z(x^2+y^2)}{\sqrt{2}z} = 0$$

and

$$\iint_{\mathbf{S_1}} \mathbf{F} = 0.$$

Since S_2 is part of the sphere of radius 2 centred at the origin the outer normal at (x,y,z) is $(x,y,z)/2$. Hence

$$\mathbf{F}\cdot\mathbf{n_2} = (xz, yz, x^2+y^2)\cdot\frac{(x,y,z)}{2} = \frac{x^2z+y^2z+(x^2+y^2)z}{2} = (x^2+y^2)z.$$

We parametrize the sphere using spherical polar coordinates (Example 10.5) and obtain

$$g\colon(\theta,\psi)\longrightarrow(2\sin\theta\cos\psi, 2\sin\theta\sin\psi, 2\cos\theta).$$

From Fig. 15.3, the domain of g is $0<\theta<\pi/4$, $0<\psi<2\pi$. We have already seen in Example 11.3 that

$$\|g_\theta\times g_\psi\| = \sqrt{EG-F^2} = 2^2\sin\theta$$

and from the above

$$\mathbf{F} \cdot \mathbf{n}_2 = 4\sin^2\theta \cdot 2\cos\theta = 8\sin^2\theta\cos\theta.$$

Hence

$$\iint_{\mathbf{S_2}} \mathbf{F} = \int_0^{\pi/4}\int_0^{2\pi} 32\sin^3\theta\cos\theta d\theta d\psi.$$

If $u = \sin\theta$ then $du = \cos\theta d\theta$ and

$$\iint_{\mathbf{S_2}} \mathbf{F} = 64\pi\int_0^{1/\sqrt{2}} u^3 du$$

$$= 64\pi \cdot \frac{u^4}{4}\Big|_0^{1/\sqrt{2}} = \frac{64\pi}{4}\cdot\frac{1}{4} = 4\pi$$

and

$$\iiint_U \operatorname{div}(\mathbf{F}) = \iint_{\mathbf{S_1}} \mathbf{F} + \iint_{\mathbf{S_2}} \mathbf{F}.$$

We have this verified the divergence theorem.

Hopefully we have developed, in the last few chapters, certain skills in handling diagrams, parametrizations and linear algebra and seen that problems in integration theory are handled by a mixture of techniques and ideas. With some practice readers will find their own preferred approach and, indeed, recognise that there are alternative approaches to a number of problems. Our next example is similar but somewhat more complicated than the previous example. We could take the same approach but instead use a few different ideas (some call them tricks) which may be useful elsewhere. This example also highlights the relationship between the change of variables for triple integrals and parametrizations of the boundary surfaces. Recall that we obtained many of the change of variables formulae for solids by "filling in" parametrization of boundary surfaces (Example 14.4). In the next example we see that restricting parametrizations of solids to the boundaries leads to parametrizations of surfaces.

Example 15.5 In this example we verify the divergence theorem on a part of a solid torus (Example 11.1). Consider the region A in the yz-plane determined by the inequalities $(y-b)^2 + z^2 \leq a$ and $b - y \leq z \leq y - b$ where $b > a > 0$. The boundary of A consists of an arc of a circle of radius a with centre $(b, 0)$ and two straight lines which pass through the centre of the circle. These are easily sketched, Fig. 15.4, and we will constantly refer to this simple diagram as it contains a great deal of information.

If we revolve A about the z-axis we obtain a wedge-shaped portion U of the solid torus (see Fig. 15.4 for a sketch of the torus). We parametrize the solid torus by filling

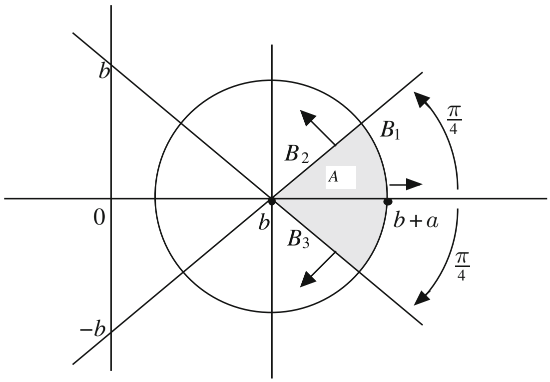


Fig. 15.4

in the toroidal polar coordinates given in Example 11.1 and use Fig. 15.4 to find the domain of the parametrization P of the wedge U.

We obtain

$$P\colon (r, \theta, \psi) \longrightarrow ((b + r\cos\theta)\cos\psi, (b + r\cos\theta)\sin\psi, r\sin\theta)$$

where $0 < r < a$, $-\pi/4 < \theta < \pi/4$ and $0 < \psi < 2\pi$. Consider the vector field

$$\mathbf{F}(x, y, z) = \left(x\left(\left(x^2+y^2\right)^{1/2} - b\right), y\left(\left(x^2+y^2\right)^{1/2} - b\right), z\left(x^2+y^2\right)^{1/2}\right).$$

In Cartesian coordinates,

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= (x^2+y^2)^{1/2} - b + x\cdot\frac{1}{2}\cdot 2x(x^2+y^2)^{-1/2} + (x^2+y^2)^{1/2} - b \\ &\quad + y\cdot\frac{1}{2}\cdot 2y(x^2+y^2)^{-1/2} + (x^2+y^2)^{1/2} \\ &= 3(x^2+y^2)^{1/2} - 2b + (x^2+y^2)\cdot(x^2+y^2)^{-1/2} \\ &= 4(x^2+y^2)^{1/2} - 2b\end{aligned}$$

and, in toroidal polar coordinates,

$$\begin{aligned}\operatorname{div}(\mathbf{F}) &= 4\left((b+r\cos\theta)^2\cos^2\psi + (b+r\cos\theta)^2\sin^2\psi\right)^{1/2} - 2b \\ &= 4(b+r\cos\theta) - 2b = 2(b+2r\cos\theta).\end{aligned}$$

Since

$$P' = (P_r,\ P_\theta,\ P_\psi) = \begin{pmatrix} \cos\theta\cos\psi & -r\sin\theta\cos\psi & -(b+r\cos\theta)\sin\psi \\ \cos\theta\sin\psi & -r\sin\theta\sin\psi & (b+r\cos\theta)\cos\psi \\ \sin\theta & r\cos\theta & 0 \end{pmatrix}$$

and P_r, P_θ and P_ψ are easily seen, by inspection, to be mutually perpendicular we have

$$|\det(P')| = 1 \cdot r \cdot (b + r\cos\theta).$$

Hence

$$\begin{aligned}
\iiint\limits_U \operatorname{div}(\mathbf{F}) &= \underset{(r)}{\int_0^a} \underset{(\theta)}{\int_{-\pi/4}^{\pi/4}} \underset{(\psi)}{\int_0^{2\pi}} 2(b + 2r\cos\theta)r(b + r\cos\theta)drd\theta d\psi \\
&= 4\pi \int_{-\pi/4}^{\pi/4} \int_0^a (b^2 r + 3r^2 b\cos\theta + 2r^3\cos^2\theta)drd\theta \\
&= 4\pi \int_{-\pi/4}^{\pi/4} \left(\frac{b^2a^2}{2} + ba^3\cos\theta + \frac{a^4}{2}\cdot\frac{1+\cos 2\theta}{2}\right)d\theta \\
&= \frac{4\pi b^2 a^2}{2}\cdot\frac{\pi}{2} + 4\pi ba^3(\sin\theta)\Big|_{-\pi/4}^{\pi/4} + \frac{4\pi a^4}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{-\pi/4}^{\pi/4} \\
&= \pi^2 b^2 a^2 + 4\sqrt{2}\pi ba^3 + \frac{\pi^2 a^4}{2} + \pi a^4.
\end{aligned}$$

To verify the divergence theorem we calculate the integral over the boundary of U oriented outwards. The boundary of U is obtained by revolving the boundary of A about the z-axis. From Fig. 15.4 we see that the boundary of A consists of an arc B_1 of a circle which revolves into a part of the torus and two straight lines, B_2 and B_3, which revolve into portions of cones. We denote by $\mathbf{S}_i$ the outwardly oriented surface obtained by revolving B_i about the z-axis, $i = 1, 2, 3$. We could proceed to parametrize each of these surfaces using the methods developed earlier, but instead we obtain our parametrizations by *appropriate restrictions* of P. For example B_1 is the boundary of A obtained by letting $r = a$ in the parametrization $(r, \theta) \to (b + r\cos\theta, r\sin\theta)$ of A and if we let $r = a$ in $P(r, \theta, \psi)$ we obtain a parametrization f of S_1 given by

$$f(\theta, \psi) = \big((b + a\cos\theta)\cos\psi, (b + a\cos\theta)\sin\psi, a\sin\theta\big)$$

where $-\pi/4 < \theta < \pi/4$ and $0 < \psi < 2\pi$. Since partial derivatives are calculated by fixing all except one of the variables we have $f_\theta(\theta, \psi) = P_\theta(a, \theta, \psi)$ and $f_\psi(\theta, \psi) = P_\psi(a, \theta, \psi)$. Thus the partial derivatives of f are the final two columns of P' and as the columns of P' are mutually perpendicular the first column is parallel to the normal of S_1. Since the first column of P' is a unit vector it follows that the unit normals to S_1 are

$$\pm(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta).$$

If $\theta = 0$ and $\psi = \pi/2$ we get the point $(0, b+a, 0)$ in S_1 with unit normals at this point given by $\pm(0, 1, 0)$. From Fig. 15.4 the outward normal is $(0, 1, 0)$. Hence the outward normal at any point of $\mathbf{S}_1$ is given by $(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$. Again, using orthogonality of the columns of P we see that

$$\|f_\theta \times f_\psi\| = \|f_\theta\| \cdot \|f_\psi\| = a(b + a\cos\theta).$$

On S_1, $x^2 + y^2 = (b + a\cos\theta)^2\cos^2\psi + (b + a\cos\theta)^2\sin^2\psi = (b + a\cos\theta)^2$ and

$$\begin{aligned}\mathbf{F} &= (a\,(b + a\cos\theta)\cos\psi\cos\theta, a\,(b + a\cos\theta)\sin\psi\cos\theta, a\sin\theta\,(b + a\cos\theta))\\ &= a(b + a\cos\theta)(\cos\psi\cos\theta, \sin\psi\cos\theta, \sin\theta)\\ &= a(b + a\cos\theta)\mathbf{n}.\end{aligned}$$

Hence

$$\begin{aligned}\iint_{\mathbf{S_1}} \mathbf{F} &= \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,\|f_\theta \times f_\psi\| d\theta d\psi\\ &= \int_{\substack{0\\(\psi)}}^{2\pi}\int_{\substack{-\pi/4\\(\theta)}}^{\pi/4} a(b + a\cos\theta)\mathbf{n}\cdot a(b + a\cos\theta)\mathbf{n}\,d\psi d\theta\\ &= 2\pi a^2 \int_{-\pi/4}^{\pi/4} (b^2 + 2ab\cos\theta + a^2\cos^2\theta)d\theta\\ &= 2\pi a^2 \cdot \frac{b^2\pi}{2} + 4\pi a^3 b(\sin\theta)\Big|_{-\pi/4}^{\pi/4} + 2\pi a^4 \int_{-\pi/4}^{\pi/4} \frac{1 + \cos 2\theta}{2} d\theta\\ &= \pi^2 a^2 b^2 + 4\pi a^3 b\sqrt{2} + \pi a^4 \frac{\pi}{2} + \frac{2\pi a^4}{2}\left(\frac{\cos 2\theta}{2}\right)\Big|_{-\pi/4}^{\pi/4}\\ &= \pi^2 a^2 b^2 + 4\sqrt{2}\pi a^3 b + \frac{\pi^2 a^4}{2} + \pi a^4.\end{aligned}$$

The boundary B_2 of A is obtained by letting $\theta = \pi/4$ in the parametrization $(r, \theta) \to (b + r\cos\theta, r\sin\theta)$ of A and hence a parametrization of S_2 is obtained by letting $\theta = \pi/4$ in $P(r, \theta, \psi)$. This gives the parametrization

$$g(r, \psi) = \left(\left(b + \frac{r}{\sqrt{2}}\right)\cos\psi, \left(b + \frac{r}{\sqrt{2}}\right)\sin\psi, \frac{r}{\sqrt{2}}\right)$$

with domain $0 \le \psi \le 2\pi$, $0 \le r \le a$. As before the partial derivatives of g are obtained from the first and third columns of P', with $\theta = \pi/4$, and again, using orthogonality of the rows of P', we see that the unit normals to S_2 are obtained by normalising the second column of P'. The normals are $\pm(\cos\psi, \sin\psi, -1)/\sqrt{2}$. From Fig. 15.4 the outer normal has a positive final coordinate and thus the outer normal at the point $g(r,\psi)$ is $(-\sin\psi, -\cos\psi, 1)/\sqrt{2}$. Figure 15.4 also shows that S_2 is part of the cone $x^2+y^2=(z+b)^2$. On S_2,

$$(x^2+y^2)^{1/2}-b=b+\frac{r}{\sqrt{2}}-b=\frac{r}{\sqrt{2}}.$$

Hence

$$\begin{aligned}\mathbf{F}\,((r,\psi)) &= \left(\left(b+\frac{r}{\sqrt{2}}\right)\cos\psi\cdot\frac{r}{\sqrt{2}}, \left(b+\frac{r}{\sqrt{2}}\right)\sin\psi\cdot\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\left(b+\frac{r}{\sqrt{2}}\right)\right)\\ &= \left(b+\frac{r}{\sqrt{2}}\right)\cdot\frac{r}{\sqrt{2}}\,(\cos\psi, \sin\psi, 1)\,.\end{aligned}$$

Hence

$$\mathbf{F}\cdot\mathbf{n}=\big(b+\frac{r}{\sqrt{2}}\big)\cdot\frac{r}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(-\sin^2\psi-\cos^2\psi+1)=0$$

and

$$\iint_{\mathbf{S_2}}\mathbf{F}=0.$$

The surface $\mathbf{S_3}$ is obtained by letting $\theta=-\pi/4$ and we obtain in the same way a parametrization

$$h\colon (r,\psi)\longrightarrow\left(\left(b+\frac{r}{\sqrt{2}}\right)\cos\psi, \left(b+\frac{r}{\sqrt{2}}\right)\sin\psi, -\frac{r}{\sqrt{2}}\right)$$

with outer normal $\left(-\frac{\cos\psi}{\sqrt{2}}, -\frac{\sin\psi}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ at the point $h(r,\psi)$. It is easily checked that

$$\mathbf{F}\,(h(r,\psi))=\left(b+\frac{r}{\sqrt{2}}\right)\cdot\frac{r}{\sqrt{2}}\,(\cos\psi, \sin\psi, -1)\,,$$

and

$$\mathbf{F}\cdot\mathbf{n}=\left(b+\frac{r}{\sqrt{2}}\right)\frac{r}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\left(-\sin^2\psi-\cos^2\psi+1\right)=0.$$

Hence $\iint_{\mathbf{S_3}}\mathbf{F}=0$. We have thus shown

$$\iiint_U \operatorname{div}(\mathbf{F})=\iint_{\mathbf{S_1}}\mathbf{F}+\iint_{\mathbf{S_2}}\mathbf{F}+\iint_{\mathbf{S_3}}\mathbf{F}$$

and verified the divergence theorem.

This completes our programme on integration theory. We have concentrated on motivating the basic definitions, the concept of orientation, the main theorems (Stokes' Theorem and the Divergence Theorem) and geometrical interpretations. We have neglected mentioning practical and theoretical applications of several-variable integration theory. The methods we have covered are not trivial and need time to be mastered. We urge the reader to have patience and to keep revising until they appear obvious. Further studies, of a pure and applied nature should then be highly rewarding. The remaining chapters of this book are devoted to the geometry of surfaces in $\mathbb{R}^3$. The examples of surfaces that arose in integration theory are both concrete and representative and should be referred to frequently as a means of appreciating and understanding the abstract concepts we meet in the final three chapters.

Exercises

15.1 Prove the divergence theorem for the domain

$$U = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

15.2 Use the divergence theorem to evaluate

$$\iint_{\mathbf{S}} \mathbf{F}$$

where $\mathbf{F}(x, y, z) = (xy^2, x^2y, y)$ and $\mathbf{S}$ is the surface, oriented outwards, of the cylinder $x^2 + y^2 \le 3$ bounded above by the plane $x + z = 0$ and below by the plane $z = 0$.

15.3 Evaluate

$$\iiint_U x dx dy dz$$

where U is the tetrahedron bounded by $x \ge 0$, $y \ge 0$, $z \ge 0$ and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

15.4 Evaluate directly and also using the divergence theorem

$$\iiint_{0 \le x, y, z \le 1} yz^2 e^{-xyz} dx dy dz.$$

15.5 Use the divergence theorem to prove

$$\iint\limits_{\mathbf{S}} (x^2, -y^2, 3xz) = 3\pi$$

where **S** is the outwardly oriented boundary of the region

$$\{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2 + y^2 + z^2 \leq 4\}.$$

15.6 Let S denote a closed surface which bounds the open set V in $\mathbb{R}^3$ and let $\mathbf{n}$ denote the outward normal to S. If f and g have continuous first- and second-order partial derivatives and $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ show that

$$\iint\limits_{\mathbf{S}} \left(f \frac{\partial g}{\partial \mathbf{n}} - g \frac{\partial f}{\partial \mathbf{n}} \right) = \iiint\limits_{V} \left(f \nabla^2 g - g \nabla^2 f \right) dxdydz.$$

15.7 Let Ω denote the region bounded by the rays $\theta = a$ and $\theta = b$ and the curve $r = f(\theta)$. Show that

$$\text{Area } (\Omega) = \frac{1}{2} \int_a^b f^2(\theta) d\theta.$$

Sketch the curve $r = 1 + \cos\theta$ and show that it encloses a region with area $3\pi/2$.

15.8 Find the area between the plane $z = 0$, the paraboloid $z = ax^2 + by^2$ and the cylinder $x^2/c^2 + y^2/d^2 = 1$.

Chapter 16
Geometry of Surfaces in $\mathbb{R}^3$

Summary *Using normal sections we define normal curvature, principal curvatures and Gaussian curvature. Geometric interpretations and a method of calculating the Gaussian curvature using parametrization are given.*

Our geometric study of surfaces in $\mathbb{R}^3$ is motivated by some very simple basic questions such as; what is curvature and how does one measure it? Is there any relationship between surface area and curvature? What is the shortest distance between two points on a surface? We start by taking an intuitive and non-rigorous look at an apparently very special case and this leads us to mathematical concepts which are both useful and natural. The surface we study is one with which we are already familiar and this simple example gives us *everything*. We have already used *all* the techniques that we require, and *all* the facts that we need are known to us—we just have to look at things in a slightly different way. The surface S we consider is the graph of the smooth function f where $f(x, y)$ is the height above sea level. We suppose f has a local maximum at (x_0, y_0) and we study S near $p = \big(x_0, y_0, f(x_0, y_0)\big)$. Since the point (x_0, y_0) is a critical point of f the tangent plane to S at p is the plane through the point $(x_0, y_0, f(x_0, y_0))$ *parallel* to the xy-plane in $\mathbb{R}^3$. The unit normals to the surface at p are $\pm(0, 0, 1)$ and, for convenience, we *choose* $(0, 0, 1)$ as our unit normal $\mathbf{n}(p)$. The tangent plane of S at p consists of all vectors of the form $(v_1, v_2, 0)$.

If our notion of curvature is meaningful it should say something when we take cross-sections of a surface. For instance, if we keep getting circles when we take cross-sections we should not be surprised if the surface is a sphere and if each cross-section is either a line or a plane we expect the surface to be a plane. We consider cross-sections of $\mathbb{R}^3$ through the point p which contain the unit normal at p, $\mathbf{n}(p)$. Since cross-sections of $\mathbb{R}^3$ are two-dimensional this will cut the tangent plane and we can find a unit tangent vector at p, $\mathbf{v}$, such that our cross-section has the form

$$p + \{x\mathbf{v} + y\mathbf{n}(p) : x, y \in \mathbb{R}^2\}.$$

We identify this cross-section of $\mathbb{R}^3$ with $\mathbb{R}^2_{(x,y)}$ by the correspondence

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_16,

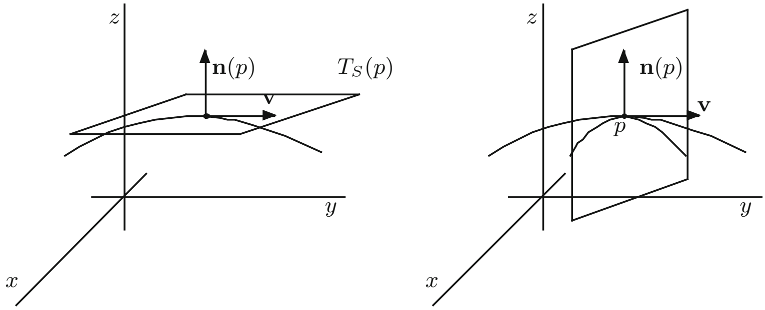


Fig. 16.1

$$p + x\mathbf{v} + y\mathbf{n}(p) \in \mathbb{R}^3 \longleftrightarrow (x, y) \in \mathbb{R}^2.$$

The intersection of this cross-section of $\mathbb{R}^3$ and the surface S is a *curve* on the surface called a *normal section* of the surface at p (Fig. 16.1).

Using this correspondence we may identify the normal section of the surface with a curve Γ in $\mathbb{R}^2$ passing through the origin. We direct Γ by requiring $(0, 1)$ to be the unit normal at the origin and calculate its curvature using the concept of plane curvature in Chap. 7. The curvature of Γ at the origin in $\mathbb{R}^2$ will depend on the unit vector $\mathbf{v}$ and the choice of normal and we denote it by $k_p(\mathbf{v})$. We call $k_p(\mathbf{v})$ the *normal curvature* of S at p in the direction $\mathbf{v}$. By examining $k_p(\mathbf{v})$ as $\mathbf{v}$ ranges over all unit tangent vectors to S at p we hope to draw conclusions on the shape of the surface near p. For example, if $k_p(\mathbf{v}) = 1/r$ for all $\mathbf{v}$ then we might expect the surface to approximate a sphere of radius r *near* p. All normal sections are paths on the surface leading to the point p (i.e. to the top of the mountain) and the different normal curvatures distinguish between the steep and the not so steep paths. Visualising circles of different radii going through p, with p as their highest point, gives us an idea of the shape of the surface near p. Of course it is very laborious to examine all these curves and to calculate all their curvatures so instead we examine them collectively to see if any particular features of the set of *all* normal curvatures captures the essence of the shape near p.

Since f has a local maximum at (x_0, y_0) and $\mathbf{n}(p) = (0, 0, 1)$ *all* normal circles of curvature will lie on the *same* side of the tangent plane and on the opposite side to the normal. This means that all normal curvatures will be negative. Choosing $(0, 0, -1)$ as unit normal at p changes the signs of all normal curvatures. If we were examining a local minimum at (x_0, y_0) and $\mathbf{n}(p) = (0, 0, 1)$ we would have found that all normal curvatures were positive.

But changing the normal or turning the surface upside down—this is equivalent to changing local maxima into local minima and conversely—does not affect the shape of the surface in any way and we conclude: if all normal curvatures are strictly positive or all are strictly negative then the shape of S near p is similar to an ellipsoid (Fig. 16.2). These points are called *elliptic* points of the surface. We can extend

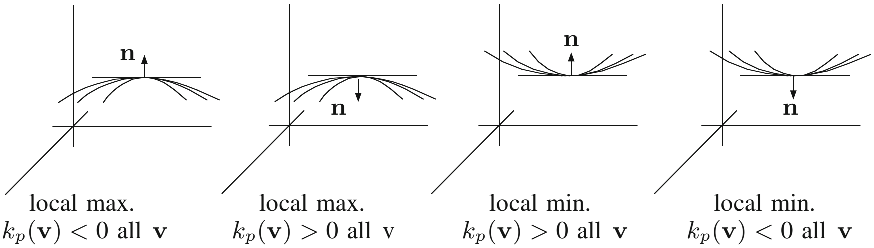


Fig. 16.2

this analysis to arbitrary critical points and clearly if, at a point p, normal curvature takes both positive and negative values then the surface near p is similar in shape to a saddle point. Such points are called *hyperbolic* points of the surface. Our preliminary investigation has shown that information on the shape of the surface can be obtained from the range of values taken by $k_p(\mathbf{v})$ as $\mathbf{v}$ varies over the set of unit tangent vectors at p. Since the range is determined by the extremal values we let $k_1(p) = \max_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$ and $k_2(p) = \min_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$ and call $k_1(p)$ and $k_2(p)$ the *principal curvatures* at p. If the maximum or minimum occurs at a vector $\mathbf{v}$ we call $\mathbf{v}$ a *principal curvature direction*.

We consider the various possibilities that may arise for $k_1(p)$ and $k_2(p)$. If $k_1(p) = k_2(p)$ then we call p an *umbilic point* and if $k_1(p) = k_2(p) = 0$ we call p a *flat spot*. If p is umbilic then $k_p(\mathbf{v}) = k_1(p)$ for all $\mathbf{v}$, $\|\mathbf{v}\| = 1$, and if $k_1(p) \neq 0$ then the surface near p is similar to a portion of a sphere of radius $1/|k_1(p)|$. If p is a flat spot then all normal curvatures are zero and the surface near p is very flat and almost like a part of a plane. At an umbilic point all directions are principal curvature directions.

We now consider non-umbilic points, i.e. $k_1(p) > k_2(p)$. If $k_2(p) > 0$ then for any $\mathbf{v}$, $\|\mathbf{v}\| = 1$, we have $0 < k_2(p) \leq k_p(\mathbf{v})$ and if $k_1(p) < 0$ then $k_p(\mathbf{v}) \leq k_1(p) < 0$ for all $\mathbf{v}$, $\|\mathbf{v}\| = 1$. Hence, in both cases, all normal curvatures have the same sign and near p the surface is shaped like an ellipsoid at one of its extreme points (i.e. like the surface $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ at the point $(0, 0, c)$). If $k_2(p) < 0 < k_1(p)$ then the surface near p is shaped like a saddle point.

To summarise our conclusions more concisely we introduce *Gaussian curvature*.

Definition 16.1 The Gaussian curvature, $K(p)$, at a point p on a surface S, is the product of the principal curvatures, $k_1(p)k_2(p)$.

We have noted already that $k_p(\mathbf{v})$ depends on the choice of normal—changing the normal changes the sign of $k_p(\mathbf{v})$—but since two changes of sign cancel one another it follows that $K(p)$ *does not depend* on the choice of normal.

Although we have only considered critical points on the graph in our analysis we will see shortly that our analysis applies to *all* points on a surface and for this reason we state the following result in its full generality.

Proposition 16.2 *At a non-umbilic point on a surface S in $\mathbb{R}^3$ we have:*

$$K(p) > 0 \iff \textit{near p, S is shaped like an ellipsoid,}$$
$$K(p) < 0 \iff \textit{near p, S is shaped like a saddle point,}$$
$$K(p) = 0 \iff \textit{near p, S is shaped like a cylinder or cone.}$$

At an umbilic point $K(p) \geq 0$ and

$$K(p) > 0 \iff \textit{near p, S is shaped like a sphere,}$$
$$K(p) = 0 \iff \textit{near p, S is very flat.}$$

Consider again the graph of a function f near a critical point (x_0, y_0). We suppose for simplicity that $(x_0, y_0) = (0, 0)$ and $f(0, 0) = 0$. The tangent plane at $p = (x_0, y_0, f(0, 0)) = (0, 0, 0)$ is the xy-plane and consists of the points $\{(v_1, v_2, 0) : v_1, v_2 \in \mathbb{R}\}$, and $\mathbf{n} = \mathbf{n}(p) = (0, 0, 1)$ is our choice of unit normal. Let $\mathbf{v} = (v_1, v_2, 0)$ be a fixed unit vector. The section of $\mathbb{R}^3$ determined by $\mathbf{n}$ and $\mathbf{v}$ is the plane

$$\{(tv_1, tv_2, s) : t, s \in \mathbb{R}\} \tag{16.1}$$

and the graph of f is the set

$$\{(x, y, f(x, y)) : (x, y) \in domain(f)\}. \tag{16.2}$$

The points which satisfy both (16.1) and (16.2) form the normal section of the surface through p defined by $\mathbf{v}$ and are easily seen near p to have the form

$$\{(tv_1, tv_2, f(tv_1, tv_2)),\ t \in [-a, a]\} = \{t\mathbf{v} + f(tv_1, tv_2)\mathbf{n} : t \in [-a, a]\} \tag{16.3}$$

for some positive a.

On identifying the plane in $\mathbb{R}^3$ spanned by $\mathbf{v}$ and $\mathbf{n}$ with $\mathbb{R}^2_{(x,y)}$ we see that the normal section can be parameterised by

$$P(t) = (t, f(tv_1, tv_2)), t \in [-a, a],$$

and may be identified with the graph of the function $h\colon [-a, a] \to \mathbb{R}, h(t) = f(tv_1, tv_2)$. If we parametrize the graph of h by $t \to \big(t, h(t)\big)$ we obtain $(1, 0)$ as unit tangent at $(0, 0)$ and $(0, 1)$ as unit normal (see Chap. 7). Hence the curvature of the graph at the origin, directed by this parametrization, is the normal curvature of S at p in the direction $\mathbf{v}$. By Example 7.1 the curvature at $(t, h(t))$ is

$$\frac{h''(t)}{(1 + h'(t)^2)^{3/2}}.$$

By the chain rule

$$h'(t) = v_1 f_x(tv_1, tv_2) + v_2 f_y(tv_1, tv_2)$$

and

$$h''(t) = f_{xx}(tv_1, tv_2)v_1^2 + 2f_{xy}(tv_1, tv_2)v_1v_2 + f_{yy}(tv_1, tv_2)v_2^2.$$

On letting $t = 0$ we obtain

$$k_p(\mathbf{v}) = f_{xx}(0,0)v_1^2 + 2f_{xy}(0,0)v_1v_2 + f_{y,y}(0,0)v_2^2 = \mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$$

where

$$H_{f(0,0)} = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{pmatrix}$$

is the *Hessian* of f at $(0,0)$. (We have taken the liberty of using $\mathbf{v}$ to denote both (v_1, v_2) and $(v_1, v_2, 0)$ and hope that this does not cause any confusion—it is a practice that we do not recommend except in special circumstances.)

This is familiar territory and we use our knowledge of the function $\mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$ as $\mathbf{v}$ ranges over all unit vectors. In Chap. 4 we showed that the maximum and minimum values of $\mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$ over $\|\mathbf{v}\| = 1$ are the *eigenvalues* of $H_{f(0,0)}$ and are achieved at the corresponding eigenvectors. Hence the eigenvectors of $H_f(0,0)$ are the principal curvature directions and p is non-umbilic if and only if $H_f(0,0)$ has two *distinct* eigenvalues. Since eigenvectors corresponding to different eigenvalues are perpendicular (Exercise 1.21) it follows that at a non-umbilic point there exist *precisely two* principal curvature directions which are perpendicular to one another—hence if we know one we can easily find the other. Note that we do not distinguish between the directions $\mathbf{v}$ and $-\mathbf{v}$.

We now discuss the case previously omitted, i.e. $K(p) = 0$, which corresponds to a degenerate critical point of the function f. If $K(p) = 0$ and p is umbilic then since $K(p) = k_1(p)k_2(p)$ and $k_1(p) = k_2(p)$ it follows that $k_1(p) = k_2(p) = 0$ and p is a *flat spot* and we have already considered this case. If $K(p) = 0$ and p is not umbilic then we have two possibilities, $k_2(p) < 0 = k_1(p)$ and $k_2(p) = 0 < k_1(p)$, and which occurs depends on the choice of normal. Since both have the same geometrical interpretation we just consider the first one. If $\mathbf{v}$ is the principal curvature direction associated with $k_1(p)$, i.e. $k_{\mathbf{v}}(p) = k_1(p)$, then the normal section near p in the direction $\mathbf{v}$ is approximately a straight line. If $\mathbf{w} \perp \mathbf{v} = 0$ then $\mathbf{w}$ is the other principal curvature direction and the normal section in this direction is approximately a circle of radius $1/|k_2(p)|$ on the opposite side of the surface to the normal (Fig. 16.3a, b).

Figure 16.3b suggests a *cylinder* as an example and indeed a cylinder has Gaussian curvature zero at *all* points. For the cylinder in Fig. 16.3c we have $k_1(p) = 0$ and $k_2(p) = -1/r$. In general, if $K(p) = 0$ then all normal circles of curvature will lie on the same side of the tangent plane and all normal curvatures will have the same sign, that is, either all non-negative or all non-positive. The cylinder and cone (see Exercise 16.2) are typical examples of surfaces without umbilics and with $K(p) = 0$. One should, however, not assume that every surface with $K(p) = 0$ is of this type as

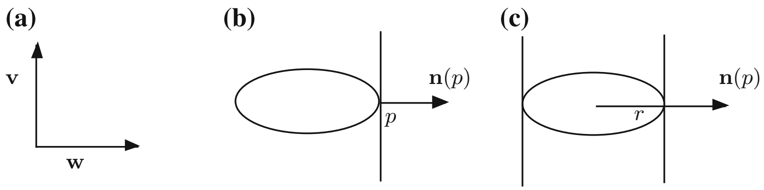


Fig. 16.3

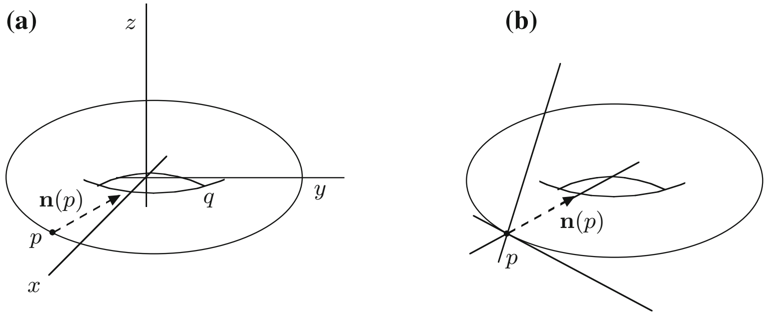


Fig. 16.4

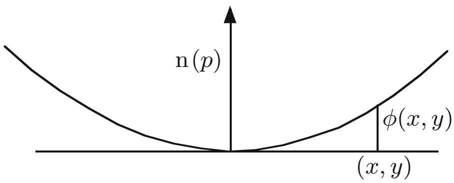


Fig. 16.5

one can find quite strange surfaces with $K(p) = 0$ at isolated points. Our geometric interpretations are meant as a rough guide and as such are reasonably useful in visualising the shape of the surface but do not, of course, explain the full subtlety of many situations—this requires further analysis.

So far we have considered a rather special situation—a critical point on a surface which is the graph of a function—and it is time to see how representative this is of the general situation.

What about non-critical points? Well, if looked at the right way, *every point is a critical point*. For example, consider the torus in $\mathbb{R}^3$ and take p as a typical point on the surface (Fig. 16.4a). To turn p into a critical point we must choose a new coordinate system for $\mathbb{R}^3$. We choose our first two coordinates so that the tangent plane to the surface at p corresponds to the (x, y)-plane and then take the z-direction as one of the normal directions (Fig. 16.4b).

We define a function ϕ on the tangent plane near p by letting $\phi(x, y)$ denote the distance squared from the tangent plane in the $\mathbf{n}(p)$ direction to the surface (Fig. 16.5). This means that the surface S near p is the graph of ϕ and that ϕ, defined on the tangent space near p, has a local minimum at p. One can carry out a similar analysis at any point on the surface, although it is not so easy to sketch at points which become saddle points, e.g. the point q in Fig. 16.4a.

We conclude that our analysis applies *everywhere* on a *graph*. Our examination of graphs, level sets, simple surfaces and surfaces at the end of Chap. 10 shows that these objects are locally equivalent and since normal curvature, principal curvatures and Gaussian curvature are locally defined our results obtained using graphs are valid at all points on all surfaces.

Gaussian curvature is the most important number that we can geometrically associate with a point on a surface. To use it effectively we need to be able to calculate it directly from any parametrization, without first finding normal curvatures or principal curvatures. To find a way to do this we turn again to normal curvatures and results from Chaps. 7 and 8 on the curvature of plane curves. If P is a unit speed parametrized curve in $\mathbb{R}^2$ then

$$P'(t) = T(t), \qquad T'(t) = \kappa(t)N(t), \qquad \kappa(t) = \langle T'(t), N(t)\rangle. \tag{16.4}$$

Let ϕ denote a parametrization of the simple surface S. Let $p \in S, \mathbf{v} \in T_p(S)$, $\|\mathbf{v}\| = 1$, and let $\mathbf{n}(p)$ denote a choice of normal to S at p. Let $P : [-a, a] \longrightarrow \Gamma$ be a parameterized curve in S with $P(0) = p$ and $P'(0) = \mathbf{v}$. If s denotes the length function on Γ defined by P and $Q = P \circ s^{-1}$ then (see Chap. 5 and (8.9)),

$$P''(t) = (s'(t))^2 Q''(s(t)) + s''(t)Q'(s(t)).$$

Since $s'(0) = 1$ and $Q'(s(0)) \perp \mathbf{n}(p)$, (16.4) implies

$$\langle P''(0), \mathbf{n}(p)\rangle = \langle Q''(s(0)), \mathbf{n}(p)\rangle = k_p(\mathbf{v}). \tag{16.5}$$

Since $\Gamma \subset S$ we may suppose

$$P : t \in [-a, a] \mapsto \phi(x(t), y(t))$$

where $t \mapsto x(t)$ and $y \mapsto y(t)$ are real-valued smooth functions on $[-a, a]$. By the chain rule

$$P'(t) = x'(t)\phi_x(x(t), y(t)) + y'(t)\phi_y(x(t), y(t))$$

and

$$\begin{aligned} P''(t) = & x''(t) \cdot \phi_x(x(t), y(t)) + y''(t) \cdot \phi_y(x(t), y(t)) \\ & + (x'(t))^2 \cdot \phi_{xx}(x(t), y(t)) + 2x'(t) \cdot y'(t) \cdot \phi_{xy}(x(t), y(t)) \\ & + (y'(t))^2 \cdot \phi_{yy}(x(t), y(t)). \end{aligned}$$

If $\overline{p} = (x(0), y(0))$, $v_1 = x'(0)$, $v_2 = y'(0)$ and we write, for simplicity, ϕ_x in place of $\phi_x(\overline{p})$, ϕ_{xx} in place of $\phi_{xx}(\overline{p})$, $\mathbf{n}$ in place of $\mathbf{n}(p)$ etc., and let $l := \langle \phi_{xx}, \mathbf{n} \rangle$, $m := \langle \phi_{xy}, \mathbf{n} \rangle = \langle \phi_{yx}, \mathbf{n} \rangle$ and $n := \langle \phi_{yy}, \mathbf{n} \rangle$. Then $P'(0) = \mathbf{v} = v_1\phi_x + v_2\phi_y$ and, since ϕ_x and ϕ_y are tangent vectors, $\langle \phi_x, \mathbf{n} \rangle = \langle \phi_y, \mathbf{n} \rangle = 0$. Hence

$$\begin{aligned}\langle P''(0), \mathbf{n}(p) \rangle = k_p(\mathbf{v}) &= v_1^2 \langle \phi_{xx}, \mathbf{n} \rangle + 2v_1 v_2 \langle \phi_{xy}, \mathbf{n} \rangle + v_2^2 \langle \phi_{yy}, \mathbf{n} \rangle \\ &= l v_1^2 + 2m v_1 v_2 + n v_2^2\end{aligned}$$

for any unit tangent vector $\mathbf{v} \in T_p(S)$. If $\mathbf{v} = v_1\phi_x + v_2\phi_y$ is any non-zero vector in $T_p(S)$ then $\|\mathbf{v}\|^2 = Ev_1^2 + 2Fv_1v_2 + Gv_2^2$ and

$$k_p\left(\frac{v_1\phi_x + v_2\phi_y}{\|v_1\phi_x + v_2\phi_y\|}\right) = \frac{lv_1^2 + 2mv_1v_2 + nv_2^2}{\|\mathbf{v}\|^2} = \frac{lv_1^2 + 2mv_1v_2 + nv_2^2}{Ev_1^2 + 2Fv_1v_2 + Gv_2^2}. \tag{16.6}$$

This is an extremely useful formula, see Exercises 16.4, 16.9 and 17.2.

The principal curvatures are the maximum and minimum values of $lv_1^2 + 2mv_1v_2 + nv_2^2$ on the set $v_1^2 E + 2v_1v_2F + v_2^2 G = 1$. To simplify the notation we let $v_1 = x$ and $v_2 = y$. Then

$$f(x, y) = lx^2 + 2mxy + ny^2 = (x, y)\begin{pmatrix} l & m \\ m & n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$g(x, y) = Ex^2 + 2Fxy + Gy^2 = (x, y)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $EG - F^2 > 0$ it is easily verified that $\{(x, y) : g(x, y) = 1\}$ is a compact subset of $\mathbb{R}^2$ and $\nabla g \neq 0$ on this level set. Hence f achieves its maximum and minimum on $\{(x, y) : g(x, y) = 1\}$ and we may apply the method of Lagrange multipliers. On writing the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ in matrix form we obtain

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{16.7}$$

and we may rewrite this as follows

$$\begin{pmatrix} l - \lambda E & m - \lambda F \\ m - \lambda F & n - \lambda G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16.8}$$

By elementary linear algebra the λ for which (16.8) has non-zero solutions (x, y) satisfy the quadratic equation

$$(l - \lambda E)(n - \lambda G) - (m - \lambda F)^2 = 0$$

or equivalently

$$\lambda^2(EG - F^2) - \lambda(En - 2Fm + Gl) + ln - m^2 = 0. \tag{16.9}$$

If λ_1 and λ_2 solve (16.9) then, since $K(p) = \lambda_1\lambda_2$, we have proved the following result.

Proposition 16.3 *The Gaussian curvature at a point p on a surface S is*

$$K(p) = \frac{ln - m^2}{EG - F^2}. \tag{16.10}$$

By (16.7) the principle curvatures satisfy

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and we see that the principle curvatures are eigenvalues of a symmetric matrix and we recover, using Exercise 1.21, a result already observed using graphs: *at a non-umbilic point the principal curvature directions are perpendicular to one another.*

The appearance of $EG - F^2$, previously encountered while calculating surface area in Proposition 16.3, suggests a relationship between Gaussian curvature and surface area. This comes to light in the Gauss–Bonnet theorem (Chap. 18).

Example 16.4 We calculate the Gaussian curvature of the surface $z = xy$. This surface is the graph of the function $f(x, y) = xy$ and we obtain a parametrization ϕ by letting $\phi(x, y) = (x, y, xy)$. We have

$$\begin{gathered} \phi_x = (1, 0, y), \qquad \phi_y = (0, 1, x) \\ E = \phi_x.\phi_x = 1 + y^2, \quad F = \phi_x.\phi_y = xy, \quad G = \phi_y.\phi_y = 1 + x^2 \\ EG - F^2 = (1 + y^2)(1 + x^2) - x^2y^2 = 1 + x^2 + y^2 \end{gathered}$$

$$\phi_x \times \phi_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1)$$

$$\begin{gathered} \mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|} = \frac{(-y, -x, 1)}{(1 + y^2 + x^2)^{1/2}} \\ \phi_{xx} = (0, 0, 0), \quad \phi_{xy} = (0, 0, 1), \quad \phi_{yy} = (0, 0, 0) \\ l = \langle \phi_{xx}, \mathbf{n} \rangle = 0, \quad m = \langle \phi_{xy}, \mathbf{n} \rangle = \frac{1}{(1 + x^2 + y^2)^{1/2}}, \quad n = \langle \phi_{yy}, \mathbf{n} \rangle = 0. \end{gathered}$$

Hence, if $p = (x, y, xy)$, then

$$K(p) = \frac{-m^2}{EG - F^2} = \frac{-1}{(1 + x^2 + y^2)^2}.$$

Since the Gaussian curvature is always strictly negative the surface $z = xy$ consists entirely of non-umbilic points which, looked at critically, are saddle points.

Although, in general, it may be difficult to find principal curvatures and principal curvature directions it is possible in special cases and this is the content of the following proposition. We say that a curve Γ on a surface is a *line of curvature* if its tangents are principal curvature directions at each point.

Proposition 16.5 *If $\phi\colon U \subset \mathbb{R}^2 \to S$ is a parametrized surface then the coordinate curves of ϕ are lines of curvature if and only if $F = m = 0$ at all non-umbilic points. In this case, at non-umbilic points, the principal curvatures are l/E and n/G.*

Proof Fix $p \in S$ and suppose the principles curvatures at p are λ_1 and λ_2, $\lambda_1 \neq \lambda_1$. Since the lines of curvature passing through p cross one another at right angles it follows, if the coordinate curves are lines of curvature, that $F = \phi_x \cdot \phi_y = 0$. Since the vectors $(1, 0)$ and $(0, 1)$ solve (16.8) we obtain after making these substitutions $l - \lambda E = m = n - \lambda G = 0$. In particular, we see that $m = 0$ and note also $\{\lambda_1, \lambda_2\} = \{\frac{l}{E}, \frac{n}{G}\}$.

Conversely, suppose $F = m = 0$ at all non-umbilic points. By (16.6)

$$k_p\left(\frac{v_1\phi_x + v_2\phi_y}{\|v_1\phi_x + v_2\phi_y\|}\right) = \frac{lv_1^2 + nv_2^2}{Ev_1^2 + Gv_2^2}.$$

when $\mathbf{v} = v_1\phi_x + v_2\phi_y \neq 0$. The maximum and minimum of $f(v_1, v_2) = lv_1^2 + nv_2^2$ on the set $Ev_2 + Gv_2^2 = 1$ are easily found using Lagrange multipliers and we see that $\{\lambda_1, \lambda_2\} = \left\{\frac{l}{E}, \frac{n}{G}\right\}$. Since $\|\phi_x\| = \sqrt{E}$ and $\|\phi_y\| = \sqrt{G}$ this implies

$$k_p\left(\frac{\phi_x}{\sqrt{E}}\right) = \frac{l}{E} \text{ and } k_p\left(\frac{\phi_y}{\sqrt{G}}\right) = \frac{n}{G}$$

and the coordinate curves are lines of curvature. This completes the proof. □

Example 16.6 The surface obtained by rotating the graph of $h\colon (a, b) \to \mathbb{R}$ (Example 10.4) is parametrized by

$$P(t, \theta) = (t, h(t)\cos\theta, h(t)\sin\theta).$$

For this parametrization $P_t = (1, h'\cos\theta, h'\sin\theta)$ and $P_\theta = (0, -h\sin\theta, h\cos\theta)$. Hence $E = 1 + (h')^2$, $F = 0$, $G = h^2$,

$$P_{tt} = (0, h''\cos\theta, h''\sin\theta)$$

$$P_{t\theta} = (0,\ -h'\sin\theta,\ h'\cos\theta)$$

and

$$P_{\theta\theta} = (0,\ -h\cos\theta,\ -h\sin\theta).$$

Moreover,

$$P_t \times P_\theta = (h'h,\ -h\cos\theta,\ -h\sin\theta)$$

and

$$\mathbf{n} = \frac{(h',\ -\cos\theta,\ -\sin\theta)}{\big(1+(h')^2\big)^{1/2}}.$$

Hence

$$\begin{aligned} l &= \langle P_{tt}, \mathbf{n}\rangle = \frac{-h''}{\big(1+(h')^2\big)^{1/2}} \\ m &= \langle P_{t\theta}, \mathbf{n}\rangle = 0 \\ n &= \langle P_{\theta\theta}, \mathbf{n}\rangle = \frac{h}{\big(1+(h')^2\big)^{1/2}}. \end{aligned}$$

By Proposition 16.5 the coordinate curves are lines of curvature and the principal curvatures and Gaussian curvature are

$$\frac{l}{E} = \frac{-h''}{\big(1+(h')^2\big)^{3/2}}, \quad \frac{n}{G} = \frac{1}{h\big(1+(h')^2\big)^{1/2}} \quad \text{and} \quad K = \frac{-h''}{h\big(1+(h')^2\big)^2}.$$

An interesting case occurs when $h(t) = c\cosh(t/c)$. This is the shape assumed by a *hanging chain* and is called a *catenary*. The surface of revolution is called a *catenoid*. Using the identities

$$\frac{d}{dt}(\cosh t) = \sinh t, \quad \frac{d}{dt}(\sinh t) = \cosh t, \quad \cosh^2 t - \sinh^2 t = 1,$$

we see that the principal curvatures for the catenoid are $\pm 1/c\cosh^2(t/c)$. Hence $k_1 + k_2 = 0$. A surface with this property is called a *minimal surface*. This terminology arose in the following way. Take a closed curve in $\mathbb{R}^3$—shaped, for instance, from a piece of wire—and place a bubble over it. This will assume a certain shape in order to *minimise* a physical quantity on the boundary called surface tension. The shape assumed by the bubble is a minimal surface. The catenoid is the only surface of revolution which is a minimal surface. On minimal surfaces $k_2(p) = -k_1(p)$ and hence

$$K(p) = k_1(p)k_2(p) = -k_1(p)^2 \le 0.$$

The quantity $\dfrac{k_1(p)+k_2(p)}{2}$ is called the *mean curvature*.

In this chapter we have covered a lot of theoretical and practical material in identifying and calculating Gaussian curvature. In the next chapter we will summarise the information we already have on this important concept and discuss further geometric implications.

Exercises

16.1 Calculate the Gaussian curvature at an arbitrary point of the helicoloid parametrized by

$$P(t, \theta) = (t \cos \theta, t \sin \theta, b\theta)$$

$t \in \mathbb{R}, \theta \in (0, 2\pi)$ and b a non-zero real number.

16.2 Find the principal curvatures and the Gaussian curvature at an arbitrary point on the cone $z^2 = x^2 + y^2$. Show that the cone has no umbilics.

16.3 Let S denote the surface parametrized by $\phi(u, v) = (u, v, u^2+v^2)$, $(u, v) \in \mathbb{R}^2$. Show that the curve Γ parametrized by

$$P(t) = \phi(t^2, t), \quad -\frac{1}{2} < t < 2,$$

lies in S. Find the unit tangent to Γ at $P(1)$ and find the (absolute) normal curvature to S at $P(1)$ in the direction $P'(1)$.

16.4 If $P: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrization of a simple surface S show that a point p is an umbilic point if and only if there exists a real number α such that $(l, m, n) = \alpha(E, F, G)$ where each term is evaluated at p. Hence find all umbilics on the surface $z = xy$.

16.5 Find the elliptic and hyperbolic points on the surface parametrized by $\phi(u, v) = (u, v, u^3 + v^3)$.

16.6 Prove that the surface parametrized by

$$f(s, t) = (\cos s, 2 \sin s, t), \quad 0 < s < 2\pi, \ t > 0$$

has constant Gaussian curvature. Describe this surface.

16.7 If the surface S is defined by the equations

$$x = a(u + v), \quad y = b(u - v), \quad z = uv$$

show that the coordinate curves are straight lines.

16.8 Find the Gaussian curvature at an arbitrary point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

16.9 Show that a surface S is a minimal surface if and only if

$$En + Gl = 2Fm.$$

Show that the level set $e^z \cos x = \cos y$ is a minimal surface.

Chapter 17
Gaussian Curvature

Summary *We define the Weingarten mapping or shape operator. This gives an intrinsic approach to Gaussian curvature.*

We first recall, from the previous chapter, the concepts introduced and the results obtained concerning the shape of a surface near a point p. All these results were obtained using plane curvature in $\mathbb{R}^2$. We defined or obtained the following:

(a) $k_p(\mathbf{v})$, the *normal curvature* at p in the direction of the unit tangent vector $\mathbf{v}$ at p

(b) *principal curvatures* at p, $k_1(p)$ and $k_2(p)$, where

$$k_1(p) = \max_{\|\mathbf{v}\|=1} k_p(\mathbf{v}) \quad \text{and} \quad k_2(p) = \min_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$$

(c) *principal curvature directions*, i.e. tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that

$$k_p(\mathbf{v}_i) = k_i(p) \quad \text{for } i = 1,\ 2$$

(d) *umbilic points*, i.e. points where $k_1(p) = k_2(p)$, and *flat spots*, i.e. where $k_1(p) = k_2(p) = 0$

(e) at an *umbilic* point all (tangential) directions are principal curvature directions; at a *non-umbilic* point there are precisely two principal curvature directions $\mathbf{v}_1$ and $\mathbf{v}_2$ which are *perpendicular* to one another

(f) *Gaussian curvature* at p

$$K(p) = k_1(p)k_2(p) = \frac{ln - m^2}{EG - F^2}$$

where ϕ is any parametrization,

$$E = \phi_x \cdot \phi_x, \quad F = \phi_x \cdot \phi_y, \quad G = \phi_y \cdot \phi_y$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_17,

$$l = \phi_{xx} \cdot \mathbf{n}, \quad m = \phi_{xy} \cdot \mathbf{n}, \quad n = \phi_{yy} \cdot \mathbf{n}$$

$$\mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}$$

(h) the coordinate curves are *lines of curvature* if and only if $F = m = 0$ at all non-umbilic points.

The Gaussian curvature contains less information than the principal curvatures, that is to say if we know the principal curvatures then we can calculate the Gaussian curvature but from the Gaussian curvature alone we cannot calculate the principal curvatures. Thus, at first glance, it appears that in using Gaussian curvature we may be neglecting important information.

However, experience and subsequent results show that the information lost is generously compensated by other gains. To begin with, Gaussian curvature is a single real number assigned to each point on a surface—the principal curvatures and directions involve two real numbers and two vectors. Thus Gaussian curvature has the advantage of *simplicity*. We have already seen that Gaussian curvature may be easily calculated from any parametrization whereas it may be difficult to calculate the principal curvatures. The principal curvatures depend, up to a factor ± 1, on the choice of normal while Gaussian curvature has the same value for *any* choice of normal. In practice this means that any parametrization may be used to calculate Gaussian curvature while only parametrizations consistent with the choice of normal may be used to find principal curvatures. Indeed, along these lines, we have a celebrated theorem of Gauss—*theorema egregium*—which asserts that Gaussian curvature is an *intrinsic* property of the surface. Roughly speaking this says that Gaussian curvature *may be calculated* directly from functions defined internally on the surface and without using such external properties as the normal or the fact that the surface lies in $\mathbb{R}^3$. Our method of calculating K uses the normal so Gauss' theorem tells us that there is another way of calculating K which does not use the normal. At first glance this may appear a rather minor point but it was this result which paved the way for the development of a very powerful and a very general type of geometry—*Riemannian geometry*—in which the key concepts are *differentiation* and the *length of tangent vectors*. Thus for simplicity, for practical and intrinsic reasons, Gaussian curvature has many advantages. However, in studying surfaces *all* of the concepts we discussed play a useful role and none should be neglected.

So far we have studied the shape of a surface by examining curves (normal sections) of the surface but there are other intuitive approaches to the same problem. Almost invariably they lead back to Gaussian curvature. For example, following the successful approach to plane curvature obtained by taking limits of circles, it is natural to regard the reciprocal of the radius of the sphere that sits closest to the surface near p as a measure of the curvature at p. For surfaces the definition of the *sphere of closest fit* is not so obvious, especially at saddle points, and an indirect approach is taken. We use a normal $\mathbf{n}$ to project the surface near p onto the unit sphere in $\mathbb{R}^3$

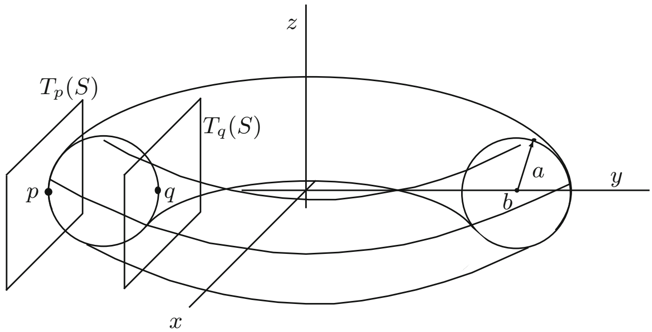


Fig. 17.1

and compare the area of the surface near p with the area of its image. When used in this way $\mathbf{n}$ is called the *Gauss map*. If B_ε is the ball with centre p and radius ε then

$$|K(p)| = \lim_{\varepsilon\to 0}\left|\frac{\text{Area}\,(\mathbf{n}(S\cap B_\varepsilon))}{\text{Area}(S\cap B_\varepsilon)}\right|.$$

Note that if S is a plane then $\mathbf{n}(S)$ consists of a single point and $K(p) = 0$ while if S is a sphere of radius r then $\mathbf{n}(x) = \pm x/r$ for all $x \in S$ and $K(p) = 1/r^2$. Further geometric interpretations appear in the final chapter.

We list now a number of results on Gaussian curvature which give some idea of its uses; more advanced and deeper results are given later. Think about these results, ask yourself if they are geometrically plausible, how they fit in with your intuition, what they say about the surfaces with which you are already familiar, and how you might go about proving them.

(i) If S is a connected surface in $\mathbb{R}^3$ consisting entirely of umbilics then S is either an open subset of a sphere or a plane.
(ii) Every compact surface in $\mathbb{R}^3$ contains a point p with $K(p) > 0$.
(iii) A compact connected surface of *constant Gaussian curvature* is a sphere.
(iv) *Hilbert's Lemma:* If p is a non-umbilic point in S, k_1 has a local maximum at p and k_2 has a local minimum at p, then $K(p) \le 0$.

Example 17.1 In this example we discuss the torus. From Fig. 17.1 we see that at the point p the surface lies on one side of the tangent plane and $K(p) > 0$ while at q it lies on both sides and $K(q) < 0$. From Figs. 17.1 and 17.2 it is clear that $1/a$ will always be a principal curvature and that $1/b - a$ (respectively $1/b + a$) is a principal curvature at p (respectively q). We confirm this in our analysis.

We use toroidal polar coordinates for our parametrization,

$$P : (\theta, \phi) \to ((b + a\cos\theta)\cos\phi,\ (b + a\cos\theta)\sin\phi,\ a\sin\theta)$$

$(\theta, \phi) \in (0, 2\pi) \times (0, 2\pi)$. From Examples 11.1 and 14.5, $E = a^2$, $F = 0$, $G = (b + a\cos\theta)^2$, and

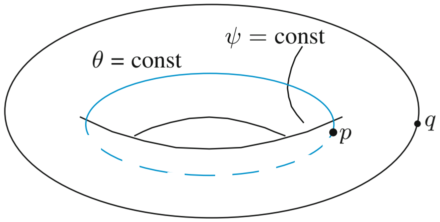


Fig. 17.2

$$\begin{aligned}
\mathbf{n} &= -(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta) \\
P_\theta &= (-a\sin\theta\cos\phi,\ -a\sin\theta\sin\phi,\ a\cos\theta) \\
P_\phi &= (-(b+a\cos\theta)\sin\phi,\ (b+a\cos\theta)\cos\phi,\ 0) \\
P_{\theta\theta} &= (-a\cos\theta\cos\phi,\ -a\cos\theta\sin\phi,\ -a\sin\theta) \\
P_{\theta\phi} &= (a\sin\theta\sin\phi,\ -a\sin\theta\cos\phi,\ 0) \\
P_{\phi\phi} &= (-(b+a\cos\theta)\cos\phi,\ -(b+a\cos\theta)\sin\phi,\ 0)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
l &= \langle P_{\theta\theta}, \mathbf{n}\rangle = a\cos^2\theta\cos^2\phi + a\cos^2\theta\sin^2\phi + a\sin^2\theta = a \\
m &= \langle P_{\theta\phi}, \mathbf{n}\rangle = -a\cos\theta\sin\theta\cos\phi\sin\phi + a\cos\theta\sin\theta\cos\phi\sin\phi = 0 \\
n &= \langle P_{\phi\phi}, \mathbf{n}\rangle = (b+a\cos\theta)(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) \\
&= (b+a\cos\theta)\cos\theta.
\end{aligned}$$

By Proposition 16.5 the coordinate curves are lines of curvature, $\frac{l}{E} = \frac{1}{a}$ and $\frac{n}{G} = \frac{\cos\theta}{b+a\cos\theta}$ are the principal curvatures and

$$K = \frac{ln - m^2}{EG - F^2} = \frac{a(b+a\cos\theta)\cos\theta}{a^2(b+a\cos\theta)^2} = \frac{\cos\theta}{a(b+a\cos\theta)}.$$

We now have a formula for Gaussian curvature and also a diagram (Fig. 17.2) and we may compare and combine them. By differentiating $K(\theta,\ \phi)$, which only depends on θ, we can locate the points of maximum and minimum Gaussian curvature, p and q, and this confirms what the sketch tells us. For the torus $F = m = 0$. Hence $(-a\sin\theta\cos\phi, -a\sin\theta\sin\psi, a\cos\theta)$ and $(-(b+a\cos\theta)\sin\psi, (b+a\cos\theta)$ $\cos\psi, 0)$ are principal curvature directions and the principal curvatures, associated with toroidal polar coordinates, are $1/a$ and $\cos\theta/(b+a\cos\theta)$.

We now consider a more intrinsic approach to Gaussian curvature. If $F : S \longrightarrow \mathbb{R}^n$, where $(S, \mathbf{n})$ is an oriented surface in $\mathbb{R}^3$, $p \in S$ and $\mathbf{v} \in T_p(S)$, let

$$D_{\mathbf{v}}F(p) = \frac{d}{dt}\Big(F\left(P(t)\right)\Big)\bigg|_{t=0} \tag{17.1}$$

where $P : [-a, a] \longrightarrow \Gamma$ is any parametrized curve in S with $P(0) = p$ and $P'(0) = \mathbf{v}$. It can be shown that this definition does not depend on Γ or P. The mapping $\mathbf{v} \longrightarrow D_{\mathbf{v}}F(p)$ is a linear mapping.

Let $\phi : U \longrightarrow S$ denote a parametrization of a part of S. Here we use ϕ_x to denote the mapping $\phi(U) \cap S \longrightarrow \mathbb{R}^3$ given by

$$\phi_x(\phi(x, y)) = \frac{d}{dt}\left(\phi(x+t, y)\right)\bigg|_{t=0}. \tag{17.2}$$

Using (17.1) and (17.2) we define $D_{\phi_x}\phi_x$, $D_{\phi_x}\phi_y$, $D_{\phi_y}\phi_x$ and $D_{\phi_y}\phi_y$. The expressions for these derivatives may initially appear cumbersome, e.g. for instance

$$\left(D_{\phi_y}\phi_x\right)(\phi(x, y)) = \frac{d}{dt}\left(\frac{d}{ds}\phi(x+s, y+t)\bigg|_{s=0}\right)\bigg|_{t=0}.$$

However, since ϕ is a parametrization all its partial derivatives exist and hence, using a two variables Taylor series expansion, we see that

$$\left(D_{\phi_y}\phi_x\right)(\phi(x, y)) = \frac{\partial^2\phi}{\partial y \partial x}(x, y). \tag{17.3}$$

We now use the notation ϕ_{xx}, ϕ_{xy}, ϕ_{yx} and ϕ_{yy}, respectively, in place of $D_{\phi_x}\phi_x$, $D_{\phi_x}\phi_y$, $D_{\phi_y}\phi_x$ and $D_{\phi_y}\phi_y$. By (17.3), $\phi_{xy} = \phi_{yx}$ (see also the introduction to Chap. 4).

The product rule for differentiation and $\langle \mathbf{n}(p), \mathbf{n}(p)\rangle = 1$ imply

$$\langle D_{\mathbf{v}}\mathbf{n}(p), \mathbf{n}(p)\rangle + \langle \mathbf{n}(p), D_{\mathbf{v}}\mathbf{n}(p)\rangle = 0$$

i.e. $\langle D_{\mathbf{v}}\mathbf{n}(p), \mathbf{n}(p)\rangle = 0$ at any point $p \in S$ and any $\mathbf{v} \in T_p(S)$. Hence $D_{\mathbf{v}}\mathbf{n}(p) \perp \mathbf{n}(p)$ and $D_{\mathbf{v}}\mathbf{n}(p)$ belongs to the tangent space at p. This allows us to define a *linear* mapping from the *tangent space* at p into itself by letting

$$L_p : \mathbf{v} \in T_p(S) \longrightarrow -D_{\mathbf{v}}\mathbf{n}(p) \in T_p(S).$$

This important mapping is called the *Weingarten mapping* or *shape operator*. We now suppose that ϕ is consistent with the orientation and write $\mathbf{n}$ in place of $\mathbf{n}\left(\phi(x, y)\right)$. Since

$$\frac{d}{dt}\mathbf{n}(\phi(x+t, y))|_{t=0} = D_{\phi_x}\mathbf{n} = -L_p\phi_x.$$

and $\langle \phi_x, \mathbf{n}\rangle = 0$ we have

$$0 = \frac{d}{dt}\langle \phi_x(x+t, y), \mathbf{n}(\phi(x+t, y))\rangle|_{t=0}$$

$$= \langle \phi_{xx}, \mathbf{n}\rangle + \langle \phi_x,\ D_{\phi_x}\mathbf{n}\rangle$$
$$= \langle \phi_{xx}, \mathbf{n}\rangle - \langle \phi_x,\ L_p\phi_x\rangle$$

and

$$\langle \phi_x,\ L_p\phi_x\rangle = \langle \phi_{xx}, \mathbf{n}\rangle = l.$$

Similarly $m = \langle \phi_y, L_p\phi_x\rangle = \langle \phi_{xy}, \mathbf{n}\rangle = \langle \phi_{yx}, \mathbf{n}\rangle = \langle \phi_x, L_p\phi_y\rangle$ and $\langle \phi_y, L_p\phi_y\rangle = \langle \phi_{yy}, \mathbf{n}\rangle = n$.

Since $\{\phi_x, \phi_y\}$ is a basis for the tangent space at p we can find scalars $\{a, b, c, d\}$ so that $L_p(\phi_x) = a\phi_x + b\phi_y$ and $L_p(\phi_y) = c\phi_x + d\phi_y$. On solving the system of linear equations,

$$\begin{aligned} L_p(\phi_x)\cdot\phi_x &= aE + bF = l\\ L_p(\phi_x)\cdot\phi_y &= aF + bG = m\\ L_p(\phi_y)\cdot\phi_x &= cE + dF = m\\ L_p(\phi_y)\cdot\phi_y &= cF + dG = n \end{aligned}$$

we see that

$$A_p = \frac{1}{EG - F^2}\begin{pmatrix} lG - mF & mE - lF\\ mG - nF & nE - mF \end{pmatrix}$$

is the matrix for L_p with respect to the basis $\{\phi_x, \phi_y\}$ for $T_p(S)$. Since

$$\begin{aligned} (EG - F^2)^2\det(A_p) &= (lG - mF)(nE - mF) - (mE - lF)(mG - nF)\\ &= (EG - F^2)(ln - m^2) \end{aligned}$$

we have proved, in view of Proposition 16.3, the following result.

Proposition 17.2 *If ϕ parametrizes a surface S in $\mathbb{R}^3$ then the determinant of the matrix of the Weingarten mapping at p with respect to the basis for the tangent space induced by ϕ is the Gaussian curvature of S at p.*

If $\mathbf{v} = v_1\phi_x + v_2\phi_y$ and $\mathbf{w} = w_1\phi_x + w_2\phi_y$ then, by the above,

$$\begin{aligned} \langle L_p(\mathbf{v}), \mathbf{w}\rangle &= \langle L_p(v_1\phi_x + v_2\phi_y), w_1\phi_x + w_2\phi_y\rangle\\ &= v_1w_1\langle L_p\phi_x, \phi_x\rangle + (v_1w_2 + v_2w_1)\langle L_p(\phi_x), \phi_y\rangle + v_2w_2\langle L_p(\phi_y), \phi_y\rangle\\ &= \langle \mathbf{v}, L_p(\mathbf{w})\rangle \end{aligned}$$

and $L_p : T_p(S) \longrightarrow T_p(S)$ is a symmetric linear operator. Moreover,

$$\langle L_p(\mathbf{v}), \mathbf{v}\rangle = lv_1^2 + 2mv_1v_2 + nv_2^2$$

and if $\mathbf{v}$ is a unit tangent vector at p then

$$\langle L_p(\mathbf{v}), \mathbf{v}\rangle = k_p(\mathbf{v}). \tag{17.4}$$

Since L_p is symmetric we can choose an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $T_p(S)$ consisting of eigenvectors for L_p. Let β_1 and β_2 denote the corresponding eigenvalues. If $x, y \in \mathbb{R}$, $x^2 + y^2 = 1$ then $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ is a unit tangent vector at p and, by (17.4),

$$\begin{aligned} k_p(\mathbf{v}) &= \langle L_p(x\mathbf{e}_1 + y\mathbf{e}_2), x\mathbf{e}_1 + y\mathbf{e}_2\rangle \\ &= \langle x\beta_1\mathbf{e}_1 + y\beta_2\mathbf{e}_2, x\mathbf{e}_1 + y\mathbf{e}_2\rangle \\ &= x^2\beta_1\langle \mathbf{e}_1, \mathbf{e}_1\rangle + 2xy\langle \mathbf{e}_1, \mathbf{e}_2\rangle + y^2\langle \mathbf{e}_2, \mathbf{e}_2\rangle \\ &= x^2\beta_1 + y^2\beta_2. \end{aligned}$$

This shows that the principal curvatures are the eigenvalues of L_p and the eigenvectors are the principal curvature directions. Since the matrix for L_p with respect to this basis is a diagonal matrix with the eigenvalues as entries we also have $K(p) = \beta_1\beta_2 = \det(L_p)$. We summarise what we have proved in the following proposition.

Proposition 17.3 *At a point p in a surface S the principal curvatures are the eigenvalues of L_p, the principal curvature directions are the eigenvectors of L_p, and the Gaussian curvature, $K(p)$, is the determinant of L_p with respect to any orthonormal basis for $T_p(S)$.*

Example 17.4 If p is umbilic then all normal curvatures are equal. If p is *non-umbilic* and k_1, k_2, $\mathbf{v}_1$, and $\mathbf{v}_2$ are the principal curvatures and the corresponding principal curvature directions then any *unit tangent vector* $\mathbf{v}$ at p has the form

$$\mathbf{v} = \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2$$

for some real number θ. By Proposition 17.3

$$L_p(\mathbf{v}_1) = k_1(p)\mathbf{v}_1 \quad \text{and} \quad L_p(\mathbf{v}_2) = k_2(p)\mathbf{v}_2$$

and, as p is non-umbilic, $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$. By (17.2),

$$\begin{aligned} k_p(\mathbf{v}) = \langle L_p(\mathbf{v}), \mathbf{v}\rangle &= \langle L_p(\cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2), \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2\rangle \\ &= \langle \cos\theta L_p(\mathbf{v}_1) + \sin\theta L_p(\mathbf{v}_2), \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2\rangle \\ &= k_1(p)\cos^2\theta + k_2(p)\sin^2\theta. \end{aligned}$$

This is known as *Euler's formula* and shows that normal curvature in any direction can be recovered from the principal curvatures and the principal curvature directions.

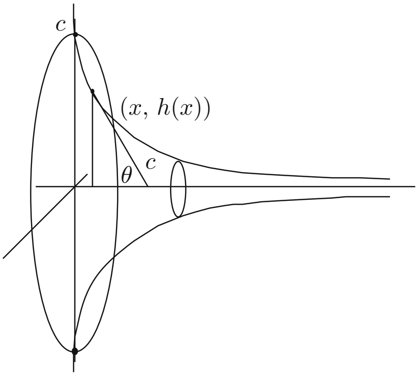


Fig. 17.3

Exercises

17.1 Find the Weingarten mapping for the torus.

17.2 Let $\mathbf{S}$ denote the graph of the function $f(u, v) = u^2 - 2v^2$ oriented by the usual parametrization. Find a curve Γ in $\mathbf{S}$ such that the normal curvature in the tangent direction to Γ is always zero. Hence find a straight line that lies in $\mathbf{S}$.

17.3 Suppose the two oriented surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ intersect in a curve Γ. Let κ denote the curvature of $\Gamma \subset \mathbb{R}^3$ and let λ_i denote the normal curvature of Γ in $\mathbf{S_i}$, $i = 1, 2$. If θ is the angle between the normals to $\mathbf{S}_1$ and $\mathbf{S}_2$ show that

$$\kappa^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos \theta.$$

17.4 Use $-D_{\mathbf{v}}(\mathbf{n})$ to give another solution to Exercise 16.3.

17.5 Show that the average of the normal curvature over all directions is the mean curvature.

17.6 Consider the plane curve h described by the following geometric condition: start at the point $(0, c)$ and move so that the tangent line always reaches the x-axis after traveling a distance c. From Fig. 17.3 deduce that

$$h' = -\frac{h}{\sqrt{c^2 - h^2}}.$$

(Note that $\sin \theta = -h/c$, $\tan \theta = h' = \dfrac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$.) Show that

$$h'' = \frac{c^2 h}{(c^2 - h^2)^2}.$$

Orient the surface of revolution, B, of this curve so that the principal curvatures are h'/c and $-1/ch'$. Hence deduce that B has constant negative Gaussian curvature. Sketch the surface B—it may help you to know that it is called the *bugle surface*.

Chapter 18
Geodesic Curvature

Summary *We define geodesic curvature and geodesics. For a curve on a surface we derive a formula connecting intrinsic curvature, normal curvature and geodesic curvature. We discuss paths of shortest distance, further interpretations of Gaussian curvature and introduce, informally and geometrically, a number of important results in differential geometry*.

Our study of normal curvature was based on identifying the normal of a curve with the normal of the surface. It is also possible to arrange things so that the normal to a curve in a surface is a tangent vector to the surface. This leads to a new type of curvature, *geodesic curvature*, that we discuss and interpret in this chapter.

Let $\mathbf{S}$ denote an oriented surface in $\mathbb{R}^3$ with smooth unit normal $\mathbf{n}$ and let $P\colon [a, b] \to \Gamma \subset \mathbf{S}$ denote a unit speed parametrized curve on the surface. Since the tangent space at each point on the surface is two-dimensional and $P'(t)$ is a tangent vector at $P(t)$ it follows that there are precisely two unit tangent vectors at $P(t)$ which are perpendicular to $P'(t)$. We distinguish between them by using the normal to the surface at $P(t)$, $\mathbf{n}\big(P(t)\big)$, and define the *surface normal* to Γ at $P(t)$, $\mathbf{n_S}(P(t))$ to be $\mathbf{n}(P(t)) \times T(t)$. To simplify our notation we sometimes write $\mathbf{n}(t)$ and $\mathbf{n_S}(t)$ in place of $\mathbf{n}\big(P(t)\big)$ and $\mathbf{n_S}(P(t))$, respectively. By construction

$$\{P'(t) = T(t), \mathbf{n}_S(t), \mathbf{n}(t)\}$$

is a right-handed orthogonal system, and in particular, an orthonormal basis for $\mathbb{R}^3$. In Chap. 7 we encountered a similar situation when we obtained the orthonormal basis $\{T, N, B\}$ at a point on a curve Γ. In that case we proceeded to obtain the Frenet–Serret equations by differentiation and using properties of the orthonormal basis. We follow *precisely* the same path to a similar end here and obtain real-valued functions $a(t)$, $b(t)$ and $c(t)$ such that

$$\begin{pmatrix} T(t) \\ \mathbf{n_S}(t) \\ \mathbf{n}(t) \end{pmatrix}' = \begin{pmatrix} 0 & a(t) & c(t) \\ -a(t) & 0 & b(t) \\ -c(t) & -b(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ \mathbf{n_S}(t) \\ \mathbf{n}(t) \end{pmatrix}. \tag{18.1}$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_18,

In particular we see that

$$\mathbf{n}'_{\mathbf{S}}(t) = -a(t)T(t) + b(t)\mathbf{n}(t) \tag{18.2}$$

and the "part" of $\mathbf{n}'_{\mathbf{S}}(t)$ which "lies" in the tangent space at $P(t)$ is parallel to $T(t)$. This equation is similar to Eq. 7.11 in Chap. 7. We define the *geodesic curvature* of Γ at $P(t)$, $\kappa_g(t)$, to be $a(t)$. By (18.2)

$$\kappa_g(t) = -\langle \mathbf{n}'_{\mathbf{S}}(t), T(t)\rangle = a(t)\,.$$

The entry $b(t)$ is called the *geodesic torsion* of Γ at $P(t)$ and written $\tau_g(t)$. Rewriting Eq. (18.2) we obtain

$$\mathbf{n}'_{\mathbf{S}}(t) = -\kappa_g(t)T(t) + \tau_g(t)\mathbf{n}(t)\,.$$

From (18.1) and the definition of normal curvature we have

$$c(t) = -\langle \frac{d}{dt}\mathbf{n}\big(P(t)\big), T(t)\rangle = k_{P(t)}\big(T(t)\big)$$

and $c(t)$ is the *normal curvature* at $P(t)$ in the direction $T(t)$. Hence, rewriting (18.1) we obtain

$$\begin{pmatrix} T(t) \\ \mathbf{n}_{\mathbf{S}}(t) \\ \mathbf{n}(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ \mathbf{n}_{\mathbf{S}}(t) \\ \mathbf{n}(t) \end{pmatrix} \tag{18.3}$$

where we have written $\kappa_n(t)$ in place of $k_{P(t)}\big(T(t)\big)$.

By (18.3)

$$T'(t) = \kappa_g(t)\mathbf{n}_{\mathbf{S}}(t) + \kappa_n(t)\mathbf{n}(t)\,. \tag{18.4}$$

but if Γ, as a parametrized curve in $\mathbb{R}^3$ has strictly positive curvature, we can add to this, by using (7.1′) and (18.4), and obtain

$$T'(t) = \kappa(t)N(t) = \kappa_g(t)\mathbf{n}_{\mathbf{S}}(t) + \kappa_n(t)\mathbf{n}(t)\,. \tag{18.5}$$

To avoid confusion between the different types of curvature we call κ the *intrinsic curvature* of Γ and use N to denote the normal to Γ in $\mathbb{R}^3$ whenever it exists. By (18.5) and Pythagoras' Theorem,

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)\,.$$

Equation (18.5) is a decomposition of the intrinsic curvature into its normal and tangential components and establishes a relationship between the three different kinds of curvature. If we consider curvature as a measure of "bending" towards the normal then, since we have chosen our normal $\mathbf{n_S}(t)$ to lie in the tangent space, this and (18.5) suggest that we consider geodesic curvature as the surface curvature of the parametrized curve. So far we have only considered a unit speed parametrized curve. We define the geodesic curvature of a parametrized curve in an oriented surface as the geodesic curvature of a unit speed reparametrization of the curve which preserves the original sense of direction. Note that our definition of geodesic curvature is based, as was normal curvature, on curvature in $\mathbb{R}^2$—see the introduction to Chap. 7. Since the two unit normals at any point on a surface are parallel, Eq. (18.4) shows that $|\kappa_g|$—the *absolute geodesic curvature*—does not depend on either the choice of normal or parametrization. As surfaces can be covered by simple, and hence orientable, surfaces it follows that absolute geodesic curvature is *well defined* for *any* curve on *any* orientable surface.

Curves with $\kappa_g = 0$ are said to have *zero geodesic curvature*. We give a geometrical interpretation of this phenomena and afterwards discuss a practical method for identifying such curves. If we are dealing with a curve in $\mathbb{R}^2$ then zero curvature implies that the curve is a straight line. If we identify $\mathbb{R}^2$ with the oriented surface $\mathbb{R}^2_{(x,y)}$ in $\mathbb{R}^3$ and consider a curve Γ in $\mathbb{R}^2$ as a curve in $\mathbb{R}^3$ then we see easily that its plane curvature in $\mathbb{R}^2$ coincides with its geodesic curvature in $\mathbb{R}^2_{(x,y)}$. In $\mathbb{R}^2$ we also note that a curve is a straight line if and only if it follows the shortest route between any pair of its points. Now the *tangent plane* is the *closest plane* to the surface near a given point p and since geodesic curvature is essentially curvature on the tangent plane it is at least plausible that zero geodesic curvature implies that Γ follows the *shortest path on the surface* between points on Γ close to p. This is indeed the case.

Formally we have the following definitions and results. A surface S is *connected* if between any two points p and q on S there exists at least one path (or directed curve) with initial point p and final point q. We define the *distance* between p and q, $d(p,q)$, as

$$\inf\{\text{length}(\gamma),\ \gamma \text{ is a path with initial point } p \text{ and final point } q\}\,.$$

A path γ joining p and q is a shortest path if

$$d(p,q) = \text{length}(\gamma)\,.$$

Shortest paths may or may not exist and if they exist they may not be unique. For instance it is easily seen that there is no shortest path on the surface $S = \{(x, y, 0) : 0 < x^2 + y^2 < 2\}$ in $\mathbb{R}^3$ joining the points $(-1, 0, 0)$ and $(1, 0, 0)$ although it is easy to see that the distance between them is 2. On the other hand there exist an infinite number of shortest paths on a sphere joining the North and South poles (any line of longitude is a shortest path). Equation 18.5 leads to a simple practical criterion for identifying unit speed curves of zero geodesic curvature since it is easily seen, using (18.5), that $P''(t)$ (or $N(t)$) is parallel to $\mathbf{n}(t)$, for any choice of normal, if and only if $\kappa_g(t) = 0$. This motivates the following definition.

Definition 18.1 A parametrized curve $P\colon [a, b] \to \Gamma \subset S$, where S is a surface in $\mathbb{R}^3$, is a geodesic in S if $P''(t)$ (the acceleration of the parametrization) is parallel to the normal to the surface at $P(t)$ for all t.

Our remarks above show that *a unit speed parametrized curve is a geodesic if and only if it has zero geodesic curvature.*

Proposition 18.2 *A parametrized curve $P\colon [a, b] \to \Gamma \in S$ is a geodesic if and only if it has constant speed and zero geodesic curvature.*

Proof We may assume, without loss of generality, that $[a, b] = [0, \alpha]$. We first suppose that the parametrized curve is a geodesic. We have

$$\frac{d}{dt}\langle P'(t), P'(t)\rangle = 2\langle P''(t), P'(t)\rangle.$$

Since the curve lies in S, $P'(t)$ is a tangent vector at $P(t)$, and as P is a geodesic $P''(t)$ is perpendicular to every tangent vector. This implies $\langle P'(t), P''(t)\rangle = 0$ and $\frac{d}{dt}\left(\|P'(t)\|^2\right) = 0$. Hence $\|P'(t)\|$ is a constant function of t and the parametrization has constant speed c. This implies that the parametrization $Q : [0, c\alpha] \to \Gamma \in S$, where $Q(t) = P(t/c)$, is unit speed. Since $Q''(t) = P''(t/c)/c^2$, we see that $Q''(t)$ is parallel to $P''(t/c)$ and hence to $\mathbf{n}(P(t/c)) = \mathbf{n}(Q(t))$.

Conversely suppose $P : [0, \alpha] \longrightarrow S$ has constant speed, c, and zero geodesic curvature. Then $Q : [0, c\alpha] \longrightarrow S$ where $Q(t) = P(t/c)$ is unit speed and has zero geodesic curvature. By (18.4), $Q''(t)$ is parallel to $\mathbf{n}(t/c)$ and since $P''(t/c)$ and $Q''(t)$ are parallel this shows that the curve parametrized by P is a geodesic. This completes the proof. □

From the above considerations it is not difficult to show that a parametrized curve is a geodesic if and only if it satisfies a certain ordinary differential equation. Existence theorems for ordinary differential equations show that any surface admits an abundance of geodesics. The following is true.

Proposition 18.3 *If S is a surface in $\mathbb{R}^3$, $p \in S$ and $\mathbf{v} \in T_p(S)$ is non-zero then there exists $\varepsilon > 0$ and a unique geodesic $P\colon [-\varepsilon, \varepsilon] \to S$ such that $P(0) = p$ and $P'(0) = \mathbf{v}$.*

Example 18.4 Let S_2 denote the unit sphere with centre at the origin in $\mathbb{R}^3$. Let $\mathbf{v}$ and $\mathbf{w}$ denote perpendicular unit vectors in $\mathbb{R}^3$ and let

$$P(t) = \cos(at)\mathbf{v} + \sin(at)\mathbf{w}, \quad t \in \mathbb{R}$$

where a is a fixed real number. By Pythagoras' Theorem P defines a parametrized curve in S_2. We have

$$P''(t) = -a^2\cos(at)\mathbf{v} - a^2\sin(at)\mathbf{w} = -a^2 P(t)\,.$$

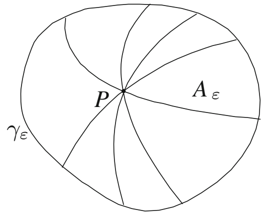


Fig. 18.1

Since $\mathbf{n}(t) = \pm P(t)$ for the unit sphere with centre at the origin this implies $P''(t) \parallel \mathbf{n}(t)$ for all t and P is a geodesic. Proposition 18.3 shows that a geodesic is completely determined by its position and velocity at a single point. Since $P(0) = \mathbf{v}$ and $P'(0) = a\mathbf{w}$ it follows that we have found *all* geodesics on the unit sphere.

We return to the problem we started with—the existence of a shortest path. Propositions 18.2 and 18.3 show that there are many curves on a surface with zero geodesic curvature. From this it is possible to prove the following result which shows, at least locally, that there are always shortest paths.

Proposition 18.5 *If p is a point on a surface S in $\mathbb{R}^3$ then there exists $\varepsilon > 0$ such that for any q in S, $d(p,q) < \varepsilon$, there is a unique shortest path in S joining p and q. This path has zero geodesic curvature and may be parametrized as a geodesic.*

Geodesics also lead to a new derivation of Gaussian curvature. Take a point p on the surface S. An extension of Proposition 18.3 shows that there exists $\varepsilon > 0$ such that for every unit tangent vector $\mathbf{v}$ at p the geodesic with initial point p and initial velocity $\mathbf{v}$ is defined on $[0, \varepsilon]$. If we consider the set of positions taken at time ε by *all* unit speed geodesics starting at p we obtain a curve γ_ε in S. We denote the inside of this curve by A_ε (Fig. 18.1).

Thus γ_ε consists of those points in S whose distance to p is ε and A_ε are the points whose distance to p is less than ε. It can be shown that γ_ε is a closed subset of S and A_ε is open. If S is flat then γ_ε is a circle of radius ε and length $2\pi\varepsilon$ and A_ε is a disc of area $\pi\varepsilon^2$. Hence the quantities $2\pi\varepsilon - l(\gamma_\varepsilon)$ and $\pi\varepsilon^2 - \mathrm{A}(A_\varepsilon)$ where l = length and A = Area are some measure of the curvature of S. In fact, taking limits we obtain the following

$$K(p) = \lim_{\varepsilon\to 0} \frac{3}{\pi\varepsilon^3}\big(2\pi\varepsilon - l(\gamma_\varepsilon)\big) = \lim_{\varepsilon\to 0} \frac{12}{\pi\varepsilon^4}\big(\pi\varepsilon^2 - A(A_\varepsilon)\big).$$

Thus both normal and geodesic curvature lead quite naturally to Gaussian curvature. It is also worth noting that the two most important geometrical concepts that we associated with a surface—Gaussian curvature and geodesics—both turned out to be local properties independent of any orientations used for calculations or motivation.

To complete our introduction to the geometry of surfaces we present without proof some rather remarkable results involving the concepts we have introduced.

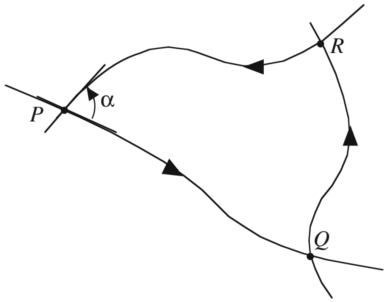


Fig. 18.2

These results are easily stated while the proofs are rather involved. Even in the absence of any ideas regarding the proofs it is well worth thinking about these results and their geometric significance. You may consider verifying them on some of the classical surfaces we have studied, e.g. the sphere, ellipsoid or torus. Trying to prove them will take some time but if you make the effort and are patient you will learn a lot regardless of how successful you are in completing the proofs.

The first result we discuss is a generalization of the well-known result in Euclidean geometry which says that the sum of the interior angles in a (plane) triangle is equal to π. This corresponds to the case where the surface is a plane and the Gaussian curvature is zero. A *triangle* in a surface is a simple closed oriented curve formed by three smooth directed curves (Fig. 18.2).

Each of these smooth curves is called an *edge*, the edges meet at a *vertex* and the interior of the triangle is called its *face*. A triangle is called a *geodesic triangle* if each edge is a geodesic. In the plane, geodesics are straight lines so the usual triangle in the plane is a geodesic triangle. At the vertices P, Q and R the tangents of the two curves which meet are in the same tangent space on the surface and we can define the *angle* between them. It is also possible to define what we mean by an *interior angle* (e.g. the angle α at the vertex P). Let A denote the face of the triangle.

Theorem 18.6 (Local Gauss–Bonnet Theorem) *In a geodesic triangle on a simple surface*

$$\iint_A K = \sum \textit{interior angles} \ -\pi.$$

Stokes' Theorem plays an important role in the proof. For example consider the sphere of radius r. By Example 18.4 the lines of longitude and the equator are geodesics. Hence the triangle formed by the lines of longitude corresponding to $\psi = 0$ and $\psi = \pi/2$ and the equator are a geodesic triangle (Fig. 18.3).

The three interior angles are all $\pi/2$ and the area of the triangle is $\frac{1}{8} \times$ (total area of the sphere) and hence equal to $4\pi r^2/8$. Since the Gaussian curvature is $1/r^2$ this implies

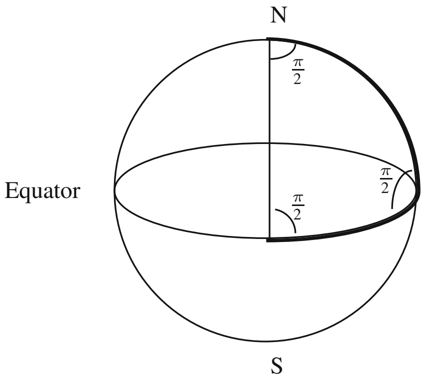


Fig. 18.3

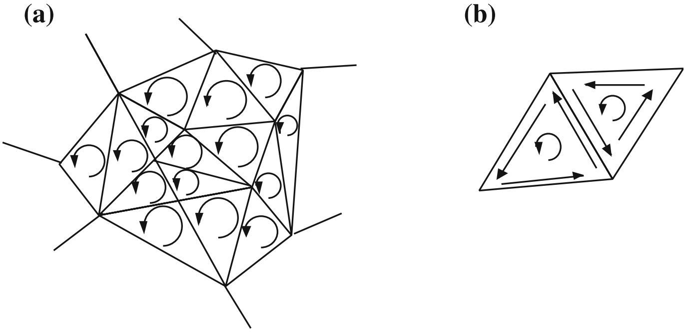


Fig. 18.4

$$\iint_A K = \frac{1}{r^2} \cdot \frac{4\pi r^2}{8} = \frac{\pi}{2}$$

while

$$\sum \text{interior angles} - \pi = \frac{3\pi}{2} - \pi = \frac{\pi}{2}$$

and we have verified the local Gauss–Bonnet Theorem for this triangle.

Any compact (i.e. closed and bounded) oriented surface **S** in $\mathbb{R}^3$ can be partitioned into a finite number of triangles each of which is oriented in an anticlockwise direction about the normal (Fig. 18.4a). This means that an edge which is in two triangles has opposite orientations in each (Fig. 18.4b). Let V denote the total number of vertices, E the total number of edges and F the total number of faces.

A remarkable result of Euler says that no matter how we partition the surface into (not necessarily geodesic) triangles the quantity $V - E + F$ remains unchanged. We call this number the *Euler–Poincaré characteristic* of S and denote it by $\chi(S)$. As

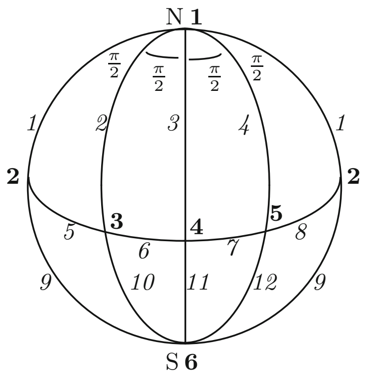


Fig. 18.5

a consequence of the local Gauss–Bonnet Theorem we have the following global result:

Theorem 18.7 (Global Gauss–Bonnet Formula) *If S is a compact oriented surface then*

$$\iint_S K = 2\pi\chi(S).$$

We call $\iint_S K$ the *total curvature* of the surface and it is remarkable that this quantity is always an integer multiple of 2π. Applying this to a sphere of radius r we find

$$\iint_S K = \frac{1}{r^2}\text{ Surface Area (sphere)} = \frac{4\pi r^2}{r^2} = 2\pi\chi(S)$$

and see that $\chi(\text{sphere}) = 2$.

On the other hand we may partition the sphere by the lines of longitude corresponding to $0, \pm\pi/2, \pi$ and the equator (Fig. 18.5). The outer edges are on the back of the sphere and coincide with 180° East or 180° West (i.e. the international date line). By counting we get $V = 6$, $E = 12$ and $F = 8$ and again

$$\chi\ (\text{sphere}) = 6 - 12 + 8 = 2\,.$$

Triangles can be replaced by rectangles in calculating the Euler-Poincaré characteristic since each rectangle can be partitioned into triangles using diagonals. This doubles the number of faces F and adds F new edges. Overall the sum $V - E + F$ is unchanged. On a box (Fig. 3.2) we have $V = 8, E = 12, F = 6$ and hence $V - E + F = 2$. Since a box can be inflated into a sphere this shows once more that the Euler-Poincaré characteristic of the sphere equals 2.

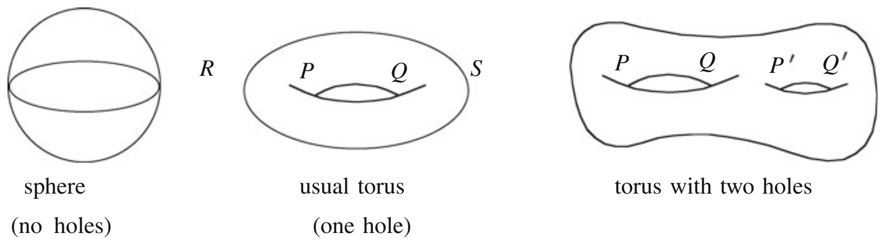


Fig. 18.6

Any compact oriented surface can be smoothly deformed into a surface with a finite number of holes (Fig. 18.6). We call the number of holes the *genus* of the surface. This smooth deformation does not change the number of faces, edges or vertices of any partition and hence the Euler–Poincaré characteristic is unchanged. It does, however, change the Gaussian curvature in many places, e.g. the sphere can be changed into an ellipsoid and we know that these do not have the same Gaussian curvature. The global Gauss–Bonnet Theorem says that the total curvature is unchanged.

For any compact surface S we have

$$\chi(S) = 2 - 2g$$

where g is the genus. This implies that the Euler–Poincaré characteristic is always an even integer and the total curvature is an integer multiple of 4π. Thus we have a remarkable set of relationships between total curvature, the number $V - E + F$ and the number of holes on a compact oriented surface **S**. This is not the end of the story as they are all equal to the index of any smooth vector field on **S**. The *index* of a vector field X, $i(X)$, is obtained by assigning an integer to each zero following a prescribed formula. Since χ (sphere) $= 2$ it follows that every smooth tangent vector field on a sphere has at least one zero and explains the existence of the bald spot which most people usually have at the point of maximal curvature on the midline of the calva.

It is interesting to speculate in a purely geometric way why these things are the way they are. For instance, we have seen that the total curvature of a sphere does not depend on the radius. Think of blowing up a balloon. The bigger the radius the larger the surface area. On the other hand the sphere is becoming less curved, i.e. the Gaussian curvature is decreasing, and over the whole surface the increase in surface area is counterbalanced by the decrease in curvature. Another simple observation: from the formula $\chi(S) = 2 - 2g$ we see that adding holes decreases the Euler–Poincaré characteristic and hence adds *negative* Gaussian curvature to the surface. Why should this be so? If we recall our study of the Gaussian curvature of the torus (Example 17.1) we noted that at the points P and Q in Fig. 18.6 we had negative Gaussian curvature while at R and S we had positive Gaussian curvature. Adding holes creates points like P' and Q' while the outside, where the Gaussian curvature is positive, is relatively unchanged and it is at least plausible that we are increasing the overall negative Gaussian curvature by adding holes.

Exercises

18.1 Show that every geodesic on the cylinder $\{(x, y, z) : x^2 + y^2 = 1\}$ has the form

$$\phi(t) = \big(\cos(at + b), \sin(at + b), ct\big), \quad t \in \mathbb{R}\,.$$

18.2 Let $\phi(t) = \big(x(t), y(t)\big)$, $a < t < b$ denote a unit speed parametrized curve in $\mathbb{R}^2$ and suppose $y(t) > 0$. Let S denote the surface obtained by rotating this curve about the x-axis. If

$$P(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$

show that the mapping

$$\varphi : t \longrightarrow P(t, \theta_0)$$

is unit speed and a geodesic for every fixed θ_0. Find the normal curvature in the direction $\varphi'(t)$ at the point $P(t, \theta_0)$.

18.3 Prove that a straight line which lies in a surface is a geodesic. By using this result find for each point P on the surface $z = x^2 - y^2$ two geodesics passing through P.

18.4 Let $P\colon [a, b] \to \Gamma$ denote a unit speed parametrized curve on the surface S. Suppose Γ has positive curvature in $\mathbb{R}^3$. Show that Γ is a line of curvature if and only if the geodesic torsion $\tau_g = 0$. If Γ is a geodesic show that $\tau_g = \tau$ where τ is the torsion of Γ as a curve in $\mathbb{R}^3$ (Chap. 7).

18.5 If Γ is a directed curve in a sphere with zero torsion show that Γ is part of a circle.

18.6 Let $P\colon [a, b] \to \Gamma$ denote a unit speed curve with strictly positive intrinsic curvature in an oriented surface **S**. Show that the normal curvature at $P(t)$ in the direction $P'(t)$ is zero if and only if the osculating plane to the curve coincides with the tangent plane to the surface. (A curve with this property at all points is called an *asymptotic curve* on the surface.)

18.7 Show that a curve in a sphere with constant geodesic curvature is part of a circle.

18.8 Find the Euler–Poincaré characteristic of the torus by partitioning it into triangles and calculating $V - E + F$. Verify your result by calculating the total curvature (see Example 17.1).

18.9 If T_1 and T_2 are two triangulations of a compact oriented surface **S** then $T_1 \subset T_2$ (or T_2 is a refinement of T_1) if every triangle in T_1 is a union of triangles from T_2. Let V_i, E_i and F_i denote, respectively, the number of vertices, edges and faces in T_i, $i = 1, 2$. Show, by induction on $V_2 - V_1$ that $V_1 - E_1 + F_1 = V_2 - E_2 + F_2$. Hence show that the Euler-Poincaré characteristic is well defined– that is that $V - E + F$ has the same value for any triangulation of $\mathbf{S}$.

18.10 Let T_i denote a triangulation of the compact oriented surface $\mathbf{S}_i$ of genus g_i, $i = 1, 2$. Let t_i denote a fixed triangle in T_i, $i = 1, 2$ and let $\mathbf{S}_3$ denote the surface obtained by taking the union of $\mathbf{S}_1$ and $\mathbf{S}_2$, identifying the edges of t_1 and t_2 (preserving orientation) and removing the face $t_1 \simeq t_2$. Show that $\mathbf{S}_3$ is a compact oriented surface of genus $g_1 + g_2$. Use this result, $\chi(Torus) = 0$ and induction to show

$$\chi(\mathbf{S}) = 2 - 2g$$

for any oriented surface $\mathbf{S}$.

Solutions

Solutions, answers, hints or relevant remarks to selected exercises are provided. Unexplained notation can be found in the text.

Chapter 1

1.1 The only non-empty open and closed subset of $\mathbb{R}^n$ is $\mathbb{R}^n$. The union of open sets is open and the intersection of closed sets is closed.

(a) Interior of a solid ellipse, open and bounded, not compact.
(b) Surface of a cone, closed, not bounded since $(n, 0, n)$ lies in the surface for all n, not compact.
(c) First octant in $\mathbb{R}^3$—like the first quadrant in $\mathbb{R}^2$. Closed, not bounded since $(n, 0, 0)$ lies in the set for all n. Hence not compact.
(d) $x^2 + y^2 + (z - 1)^2 = 1$ is the surface of a sphere with centre $(0, 0, 1)$ and radius 1. Closed, bounded and hence compact.
(e) Intersection of sphere of radius 2 and centre $(0, 0, 2)$ and cylinder parallel to z-axis based on circle in (x, y)-plane with centre $(0, 0)$ and radius 2. Intersection is circle in the plane $z = 2$ of radius 2 with centre $(0, 0, 2)$, compact.

1.2 (a) $\frac{\partial f}{\partial x} = 2x\log(1+x^2y^2)+\frac{2(z^2+x^2)xy^2}{1+x^2y^2}$, $\frac{\partial f}{\partial y} = \frac{2(z^2+x^2)x^2y}{1+x^2y^2}$, $\frac{\partial f}{\partial z} = 2z\log(1+x^2y^2)$.
(b) $\frac{\partial g}{\partial x} = y\tan^{-1}(xz) + \frac{xyz}{1+x^2z^2}$, $\frac{\partial g}{\partial y} = x\tan^{-1}(xz)$, $\frac{\partial g}{\partial z} = \frac{x^2y}{1+x^2z^2}$.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7,

1.3 $F'(x,y,z,w) = \begin{pmatrix} 2x & -2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ z & 0 & x & 0 \\ 2xz^2w^2 & 0 & 2zw^2x^2 & 2z^2wx^2 \end{pmatrix}$,

$$D_{\mathbf{v}}F(1,2,-1,-2) = \begin{pmatrix} 2 & -4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 8 & 0 & -8 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -4 \\ 36 \end{pmatrix}.$$

1.4 $\nabla f(x,y,z) = (2x-y, -x+z^3, 3yz^2-6)$. $\nabla f(x,y,z) = (0,0,0) \Leftrightarrow 2x-y = 0,\ -x+z^3 = 0,\ 3yz^2-6 = 0 \Leftrightarrow y = 2x,\ z^3 = x,\ yz^2 = 2 \Leftrightarrow y = 2x,\ z^3 = x,\ xz^2 = 1 \Leftrightarrow y = 2x,\ z^3 = x,\ z^5 = 1 \Leftrightarrow z = 1,\ x = 1,\ y = 2$. Solution $(1,2,1)$.

1.5 $\nabla f = (2xe^y, x^2e^y, 0)$, $\nabla g = (zy^2e^{xz}, 2ye^{xz}, xy^2e^{xz})$, $fg = x^2y^2e^{y+xz}$. $\nabla(fg) = (2xy^2e^{y+xz} + x^2y^2ze^{y+xz},\ 2x^2ye^{y+xz} + x^2y^2e^{y+xz},\ x^3y^2e^{y+xz}) = x^2e^y(zy^2e^{xz}, 2ye^{xz}, xy^2e^{xz}) + y^2e^{xz}(2xe^y, x^2e^y, 0) = f\nabla g + g\nabla f$.

1.6 If $P(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ then $\|P(t)\|^2 = x_1^2(t) + \cdots + x_n^2(t)$ and $\frac{d}{dt}(\|P(t)\|^2) = 2x_1(t)x_1'(t) + \cdots + 2x_n(t)x_n'(t) = 2\sum_{i=1}^n x_i(t)x_i'(t) = 2P(t)\circ P'(t) = 2\langle P(t), P'(t)\rangle$ where we supposed for convenience that $P(t)$ and $P'(t)$ are in the same space. If $\|P(t)\|$ does not depend on t then $\frac{d}{dt}(\|P(t)\|^2) = 0$ and $\langle P(t), P'(t)\rangle = 0$. The inner product of two vectors is zero if and only if they are perpendicular. In this exercise, which is extensively used in Chaps. 5–8 and 16–18, we used the inner product notation. If we use matrix notation then $P(t)$ is a $1\times n$ matrix, $P'(t)$ is an $n\times 1$ matrix and $P(t)\circ P'(t)$ is a scalar.

1.7 $F'(X) = \begin{pmatrix} 2x & 0 & 0 \\ 0 & 2y & 2z \\ yz & xz & xy \end{pmatrix}$, $G'(X) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & 2y & -2z \\ yz & xz & xy \end{pmatrix}$

$H(x,y,z) = x^2e^x + y^4 - z^4 + x^2y^2z^2$,
$\nabla H(x,y,z) = (2xe^x + x^2e^x + 2xy^2z^2, 4y^3 + 2x^2yz^2, -4z^3 + 2x^2y^2z)$.
Note $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ hence F' is a 3×3 matrix, $G\colon \mathbb{R}^3 \to \mathbb{R}^3$ hence G' is a 3×3 matrix and $\langle F, G\rangle\colon \mathbb{R}^3 \to \mathbb{R}^1$ hence $\nabla(\langle F, G\rangle)$ is a 1×3 matrix. This means $G\circ F' + F\circ G'$ is a 1×3 matrix and $F'\circ G + F\circ G'$ would *not* give the correct answer since it is not possible to add a 1×3 matrix and a 3×1 matrix.

1.8 $\frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$.

1.9 Let $F = (f_1, \ldots, f_m)$. Then $\|F\| = (f_1^2 + \cdots + f_m^2)^{1/2}$ and $\frac{\partial}{\partial x_i}(\|F\|) = \frac{1}{2}(f_1^2 + f_2^2 + \cdots + f_m^2)^{-1/2}\cdot\frac{\partial}{\partial x_i}(\sum_{j=1}^m f_j^2) = \frac{1}{\|F\|}\cdot\sum_{j=1}^m f_j\frac{\partial f_j}{\partial x_i} = \langle F, \frac{\partial F}{\partial x_i}\rangle/\|F\|$, $\nabla_{\mathbf{v}}(\|F\|) = \sum_{i=1}^n v_i\frac{\partial}{\partial x_i}(\|F\|) = \frac{1}{\|F\|}\sum_{i=1}^n v_i\langle F, \frac{\partial F}{\partial x_i}\rangle = \frac{1}{\|F\|}\langle F, \sum_{i=1}^n v_i\frac{\partial F}{\partial x_i}\rangle = \frac{\langle F, D_{\mathbf{v}}F\rangle}{\|F\|}$. We require $F(P) \neq 0$ since we divide by $\|F\|$. The result is not true otherwise, e.g. $f(x) = |x|$, $x \in \mathbb{R}$, is not differentiable at the origin. Note the use of the notation $\nabla_{\mathbf{v}}$ in place of $D_{\mathbf{v}}$ for scalar-valued functions.

1.10 Let $F(x_1,\ldots,x_n) = (x_1,\ldots,x_n)$ in Exercise 1.9. Then $D_{\mathbf{e}_i}F = \mathbf{e}_i$ and $D_{\mathbf{e}_i}(1/\|X\|) = -\frac{1}{\|X\|^2}D_{\mathbf{e}_i}(\|X\|) = -\frac{1}{\|X\|^2}D_{\mathbf{e}_i}(\|F\|) = -\frac{\langle F, D_{\mathbf{e}_i}F\rangle}{\|X\|^2\|F\|}$ $= -\frac{\langle \mathbf{e}_i, F\rangle}{\|X\|^3} = -x_i/\|X\|^3$. Hence $\nabla(1/\|X\|) = -X/\|X\|^3$.

1.11 $H(x,y,z) = \big((xyz)^2 + (x^2+y^2)^2, (xyz)^2 - (x^2+y^2)^2, (x^2-y^2)^2 - z^4, (x^2-y^2)^2 + z^4\big)$ and $H_2(x,y,z) = (xyz)^2 - (x^2+y^2)^2$. Hence $\frac{\partial H_2}{\partial x} = 2xy^2z^2 - 4x(x^2+y^2)$.

$$G' \circ F' = \begin{pmatrix} 2u & 2v & 0 & 0 \\ 2u & -2v & 0 & 0 \\ 0 & 0 & 2w & -2t \\ 0 & 0 & 2w & 2t \end{pmatrix} \begin{pmatrix} yz & xz & xy \\ 2x & 2y & 0 \\ 2x & -2y & 0 \\ 0 & 0 & 2z \end{pmatrix} = \begin{pmatrix} - & - & - \\ \frac{\partial H_2}{\partial x} & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

$\frac{\partial H_2}{\partial x} = 2uyz - 2v\cdot 2x = 2xyz\cdot yz - 2(x^2+y^2)2x = 2xy^2z^2 - 4x(x^2+y^2)$.

1.12 h is the composition

$$\begin{array}{ccc} (x_1,\ldots,x_n) \to & (e^{x_1},\ldots,e^{x_n}) & \\ & \| & \\ & (y_1,\ldots,y_n) & \to f(y_1,\ldots,y_n) \end{array}$$

$\frac{\partial h}{\partial x_i} = e^{x_i}\frac{\partial f}{\partial y_i}$, $\frac{\partial^2 h}{\partial {x_i}^2} = e^{x_i}\frac{\partial f}{\partial y_i} + e^{x_i}\frac{\partial}{\partial y_i}(\frac{\partial f}{\partial y_i}) = e^{x_i}\frac{\partial f}{\partial y_i} + e^{2x_i}\frac{\partial^2 f}{\partial {y_i}^2} = y_i\frac{\partial f}{\partial y_i} + y_i^2\frac{\partial^2 f}{\partial {y_i}^2}$. Hence $\sum_{i=1}^n \frac{\partial^2 h}{\partial {x_i}^2} = \sum_{i=1}^n y_i\frac{\partial f}{\partial y_i} + \sum_{i=1}^n y_i^2\frac{\partial^2 f}{\partial {y_i}^2} = 0$.

1.14 $P = (1,1,1)$, $\Delta X = (0.1, 0.05, -0.05)$, $f(P) = 3$, $f(P+\Delta X) = 3.42628125$, $\nabla f(P) = (3,5,3)$, $f(P) + \nabla f(P)\cdot\Delta X = 3.40$, Error$=$ 0.02628125, Error$\times 100/f(P) = 0.87604166\,\%$.

1.15 $F^{-1}(C)$ is the intersection of the cone $z^2 - x^2 - y^2 = 1$ and the plane $2x - y = 2$. Solving these equations yields $y = 2x-2$ and $z^2 = 1 + x^2 + 4(x-1)^2 = 1 + 5x^2 - 8x + 4 = (9/5) + 5(x-4/5)^2$. This shows that the level set is a hyperbola.

1.19 The level set can be rewritten as $y^6 + (x-y)^2 + (xy-4z)^2 = 51$, hence $y^6 < 64$ and $|y| < 2$, $(x-y)^2 < 64$ implies $|x| < |y| + 8 < 10$. Hence $|xy| < 20$ and $(xy-4z)^2 < 64$ implies $|4z| < |xy| + 8 < 28$. Hence $|z| < 7$. This shows that the level set is bounded. It is also closed.

1.21 $AX = \lambda X$, $AY = \mu Y$, $\lambda \neq \mu \Rightarrow \lambda\langle X,Y\rangle = \langle\lambda X, Y\rangle = \langle AX, Y\rangle = \langle X, AY\rangle = \langle X, \mu Y\rangle = \mu\langle X,Y\rangle \Rightarrow (\lambda-\mu)\langle X,Y\rangle = 0 \Rightarrow X \perp Y$ since $\lambda \neq \mu$.

Chapter 2

2.1 $F_1'(X) = \nabla F_1(X) = (2x_1, -2x_2, 0, 0)$. F_1 has full rank except when $x_1 = x_2 = 0$. $x_1 \neq 0 \Rightarrow \{x_2, x_3, x_4\}$ can be taken as the independent variables, $x_2 \neq 0 \Rightarrow \{x_1, x_3, x_4\}$ can be taken as the independent variables. $F_1(P_1) = -3$. If $x_1 \neq 0$ let $\phi_1(x_2, x_3, x_4) = (x_2^2 - 3)^{1/2}$ near $(x_2, x_3, x_4) = (2, 0, -1)$

$$F_2'(X) = \begin{pmatrix} 2x_1 & -2x_2 & 0 & 0 \\ 0 & 0 & 2x_3 & -2x_4 \end{pmatrix},$$

full rank $\Leftrightarrow$ x_1 or $x_2 \neq 0$ and x_3 or $x_4 \neq 0$. Pairs of independent variables (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4). $F_2(P_2) = (1, 3)$, $\phi_2(x_2, x_3) = \left((1 + x_2^2)^{1/2}, -(x_3^2 - 3)^{1/2}\right)$ near $(x_2, x_3) = (0, 2)$, $F_3'(X) = \begin{pmatrix} 2x_1 & -2x_2 & 0 & 0 \\ 0 & 0 & 2x_3 & -2x_4 \\ -2x_1 & 0 & 0 & 2x_4 \end{pmatrix}$, full rank $\Leftrightarrow$ any three of the variables x_1, x_2, x_3, x_4 are non-zero. $F_3(P_3) = (-3, -7, 15)$, $\phi_3(x_1) = \left(\sqrt{x_1^2 + 3}, \sqrt{x_1^2 + 8}, \sqrt{x_1^2 + 15}\right)$ near $x_1 = 1$. Other solutions also exist.

2.2 (i) $x^2 + y^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 \Rightarrow u = \pm\sqrt{x^2 + y^2} \Rightarrow \frac{\partial u}{\partial x} = \pm x/\sqrt{x^2 + y^2}$, $y/x = u \sin v / u \cos v = \tan v \Rightarrow v = \tan^{-1}(y/x) \Rightarrow \dfrac{\partial v}{\partial x} = -y/(x^2 + y^2)$.

(ii) $\dfrac{\partial u}{\partial x} \cos v - u \sin v \dfrac{\partial v}{\partial x} = 1$, $\dfrac{\partial u}{\partial x} \sin v + u \cos v \dfrac{\partial v}{\partial x} = 0$. Solving these two linear equations for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ gives us $\frac{\partial u}{\partial x} = \cos v$ and $\frac{\partial v}{\partial x} = -\sin v/u$ which agree with (i).

2.3 $F'(X) = \begin{pmatrix} 2x_1x_2^2 & 2x_1^2x_2 & 0 & 0 \\ x_2x_3 & x_1x_3 & x_1x_2 & 0 \\ 0 & 0 & 0 & 2x_1 \end{pmatrix}$, $F'(1, 2, 3, 4) = \begin{pmatrix} 8 & 4 & 0 & 0 \\ 6 & 3 & 2 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$, hence $8x_1 + 4x_2 = 0$, $6x_1 + 3x_2 + 2x_3 = 0$, $8x_4 = 0$. Solution set $\{(x_1, -2x_1, 0, 0) : x_1 \in \mathbb{R}\}$, Basis $= \{(1, -2, 0, 0)\}$, Tangent line $= \{(1 + t, 1 - 2t, 3, 4) : t \in \mathbb{R}\}$.

2.4 (a) Let $f(x, y, z) = xe^y - z$. Surface $= f^{-1}(0)$, $\nabla f(x, y, z) = (e^y, xe^y, -1)$, $\nabla f(1, 0, 1) = (1, 1, -1)$, Normal line $= \{(1 + t, t, 1 - t) : t \in \mathbb{R}\}$, Tangent plane $= \{(x, y, z) : (x - 1) \cdot 1 + y \cdot 1 + (z - 1) \cdot (-1) = 0\} = \{(x, y, z) : x + y - z = 0\}$.

(b) Let $F(x, y, z) = (x^2 + y^2 - z^2, x + y + z)$. Γ is the set $F^{-1}(1, 5)$.

$$F'(x, y, z) = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{pmatrix}, \quad F'(1, 2, 2) = \begin{pmatrix} 2 & 4 & -4 \\ 1 & 1 & 1 \end{pmatrix}$$

Tangent line $= (1, 2, 2) + \{(x, y, z) : 2x + 4y - 4z = 0,\ x + y + z = 0\} = (1, 2, 2) + \{(x, y, z) : y = 3z,\ x = -4z\} = \{(1 - 4t, 2 + 3t, 2 + t) : t \in \mathbb{R}\}$.

2.5 Equation of plane is $ax + by + cz = d$. Using $(1, 2, 3)$ and $(4, 5, 6)$ we obtain $a + 2b + 3c = d$ and $4a + 5b + 6c = d$. Since the plane is perpendicular to the plane $7x + 8y + 9z = 10$ it follows that $(a, b, c) \cdot (7, 8, 9) = 0$, i.e. $7a + 8b + 9c = 0$. Solving for a, b, c and d gives $b = -2a$, $c = a$ and $d = 0$. Solution is $\{(x, y, z) : x - 2y + z = 0\}$.

2.6 Let $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$, $\nabla f(1, 4, 1) = (1/2, 1/4, 1/2)$. Tangent plane to $f^{-1}(4)$ at $(1, 4, 1)$ is $\{(x, y, z) : (x - 1) \cdot (1/2) + (y - 4) \cdot (1/4) + (z - 1) \cdot (1/2) = 0\} = \{(x, y, z) : 2x + y + 2z = 8\}$.

2.7 Let $f(x, y, z) = x^2 + 4y^2 + 4z^2$, $\nabla f(x, y, z) = (2x, 8y, 8z)$. Tangent planes to $f^{-1}(1)$ at $(1/\sqrt{2}, 1/4, 1/4)$ and $(\sqrt{3}/2, 0, 1/4)$ are $\sqrt{2}x + 2y + 2z = 2$ and $\sqrt{3}x + 2z = 2$. Line of intersection $= \{(t, (\sqrt{3} - \sqrt{2})t/2, 1 - t\sqrt{3}/2) : t \in \mathbb{R}\}$. $k =$ distance squared of line to the origin $= (9 - 2\sqrt{6})/(12 - 2\sqrt{6}) = (14 - \sqrt{6})/20$.

2.8 Substitute $x^2 = 1 + y^2$ into $x^2 + 2y^2 = 4$ to get $1 + 3y^2 = 4$. Hence $3y^2 = 3$ and $y = \pm 1$, $x = \pm\sqrt{2}$. Four points are $(\pm\sqrt{2}, \pm 1)$, $(a, b) = (\sqrt{2}, 1)$. Tangent line to hyperbola at $(\sqrt{2}, 1)$, $\sqrt{2}x - y = 1$ has slope $\sqrt{2}$. Normal line to ellipse points in direction $(2\sqrt{2}, 4)$ and hence has slope $4/2\sqrt{2} = \sqrt{2}$. Both lines pass through $(\sqrt{2}, 1)$ and hence coincide. Tangent lines meet at $(0, \pm 1)$ and at $(\pm 1/\sqrt{2}, 0)$. Area$= \sqrt{2}$.

2.9 Direction of normal line to paraboloid at $(1, 1, 4)$ is $(2, 2, -1)$. Tangent plane at $(1, 1, 4)$ is $\{(x, y, z) : 2x + 2y - z = 0\}$. Normal line through $(1, 1, 4)$ is $\{(1 + 2t, 1 + 2t, 4 - t) : t \in \mathbb{R}\}$, $t = -9/8$ gives the point $(-5/4, -5/4, 41/8)$ on paraboloid and on normal. Normal line through $(-5/4, -5/4, 41/8)$ has direction $(-5/2, -5/2, -1)$, $\cos\theta = \frac{(-5/2, -5/2, -1) \cdot (2, 2, -1)}{(25/4 + 25/4 + 1)^{1/2}(4 + 4 + 1)^{1/2}} = \frac{-9}{(27.9/2)^{1/2}} = -\sqrt{\frac{2}{3}}$.

2.10 Let $f(x, y, z) = \log(x^2 + y^2) - 2z$, $S = f^{-1}(0) = \text{Graph}(g)$, where $g(x, y) = \frac{1}{2}\log(x^2 + y^2)$, $\nabla f(1, -1, \frac{1}{2}\log 2) = (1, -1, -2)$. Tangent plane $= \{(x, y, z) : x - y - 2z = 2 - \log 2\}$, Normal line $= \{(1 + t, -1 - t, \frac{1}{2}\log 2 - t) : t \in \mathbb{R}\}$.

2.11 At points of contact normals coincide. Hence $(f'(x), -1, 0) = \lambda(2z, 1, 2z + 2x) \Rightarrow \lambda = -1 \Rightarrow z = -x$ and $f'(x) = (-1)(-2x) \Rightarrow f(x) = x^2$.

Chapter 3

3.1 Let $g(x, y, z) = xy + yz$, $f_1(x, y, z) = x^2 + y^2 - 1$, $f_2(x, y, z) = x - yz$. $\nabla g = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2 \Rightarrow (y, x + z, y) = \lambda_1(2x, 2y, 0) + \lambda_2(1, -z, -y)$, $y = -\lambda_2 y \Rightarrow y = 0$ or $\lambda_2 = -1$, $y = 0 \Rightarrow x = 0$ which contradicts $x^2 + y^2 = 1$. $\lambda_2 = -1 \Rightarrow x + z = \lambda_1 2y + z \Rightarrow x = \lambda_1 2y = yz \Rightarrow z = 2\lambda_1$, $y = 2\lambda_1 x - 1 = zx - 1 = (x^2/y) - 1 \Rightarrow y^2 = x^2 - y = 1 - y^2 - y \Rightarrow 2y^2 + y - 1 = 0$. Hence $y = -1$ or $y = 1/2$, $y = -1 \Rightarrow x = 0$ and $z = 0$. Solution $(0, -1, 0)$ and $g(0, -1, 0) = 0$. If $y = 1/2$, $x = \pm\sqrt{3}/2$, $z = \pm\sqrt{3}$. Maximum $= 3\sqrt{3}/4$, minimum $= -3\sqrt{3}/4$.

3.2 Maximize z on $\{x^2 + y^2 = 1\} \cap \{x + y + z = 1\}$, $(0, 0, 1) = \lambda_1(2x, 2y, 0) + \lambda_2(1, 1, 1) \Rightarrow \lambda_2 = 1$, $2x\lambda_1 = -1 = 2y\lambda_1 \Rightarrow x = y = \pm 1/\sqrt{2}$, $z = 1 \mp \sqrt{2}$. Maximum $1 + \sqrt{2}$, minimum $1 - \sqrt{2}$.

3.3 Maximize $ab + (1/2)bd$ subject to $b + 2a + 2c = P$, $b^2/4 + d^2 = c^2$. $(b, a + (1/2)d, 0, (1/2)b) = \lambda_1(2, 1, 2, 0) + \lambda_2(0, b/2, -2c, 2d) \Rightarrow c = 2d$, $b = \sqrt{12}d$, $a = (\sqrt{3} + 1)d \Rightarrow P = (4\sqrt{3} + 6)d$. Maximum $= (2 - \sqrt{3})P^2/4$.

3.4 $(2x, 2y, 2z) = \lambda_1(1, 1, -1) + \lambda_2(1, 3, 1) \Rightarrow 2x = \lambda_1 + \lambda_2$, $2y = \lambda_1 + 3\lambda_2$, $2z = -\lambda_1 + \lambda_2 \Rightarrow 2x - y + z = 0$. Together with constraints this gives solution $(0, 1/2, 1/2)$. Minimum $1/4$. The two constraints are planes which intersect in a line. Question asks to find minimum and maximum distance squared from line to the origin. Line contains points which tend to infinity so no maximum.

3.5 The two constraints are planes which intersect in a line. This line contains points which tend to infinity. f consists of positive terms added together and tends to infinity as either x or y or z goes to infinity. Since f is always positive it must have an absolute minimum. $(2x, 2y, 4z) = \lambda_1(1, 1, 1) + \lambda_2(1, -1, 3) \Rightarrow \ldots \Rightarrow x = 17/14$, $y = 16/14$, $z = 9/14$. Minimum $= 707/196$. $z = 3 - x - y = \phi(x, y)$, $f(x, y, \phi(x, y)) = x^2 + y^2 + 2(3 - x - y)^2$ and constraint becomes $2x + 4y = 7$.

$\phi_1(x) = (7 - 2x)/4$, $\phi_2(x) = (1/4)(5 - 2x)$, $f(x, \phi_1(x), \phi_2(x)) = x^2 + (7 - 2x)^2/16 + (5 - 2x)^2/8$.

3.6 $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}) = \lambda(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}) \Rightarrow \lambda \neq 0$, $\frac{1}{a} = \frac{2\lambda x}{a^2} \Rightarrow 1 = \frac{2\lambda x}{a} = \frac{2\lambda y}{b} = \frac{2\lambda z}{c} \Rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \Rightarrow x = \pm\frac{a}{\sqrt{3}}$, $y = \pm\frac{b}{\sqrt{3}}$, $z = \pm\frac{c}{\sqrt{3}}$.

3.7 $(yz, xz, xy) = \lambda(-1/x^2, -1/y^2, -1/z^2) \Rightarrow \lambda \neq 0$, $x^2yz = -\lambda = xy^2z = xyz^2 \Rightarrow x = y = z$ and $3/x = 1 \Rightarrow x = y = z = 3$. Minimum 27.

3.8 $V(x, y, z) = xyz$, $(yz, xz, xy) = \lambda(1/a, 1/b, 1/c) \Rightarrow \ldots \Rightarrow x/a = y/b = z/c$. Maximum $= abc/27$.

3.9 Join the vertices to the centre and let x, y, z be the angles at the centre. Products of lengths of sides $= 8R^3 \sin\frac{x}{2} \sin\frac{y}{2} \sin\frac{z}{2} = f(x, y, z)$. Sum of squares of lengths $= 4R^2(\sin^2\frac{x}{2} + \sin^2\frac{y}{2} + \sin^2\frac{z}{2}) = g(x, y, z)$. R is the radius of the circle. $h(x, y, z) = x + y + z - 2\pi$.

(a) $\nabla f = \lambda \nabla h \ldots \Rightarrow \cos(x/2) = \cos(y/2) = \cos(z/2) \Rightarrow x = y = z$.
(b) $\nabla g = \lambda \nabla h \ldots \Rightarrow \sin(x/2) = \sin(y/2) = \sin(z/2) \Rightarrow x = y = z$.

3.10 Since $x^2 \geq 0$, $3x^2 - y^5 = 0 \Rightarrow y^5 \geq 0 \Rightarrow y \geq 0$. Hence $f(x, y) = 2y \geq 0$ and $f(0, 0) = 0$ implies f has minimum value 0 at $(0, 0)$. The method of Lagrange multipliers does not work since the surface $3x^2 - y^5 = 0$ does not have full rank at $(0, 0)$.

3.11 Minimize $g(\mathbf{v}) = \langle \nabla f(P), \mathbf{v} \rangle$ over $\langle \mathbf{v}, \mathbf{v} \rangle = h(\mathbf{v}) = 1$. $\nabla g(\mathbf{v}) = \lambda \nabla h(\mathbf{v}) \Rightarrow \nabla f(P) = \lambda \mathbf{v}$ and $\|\mathbf{v}\| = 1 \Rightarrow \lambda = \pm\|\nabla f(P)\|/2$. Hence $g(\mathbf{v}) = \ldots = \pm\|\nabla f(P)\|$. Maximum increase in direction $\nabla f(P)/\|\nabla f(P)\|$.

3.12 Let $y_i = x_i/i$. Maximize $n! y_1 \cdots y_n$ on $\sum_{i=1}^{n} y_i^2 = 1$. Use the method in Example 3.2 to get $y_1 = y_2 = \ldots = y_n = 1/\sqrt{n}$ at maximum.

3.13 If x, y and z are perpendicular distances to the sides, of length a, b and c, then it is necessary to minimize d, where $d^2 = x^2 + y^2 + z^2$, subject to constraint Area $= A = (1/2)(ax + by + cz)$. Minimum $= 4A^2/(a^2 + b^2 + c^2)$.

3.14 Nearest point on line, $(11, 2, -4)$. Distance $= \sqrt{6}$.

3.15 Since $-1 \leq \cos\theta \leq +1$, the result in Exercise 2.12 is equivalent to the Cauchy–Schwarz inequality. We get the equality case by considering when $\cos\theta = \pm 1$.

3.16 The constraints form a compact subset of $\mathbb{R}^4$ so the function has a maximum and a minimum. If $(x, y, u, v) = (a, 0, b, 0)$ then $(xv - yu)^2 = 0$ and the minimum is 0. Let $c = xv - yu$. It suffices to consider $c \neq 0$. If $(x, y, u, v) = (a, 0, 0, b)$ then $(xv - yu)^2 = a^2b^2$ and the maximum is positive. $\nabla((xv - yu)^2) = 2c(v, -u, -y, x) = \lambda_1(2x, 2y, 0, 0) + \lambda_2(0, 0, 2u, 2v)$, $cv = \lambda_1 x, c(-u) = \lambda_1 y, c(-y) = \lambda_2 u, cx = \lambda_2 v$. $c^2v^2 + c^2u^2 = \lambda_1^2 x^2 + \lambda_1^2 y^2 \Longrightarrow c^2b^2 = \lambda_1^2 a^2, c^2 = cvx - cuy = c(vx - uy) = \lambda_1(x^2 + y^2) = \lambda_1 a^2$. Hence $\lambda_1 = b^2$ and $c^2 = a^2b^2 =$ Maximum.

Alternatively, let $x = a\cos\theta, y = a\sin\theta, u = b\cos\phi, v = b\sin\phi$ then $(xv - yu)^2 = a^2b^2\sin^2(\theta - \phi)$ and as $\sin^2(\theta - \phi)$ ranges between 0 and 1 we get the above result. Also this exercise is a special case of the Cauchy-Schwarz inequality.

Chapter 4

4.1 (a) $\nabla f(x, y) = (2x + y + 2, x + 2) = (0, 0) \Rightarrow (-2, 2)$ is the only critical point. $H_{f(x,y)} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $\det(H_{f(-2,2)}) = -1 < 0 \Rightarrow f$ has a saddle point at $(-2, 2)$.

(b) local minimum at $(1, 1)$, saddle point at $(0, 0)$. To show the local minimum is an absolute minimum over $\{(x, y); x > 1/2, y > 1/2\}$ note that $\det(H_{f(x,y)}) = 36xy - 9 > 0$ when $x > 1/2$ and $y > 1/2$.

(c) $\nabla f(x, y, z) = (3x^2z - 192, 2y - z, x^3 - y)$,

$$f''(x, y, z) = H_{f(x,y,z)} = \begin{pmatrix} 6xz & 0 & 3x^2 \\ 0 & 2 & -1 \\ 3x^2 & -1 & 0 \end{pmatrix}.$$

Critical point $(2, 8, 16)$.

$$H_{f(2,8,16)} = \begin{pmatrix} 192 & 0 & 12 \\ 0 & 2 & -1 \\ 12 & -1 & 0 \end{pmatrix}$$

saddle point.

(d) saddle points at $(2, 4)$, $(-1, 4)$, $(2, 1)$, local maximum at $(1, 3)$ (the change of variables $u = 2 - x$, $v = 4 - y$ simplifies the calculations).

(e) local maxima at $(\pm 1, \pm 1, 1)$, $(\pm 1, 1, \pm 1)$, $(1, \pm 1, \pm 1)$.

(f) critical points at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, local maxima if even number of negative signs otherwise local minima. Using one variable calculus, max of xe^{-x^2} over $\mathbb{R}$ is $1/\sqrt{2}e^{1/2}$ and min is $-1/\sqrt{2}e^{1/2}$. Hence max of $xyze^{-x^2-y^2-z^2}$ over $\mathbb{R}^3$ is $1/2\sqrt{2}e^{3/2}$ and min is $-1/2\sqrt{2}e^{3/2}$.

(g) saddle point at $(1, 1, 1/2)$.

(h) saddle point at $(2\sqrt{2}, 2, -2)$.

(i) saddle point at $(-1, 1/2, 1/2)$.

(j) saddle point at $(1, 2, 1)$.

4.2 We always have $f(x, y, z) \geq 0$. Since $f(0, 0, 0) = 0$, f has an absolute minimum at $(0, 0, 0)$. Critical points satisfy $2ax = 2x(ax^2+by^2+cz^2)$, $2by = 2y(ax^2+by^2+cz^2)$, $2cz = 2z(ax^2+by^2+cz^2)$. Critical points are $(0, 0, 0)$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ since if (x, y, z) is a critical point, $x \neq 0$ and $y \neq 0$ then $2ax/2by = 2x/2y \Rightarrow axy = bxy \Rightarrow a = b$ contradiction. Similarly all critical points can have only one non-zero component and this must be 1. Local minimum at $(0, 0, 0)$, local maxima at $(\pm 1, 0, 0)$, saddle points at $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. Since $2e^{x^2} > x^4$, $x^2e^{-x^2} \longrightarrow 0$ as $x \longrightarrow \pm\infty$ and hence $f(x, y, z) \longrightarrow 0$ if any one of $x, y, z \longrightarrow \pm\infty$. Use the method in Example 3.3 to show that f has an absolute maximum over $\mathbb{R}^3$ at a critical point. Absolute maximum a/e.

4.3 Local minimum at $(1/4, 1/4, 1/4)$, degenerate critical points at $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$, $(0, y, 1-y)$, $(x, 1-x, 0)$, $(x, 0, 1-x)$, $x,\ y,\ z \in \mathbb{R}$.

4.4 The function is only defined when $x \neq 0$, $y \neq 0$ and $z \neq 0$. At a critical point $2x^3 = y^3 + z^3 \Rightarrow 3x^3 = x^3 + y^3 + z^3$. Similarly $2y^3 = x^3 + z^3$ and $2z^3 = x^3 + z^3 \Rightarrow x^3 = y^3 = z^3 \Rightarrow x = y = z$ and $(*)\{(x, x, x) : x \neq 0\}$ is a set of critical points.

$$H_{f(x,x,x)} = \frac{1}{x^6}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

and $\det(H_{f(x,x,x)}) = 0$.

4.5 Minimize $d^2 = (x+1)^2 + (y-1)^2 + (xy-1)^2$. Critical points satisfy $x + 1 + (xy-1)y = 0$ and $y - 1 + (xy-1)x = 0$. Adding we obtain $x^2y + y^2x = xy(x+y) = 0$. If $x = 0$ then $y = 1$ and if $y = 0$ then $x = -1$. If $y = -x \Rightarrow x^3 + 2x + 1 = 0 \Rightarrow x < 0$ and $d^2 = 2(x+1)^2 + (x^2+1)^2 = x^4 + 4(x + \frac{1}{2})^2 + 2 > 2$. Minimum $= \sqrt{2}$. Lagrange multipliers can also be used for this problem.

4.7 $\nabla\left(\sum_{i=1}^{m} \|X - Y_i\|^2\right) = 2\left(mX - \sum_{i=1}^{m} Y_i\right)$.

4.8 $2x + 6\phi\phi_x - 2y - 2y\phi_x = 0$, $4y + 6\phi\phi_y - 2x - 2\phi - 2y\phi_y = 0$ and $\phi_x = \phi_y = 0 \Rightarrow x = y = \phi \Rightarrow x = \pm 1$. Critical points of ϕ at $\pm(1, 1)$, $\det(H_{\phi(1,1)}) = \det(H_{\phi(-1,-1)}) = 1/4$. $\phi_{xx}(1, 1) < 0$, local maximum at $(1, 1)$, $\phi_{xx}(-1, -1) > 0$, local minimum at $(-1, -1)$. The equation of the level set yields two solutions; $\phi_1(x, y) = (y + \sqrt{6 - 3x^2 - 5y^2 + 6xy})/3$, $\phi_2(x, y) = (y - \sqrt{6 - 3x^2 - 5y^2 + 6xy})/3$. Both have a local maximum at $(1, 1)$ and both have a local minimum $(-1, -1)$.

4.10 $|x| \leq 2$ and $|y| \leq 2$ since $x^2 + y^2 = 2$ and $|z| = |1 - x| \leq 1 + |x| \leq 3$ and $F^{-1}(2, 1)$ is bounded and hence compact since it is clearly closed.

Chapter 5

5.1 $P'(t) = (6\sinh 3t, -6\cosh 3t, 6)$, $\|P'(t)\| = 6\sqrt{2}\cosh 3t$, length $= 2\sinh 15$.

5.2 (c) On [0, 1], $\left(\cos^{-1}(s)\right)' = -(1-s^2)^{-1/2}$, $P'(s) = (-(1-s^2)^{1/2}, -s, 0)$.

5.3 $s(t) = t\sqrt{r^2+h^2}$, $P(s^{-1}(t)) = \left(r\cos\left(\frac{t}{\sqrt{r^2+h^2}}\right), r\sin\left(\frac{t}{\sqrt{r^2+h^2}}\right), \frac{ht}{\sqrt{r^2+h^2}}\right)$.

5.4 (a) $s(t) = \sqrt{3}(e^t - 1)$, $s^{-1}(t) = \log(\frac{t}{\sqrt{3}} + 1)$. Unit speed parametrization $t \to (\frac{t}{\sqrt{3}} + 1)\left(\cos(\log(\frac{t}{\sqrt{3}} + 1)), \sin(\log(\frac{t}{\sqrt{3}} + 1)), 1\right)$ for $0 \le t \le \sqrt{3}(e-1)$.

(b) $s^{-1}(t) = \sinh^{-1}(t/\sqrt{2})$. Unit speed parametrization

$$t \to \left(\left(1 + \frac{t^2}{2}\right)^{1/2}, \frac{t}{\sqrt{2}}, \sinh^{-1}\left(\frac{t}{\sqrt{2}}\right)\right)$$

where $0 \le t \le \sqrt{2}\sinh(1)$.

5.5 Let $P(t) = \left(\sqrt{t^2 - \frac{t^4}{16}}, 4 - \frac{t^2}{4}, t\right)$ for $0 \le t \le 4$.

5.6 We need ϕ: $[0, 1] \to [0, 1]$, $\phi'(t) > 0$ for $0 < t < 1$, $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = 0$, $\phi'(1) = 0$. Take $\phi(t) = 3t^2 - 2t^3$.

$$P(t) = \begin{cases} (1,2) + (3t^2 - 2t^3)(-2,-4), & 0 \le t \le 1 \\ (-1,-2) + (3(t-1)^2 - 2(t-1)^3)(5,2), & 1 \le t \le 2 \\ (4,0) + (3(t-2)^2 - 2(t-2)^3)(-3,2), & 2 \le t \le 3. \end{cases}$$

5.8 This exercise shows that the rate of change of f at X_0 along two curves, which pass through X_0, depends only on the tangents to the curves at X_0.

5.9 To show $\langle TX, TY\rangle = \langle X, Y\rangle$ expand $\|T(X+Y)\|^2$ and $\|T(X-Y)\|^2$ and consider the difference. Use exercise 2.12 to show that angles are preserved. To show that area is preserved it suffices (?) to show that the area of rectangles is preserved. (Hint: think of Riemann sums.)

Chapter 6

6.1 (a) $F(P(t)) \cdot P'(t) = t,\ 2\pi^2$; (b) $F(P(t)) \cdot P'(t) = t^2 + 2t^7 + 3t^6,\ 85/84$; (c) $F(P(t)) \cdot P'(t) = (\cos e^t, e, e^t) \cdot (0, 1, e^t) = e + e^{2t},\ (e^8 + 8e - 1)/2$.

6.2 (a) $x^2 + yz + e^{xz}$, (b) does not have a potential, (c) $xy + z^2 + \sin(xyz)$, (d) $\sin(x^2 + yz)$.

6.4 (a) Use Exercise 1.10, $(-2y+z, 2x-3z, -x+3y)/\|X\|^3$; (b) $(xz^2-xy^2, yx^2-yz^2, zy^2-zx^2)/\|X\|^3$; (c) $(0, 0, 0)$.

6.6 This exercise shows that div and $curl$ operate like derivatives. In (b) the minus sign is, perhaps, unexpected. A very careful application of the definitions is needed to verify these formulae.

6.9 Use Exercise 6.7, $\nabla(f(\|X\|)) = \nabla(f(\sqrt{x^2+y^2+z^2})) = f'(\|X\|)X/\|X\|$ and

$$\begin{aligned}\nabla^2(f(\|X\|)) &= \frac{\partial}{\partial x}\left(f'(\|X\|)\cdot\frac{x}{\|X\|}\right) + \frac{\partial}{\partial y}\left(f'(\|X\|)\cdot\frac{y}{\|X\|}\right)\\ &\quad + \frac{\partial}{\partial x}\left(f'(\|X\|)\cdot\frac{z}{\|X\|}\right)\\ &= f''(\|X\|)\frac{x^2+y^2+z^2}{\|X\|} + f'(\|X\|)\cdot\frac{2(x^2+y^2+z^2)}{\|X\|^3}.\end{aligned}$$

Then g harmonic $\Leftrightarrow \nabla^2(f(\|X\|)) = 0 \Leftrightarrow 2f'(r)/r + f''(r) = 0 \Leftrightarrow r^2 f''(r) + 2rf'(r) = \left(r^2 f'(r)\right)' = 0 \Leftrightarrow f'(r) = C/r^2 \Leftrightarrow f(r) = B + (A/r)$.

Chapter 7

7.1 $P(t) = (3\cos t, 9\sin t), 0 \le t \le 2\pi$, ellipse, Curvature $27(9\sin^2 t + 81\cos^2 t)^{-3/2}$, maximum curvature at $(0, \pm 9)$.

7.2 $|\kappa| = \left| \dfrac{\frac{2}{a^2}(\frac{2y}{b^2})^2 + \frac{2}{b^2}(\frac{2x}{a^2})^2}{\left((\frac{2x}{a^2})^2 + (\frac{2y}{b^2})^2\right)^{3/2}} \right| = \dfrac{ab}{\left(\frac{x^2b^2}{a^2} + \frac{a^2y^2}{b^2}\right)^{3/2}}$. If the ellipse has an anticlockwise orientation then $|\kappa| = \kappa$, otherwise $\kappa = -|\kappa|$.

7.4 Let $P(t) = (x(t), y(t), z(t))$ be unit speed. $P'(t) = (x'(t), y'(t), z'(t))$ is independent of $t \Leftrightarrow$ all tangents are parallel $\Leftrightarrow P(t) = (a + bt, c + dt, e + ft)$.

7.5 (a) $T(t) = ((1 + t)^{1/2}, -(1 - t)^{1/2}, \sqrt{2})/2, N(t) = ((1 - t)^{1/2}, (1 + t)^{1/2}, 0)/\sqrt{2}$, $B(t) = (-(1 + t)^{1/2}, (1 - t)^{1/2}, \sqrt{2})/2, \kappa(t) = \tau(t) = \sqrt{2}/4\sqrt{1 - t^2}$. (b) $T(t) = (-(1 - t^2)^{1/2}, -t, 0), N(t) = (t, -(1 - t^2)^{1/2}, 0)$, $B(t) = (0, 0, 1)$, $\kappa(t) = 1/\sqrt{1 - t^2}$, $\tau(t) = 0$. (c) $T(t) = (t(1 + t^2)^{-1/2}, 2, (1 + t^2)^{-1/2})/\sqrt{5}$, $N(t) = ((1 + t^2)^{-1/2}, 0, -t(1 + t^2)^{-1/2})$, $B(t) = (-2t(1 + t^2)^{-1/2}, 1, -2(1 + t^2)^{-1/2})/\sqrt{5}$, $\kappa(t) = 1/\sqrt{5}(1 + t^2)$, $\tau(t) = 2/\sqrt{5}(1 + t^2)$. (The point of this exercise was to calculate the Frenet-Serret apparatus directly. The curve in (b) lies in the xy-plane. By Proposition 8.1, $\tau(t) = 0$ and $B(t) = \pm(0, 0, 1)$. It is necessary to calculate $B(t)$ in order to check which sign to take but the above advance information is useful in order to know what to expect.)

7.6 $\|P'(t)\| = a \sec\alpha$. Let $ab = \cos\alpha$. Then $Q(t) := (a\cos bt, a\sin bt, t\sin\alpha)$ is unit speed. $T'(t) = (-ab^2\cos bt, -ab^2\sin t, 0)$, $\kappa(t) = ab^2$, $N(t) = (-\cos bt, -\sin bt, 0)$, $N(0) = (-1, 0, 0)$, and normal points into cylinder, hence centre of curvature $= Q(t) + \kappa(t)N(t) = Q(t) + T'(t) = ((a - ab^2)\cos bt, (a - ab^2)\sin bt, t\sin\alpha)$, a helix, on the cylinder $x^2 + y^2 = a^2(1 - b^2)^2$.

7.7 $Q(t) = (a\cos(\frac{b}{a}\sinh^{-1}(\frac{t}{b})), a\sin(\frac{b}{a}\sinh^{-1}(\frac{t}{b})), \sqrt{b^2 + t^2})$ is a unit speed parametrization. Osculating plane at $Q(t) := (q_1(t), q_2(t), q_3(t))$ is perpendicular to $Q'(t) \times Q''(t)$ and tangent plane at $Q(t)$ is perpendicular to $\widetilde{Q}(t) := (q_1(t), q_2(t), 0)$. It suffices to show $\left[Q'(t) \times Q''(t) \cdot \widetilde{Q}(t)\right]/\|Q''(t)\| \cdot \|\widetilde{Q}(t)\|]$ $= \alpha(t)$ is constant. $\|\widetilde{Q}(t)\| = a$, $Q'(t) \times Q''(t) \cdot \widetilde{Q}(t) = ab/(b^2 + t^2)$, $\|Q''(t)\| = b\sqrt{a^2 + b^2}/a(t^2 + b^2)$ and $\alpha(t) = a/\sqrt{a^2 + b^2}$. Alternatively, if $P(t) = (a\cos t, a\sin t, b\cosh\frac{at}{b})$ and $\overline{P}(t) = (a\cos t, a\sin t, 0)$ then results in Chap. 8 show it suffices to prove $\left[P'(t) \times P''(t) \cdot \overline{P}(t)\right] \|P'(t) \times P''(t)\| \cdot \|\overline{P}(t)\|$ is constant. $\|\overline{P}(t)\| = a$, $P'(t) \times P''(t) \cdot \overline{P}(t) = [a^4\cosh(at/b)]/b$, and

$$\|P'(t) \times P''(t)\| = [a^2(a^2 + b^2)^{1/2}\cosh(at/b)]/b.$$

7.11 $Q'(t) = B(t)$ implies Q is unit speed and $Q' = T_{\widetilde{\Gamma}} = B$, $Q'' = T'_{\widetilde{\Gamma}} = B' = -\tau N = \kappa_{\widetilde{\Gamma}} N_{\widetilde{\Gamma}}$. Hence $\kappa_{\widetilde{\Gamma}} = \tau$ (since $\tau \ge 0$ and $\kappa_{\widetilde{\Gamma}}$ is positive) and $N_{\widetilde{\Gamma}} = -N$. $B_{\widetilde{\Gamma}} = T_{\widetilde{\Gamma}} \times N_{\widetilde{\Gamma}} = B \times (-N) = T$, $B'_{\widetilde{\Gamma}} = -\tau_{\widetilde{\Gamma}} N_{\widetilde{\Gamma}} = T' = \kappa N$. Hence $\tau_{\widetilde{\Gamma}} = \kappa$.

7.12 $X = \tau T + \kappa B$.

Chapter 8

8.1 By inspection one can take $\mathbf{u} = (0, 0, 1)$ in (a) and (b) and $\mathbf{u} = (0, 1, 0)$ in (c).

8.2 $T(t) = (1, -t^{-2}, -t^{-2} - 1)/(2 + 2t^{-2} + 2t^{-4})^{1/2}$, $N(t) = (1 + 2t^{-2}, 2 + t^{-2}, 1 - t^{-2})/(6 + 6t^{-2} + 6t^{-4})^{1/2}$, $B(t) = (1, -1, 1)/\sqrt{3}$, $\kappa(t) = 2t^{-3}\sqrt{3}(2 + 2t^{-2} + 2t^{-4})^{-3/2}$, $\tau(t) = 0$.

Note By inspection $P(t) \cdot (1, -1, 1) = -1$ so Γ lies in the plane $x - y + z = -1$. By Proposition 8.1, $\tau(t) = 0$ and $B(t) = \pm(1, -1, 1)/\sqrt{3}$. To know which sign to take it was necessary to do the above calculations.

8.3 $P'(\theta) = (-\tan\theta, \cot\theta, \sqrt{2})$, $\|P'(\theta)\| = \frac{2}{\sin 2\theta}$, $P''(\theta) = (-\sec^2\theta, -\mathrm{cosec}^2\theta, 0)$, $\|P'(\theta) \times P''(\theta)\| = 4\sqrt{2}/(\sin^2 2\theta)$, $\kappa(\theta) = \sin 2\theta/\sqrt{2}$.

8.4 $P'(t) = 3(2t, 1 - t^2, 1 + t^2)$, $\|P'(t)\| = 3\sqrt{2}(1 + t^2)$, $T(t) = \left(\frac{\sqrt{2}t}{1+t^2}, \frac{1-t^2}{\sqrt{2}(1+t^2)}, \frac{1}{\sqrt{2}}\right)$, $T(t) \cdot (0, 0, \pm 1) = \pm\frac{1}{\sqrt{2}}$, $P'(t) \times P''(t) = 18(2t, 1 - t^2, -1 - t^2)$, $\|P'(t) \times P''(t)\| = 18\sqrt{2}(1 + t^2)$, $\kappa(t) = 1/3(1 + t^2)^2$, $P'''(t) = 6(0, -1, 1)$, $\langle P'''(t), P'(t) \times P''(t)\rangle = -18 \cdot 6 \cdot 2$, $\tau(t) = -1/3(1 + t^2)^2$.

8.5 $\kappa(t) = (\sqrt{2}/3)\mathrm{e}^{-t}$, $\tau(t) = (1/3)\mathrm{e}^{-t}$.

8.6 The normal plane at $P(t)$ is $\{X : \langle P(t) - X, P'(t)\rangle = 0\}$. If X_0 lies in every normal plane then $\langle P(t) - X_0, P'(t)\rangle = 0$ for all t. Hence $\frac{d}{dt}\langle P(t) - X_0, P(t) - X_0\rangle = 0$ and $\|P(t) - X_0\|$ is independent of t, and the curve lies on a sphere. $\langle P(\theta), P'(\theta)\rangle = -2\sin 2\theta$, $\langle (a, b, c), P'(\theta)\rangle = 2a\sin 2\theta + 2b\sin\theta + 2c\cos 2\theta$. Hence $a = -1$, $b = 0$ and $c = 0$ imply $\langle P(\theta) - (a, b, c), P'(\theta)\rangle = 0$. Centre $(-1, 0, 0)$, radius 2.

8.7 $P'(t) = (a, 2bt, 3t^2)$, $\|P'(t)\|^2 = a^2 + 4b^2t^2 + 9t^4$, $P''(t) = (0, 2b, 6t)$, $P'(t) \times P''(t) = (6bt^2, -6at, 2ab)$, $\|P'(t) \times P''(t)\|^2 = 36b^2t^4 + 36a^2t^2 + 4a^2b^2$, $P'''(t) = (0, 0, 6)$, $P'''(t) \cdot P'(t) \times P''(t) = 12ab$,

$$\begin{aligned}\tau(t)/\kappa(t) &= \|P'(t)\|^3(P'''(t) \cdot P'(t) \times P''(t))/\|P'(t) \times P''(t)\|^3 \\ &= 12ab(a^2 + 4b^2t^2 + 9t^4)^{3/2}/(36b^2t^4 + 36a^2t^2 + 4a^2b^2)^{3/2}.\end{aligned}$$

P parametrizes a generalised helix $\Leftrightarrow a^2 + 4b^2t^2 + 9t^4 = \alpha(36b^2t^4 + 36a^2t^2 + 4a^2b^2)$ for some $\alpha \in \mathbb{R}$ $\Leftrightarrow \alpha = 1/4b^2$ and $4b^2 = \alpha 36a^2 \Leftrightarrow 4b^4 = 9a^2$.

Chapter 9

9.1 $\int_0^1 x^2 dx \cdot \int_0^{\pi/4} \sin^2 y dy = [x^3/3]_0^1 [x/2 - (\sin 2x)/4]_0^{\pi/4} = (\pi - 2)/24.$

9.2 (a) $\int_0^{\pi} \left\{\int_0^{x} x\cos(x+y)dy\right\} dx = \frac{\pi}{2}$,

(b) $\int_{-1}^{+1} \left\{\int_{1-\sqrt{(1-x^2)}}^{1+\sqrt{(1-x^2)}} (x^2+y^2)dy\right\} dx = 8/3 \int_0^1 (1-x^2)^{1/2}(2+x^2)dx = 3\pi/2$; or alternatively, let $x = r\cos\theta$, $y = 1 + r\sin\theta$, $dxdy = rdrd\theta$ and $\int_0^1 \int_0^{2\pi} (r^2 + 1 + 2r\sin\theta)rdrd\theta = 3\pi/2$,

(c) $\int_1^2 \{\int_{1/y}^{y} \frac{y^2}{x^2} dx\} dy = \int_1^2 (-y + y^3)\, dy = 9/4.$

9.3 $\int_0^2 \left\{\int_{y^2}^{4y-y^2} dx\right\} dy = 8/3.$

9.4 Let $\mathbf{F} = (Q, -P)$. If $t \to (x(t), y(t))$ is a unit speed parametrization then $\mathbf{n} = (y', -x')$. Now apply Green's Theorem.

9.5 (a) $\int_0^2 \left\{\int_0^{3y} e^{y^2} dx\right\} dy = 3(e^4 - 1)/2$,

(b) $\int_0^2 \left\{\int_{x^2}^{2x} e^{y/x} dy\right\} dx = e^2 - 1.$

9.6 $\iint\limits_{\substack{x^2+y^2\le 1\\ x\ge 0, y\ge 0}} (1 - xy)dx\,dy = \int_0^1 \left\{\int_0^{\sqrt{1-x^2}} (1 - xy)dy\right\} dx = (2\pi - 1)/8.$ The calculations are easier using polar coordinates.

9.7 20π.

9.8 (a) Once Green's theorem has been used, symmetry implies that the answer is 0.

(b) By Green's theorem $\int_\Gamma = \iint\limits_{(x-2)^2+y^2<4} 2x dx\, dy$. This can be evaluated in the usual fashion but some geometry avoids all the calculations:

$$\int_\Gamma = \iint\limits_{(x-2)^2+y^2<4} 2(x-2)dx\,dy + 4 \iint\limits_{(x-2)^2+y^2<4} dx\,dy = 0 + 4\pi.4 = 16\pi.$$

The first integral is zero since $x - 2$ has average value 0 on the disc and the second integral is $4\times$ (area of disc).

(c) $\int_\Gamma = \iint\limits_{\Omega} (-3y - 4x^2 y)dx\,dy = \int_3^5 dx \cdot \int_1^4 -3ydy - \int_3^5 4x^2 dx \cdot \int_1^4 ydy = -1025.$

Chapter 10

10.1 (a) $(u, v) \to (a \cosh u \cos v, b \cosh u \sin v, c \sinh u), -\infty < u < +\infty, 0 < v < 2\pi$.

(b) $(u, v) \to (a \cosh u, b \sinh u \cos v, c \sinh u \sin v), -\infty < u < +\infty, 0 < v < 2\pi$;

10.2 (a) P_1 is easily seen to be injective. For the remainder see Example 13.5.

(b) $(P_2)_x = (\cos\theta, 0, 1)$, $(P_2)_\theta = (-x \sin\theta, y\cos\theta, 0)$ and

$$(P_2)_x \times (P_2)_\theta = (-y \cos, -x \sin, y \cos^2\theta).$$

If $(P_2)_x \times (P_2)_\theta = 0$ then $\cos\theta = \sin\theta = 0$. This is impossible since $\cos^2\theta + \sin^2\theta = 1$. If $P_2(x, \theta) = P_2(x_1, \theta_1)$ then $x + a = x_1 + a$ and $\cos\theta = \cos\theta_1$, $\sin\theta = \sin\theta_1$. Hence $x = x_1$ and $\theta = \theta_1$.

(c) $(P_3)_u \times (P_3)_v = (u + v, v - u, -2) \neq (0, 0, 0)$, $P_3(u_1, v_1) = P_3(u_2, v_2) \Rightarrow u_1 + v_1 = u_2 + v_2$ and $u_1 - v_1 = u_2 - v_2 \Rightarrow u_1 - u_2 = v_2 - v_1 = -(v_2 - v_1) \Rightarrow v_1 = v_2$ and $u_1 = u_2$.

10.3 Parametrize the ellipsoid $((x/\sqrt{2}tr)^2 + (y/\sqrt{2}tr)^2 + (z/\sqrt{2}th)^2 = 1$ using ellipsoidal polar coordinates and take the $\theta = \pi/4$ cross section.

10.4 Parametrization formula unchanged. Range $0 < \theta < \pi/2$, $0 < \psi < \pi/2$.

10.5 $\|(0, 0, 1) + t((u, v, 0) - (0, 0, 1))\|^2 = 1 \Rightarrow t^2u^2 + t^2v^2 + (1 - t)^2 = 1$. Hence $t = 0$ or $t = 2/(1 + u^2 + v^2)$, $\phi(u, v) = \big(2u/(1 + u^2 + v^2), 2v/(1 + u^2 + v^2), 1 - 2/(1 + u^2 + v^2)\big)$. $\phi(u_1, v_1) = \phi(u_2, v_2) \Rightarrow u_1^2 + v_1^2 = u_1^2 + v_1^2$ (from third coordinate) $\Rightarrow u_1 = u_2$ and $v_1 = v_2$ from the first and second coordinates and ϕ is injective. $\phi_u \times \phi_v = \big(-2u(1 + u^2 + v^2)^{-2}, -2v(1 + u^2 + v^2)^{-2}, (1 - (u^2 + v^2)^2)(1 + u^2 + v^2)^{-2}\big) \neq (0, 0, 0)$ for all $(u, v) \in \mathbb{R}^2$. Also $\phi(\mathbb{R}^2) = S\backslash(0, 0, 1)$.

10.6 Let $P(u) = (u \cos u, u \sin u, u\sqrt{3})$, $0 \le u \le 2$, Length $= \int_0^2 (4 + u^2)^{1/2} du = 2(\sqrt{2} + \sinh^{-1}(1))$. (Use $u = 2 \sinh\theta$, $\sinh^{-1}(1) = \log(1 + \sqrt{2})$ and $\sinh(2 \sinh^{-1}(1)) = \Big((1 + \sqrt{2})^2 - (1 + \sqrt{2})^{-2}\Big)/2 = 2\sqrt{2}$.

10.7 $\nabla f(x, y, z) = (1 + y, x + z, y) \neq (0, 0, 0)$ since either $y \neq -1$ or $y \neq 0$. $U_0 = \{(x, y, z) : y \neq 0\}$ and $\phi_0(x, y) = (x, y, \frac{1-x-xy}{y})$, $(x, y) \in \mathbb{R}^2 \setminus \{(x, y); y \neq 0\}$. $U_1 = \{(x, y, z) \in S,\ y \neq -1\}$ and $\phi_1(y, z) = (\frac{1-yz}{1+y}, y, z)$ for $(y, z) \in \mathbb{R}^2 \setminus \{(y, z); y \neq -1\}$. (U_0, ϕ_0) and (U_1, ϕ_1) are graphs and hence parametrized surfaces.

10.8 At time t, L is at units above the xy-plane. The (x, y) coordinates of a point u units along L are $(u \sin bt, u \cos bt)$ after time t.

Chapter 11

11.1 $P(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$, $0 < r < 2$, $0 < \theta < 2\pi$, $EG - F^2 = r^2(1+4r^2)$, surface area $= \pi(17^{3/2} - 1)/6$.

11.2 (a) $P(x,\theta) = (x, x^3\cos\theta, x^3\sin\theta)$, $0 < x < 1$, $0 < \theta < 2\pi$, $EG - F^2 = x^6(1 + 9x^4)$. Use substitution $u = 1 + 9x^4$, surface area $= \pi(10^{3/2} - 1)/27$.

(b) $P(x,\theta) = (x, x^2\cos\theta, x^2\sin\theta)$, $0 < x < 1$, $0 < \theta < 2\pi$, $EG - F^2 = x^4(1 + 4x^2)$. Surface area $= \int_0^{2\pi}\int_0^1 x^2\sqrt{1+4x^2}dx = \frac{2\pi}{8}\int_0^{\sinh^{-1}(2)} \sinh^2(\theta)\cosh^2(\theta)d\theta = \frac{\pi}{32}\int_0^{\sinh^{-1}(2)}(\cosh(4\theta) - 1)d\theta = \pi(18\sqrt{5} - \log(2+\sqrt{5}))/32$ (use substitution $2x = \sinh\theta$ and identity $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$).

11.3 $E = 1$, $F = 0$, $G = r^2 + 1$, surface area $= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\,drd\theta = 2\pi\int_0^{\sinh^{-1}(1)}\cosh^2\phi\,d\phi = \pi(\sqrt{2} + \log(1+\sqrt{2}))$. Substitution $r = \sinh\phi$.

11.4 If $P(x,y) = (x, y, f(x,y))$ then $E = 1 + f_x^2$, $F = f_x f_y$ and $G = 1 + f_y^2$. Hence $EG - F^2 = (1 = f_x^2)(1 + f_y^2) - f_x^2 f_y^2 = 1 + f_x^2 + f_y^2 = 1 + \|\nabla f\|^2$.

11.5 Use parametrization $P(r,\theta) = (r\cos\theta, r\sin\theta, r^2/3)$, $0 < \theta < 2\pi$, $0 < r < \sqrt{3}$. Surface area $= \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{1+(4r^2/9)}\,drd\theta = (7\sqrt{21} - 9)\pi/6$.

11.6 Using $P(x,y) = (x, y, xy)$. By Exercise 11.4, surface Area $= \iint\limits_{x^2+y^2\le a^2} (1 + x^2 + y^2)^{1/2}dx\,dy$. Change to polar coordinates, $(x,y) \longrightarrow (r\cos\theta, r\sin\theta)$. Alternatively, let $P_1(r,\theta) = (r\cos\theta, r\sin\theta, \frac{r^2}{2}\sin 2\theta)$, $0 < \theta < 2\pi$, $0 < r < a$. Then $E = 1 + r^2\sin^2 2\theta$, $F = \frac{r^3}{2}\sin 4\theta$, $G = r^2(1 + r^2\cos^2 2\theta)$, $EG - F^2 = r^2 + r^4$. Surface area $= \displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\sqrt{1+r^2}drd\theta = 2\pi((1+a^2)^{3/2} - 1)/3$ (substitution $u = 1 + r^2$).

11.7 First octant $\Rightarrow 0 < \theta < \pi/2$, between the planes $\Rightarrow 0 < r < 8\sqrt{3}$, inside cylinder $\Rightarrow r^2\cos^2\theta < r^2/4 \Rightarrow \cos\theta < \frac{1}{2} \Rightarrow \pi/3 < \theta < \pi/2$. $E = (r^2/64) + 1$, $F = 0$, $G = r^2$. Surface area $= \int_{\pi/3}^{\pi/2}\int_0^{8\sqrt{3}} r\sqrt{1+(r^2/64)}\,drd\theta = 224\pi/9$.

Chapter 12

12.1 $S \subset \{(x, y, z) : 6x + 3y + 2z = 6\}$, $\mathbf{n}(p) = (6, 3, 2)/7$, $\mathbf{F} \cdot \mathbf{n} = \frac{18}{7}$, $\iint_{\mathbf{S}} \mathbf{F} = \frac{18}{7} \iint_{S} \sqrt{EG - F^2} dx\, dy = \frac{18}{7}(\text{Area } S) = 9$. Area of triangle in $\mathbb{R}^3$ with vertices at $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{2}\|(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{c})\|$. Alternatively, using projections onto coordinate planes, $\iint_{\mathbf{S}} \mathbf{F} = \iint_{\mathbf{S}} (f, g, h) = \iint_{P_{yz}} f + \iint_{P_{zx}} g + \iint_{P_{xy}} h = 1.\frac{1}{2}.2.3 + 2.\frac{1}{2}.1.3 + 3.\frac{1}{2}.1.2 = 9$. Also, $P(x, y) = (x, y, (6 - 6x - 3y)/2)$, $(x, y) \in$ triangle Δ in xy-plane with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$ parameterizes S and $P_x \times P_y = (3, 3/2, 1)$ has the correct orientation. $\iint_{\mathbf{S}} \mathbf{F} = \iint_{\Delta} \langle \mathbf{F}, P_x \times P_y \rangle dxdy = 9(\text{Area } \Delta) = 9$.

12.3 $\phi(u, v) = (u, v, uv^3)$, $\phi_u = (1, 0, v^3)$, $\phi_v = (1, 0, 3uv^2)$ and $\phi_u \times \phi_v = (-v^3, -3uv^2, 1)$. Hence $\mathbf{F}(\phi(u, v)) \cdot \phi_u(u, v) \times \phi_v(u, v) = 2 + 18u^2v^4 + 2v^9$. Answer 467.

12.4 $P(\theta, \psi) = ((b + a\cos\theta)\cos\psi, (b + a\cos\theta)\sin\psi, a\sin\theta)$, $0 < \theta, \psi < 2\pi$. This surface is a torus (see Table 11.1). $\mathbf{F}(P(\theta, \psi)) = (a\cos\theta\cos\psi, a\cos\theta \sin\psi, a\sin\theta)$. $\langle \mathbf{F}, P_\theta \times P_\psi \rangle = -a^2(b + a\cos\theta)$, $\iint_{\mathbf{S}} \mathbf{F} = -4\pi^2 a^2 b$. Since $P_\theta \times P_\psi(0, 0) = -a(b + a)(1, 0, 0)$ points inwards, P is not consistent with the orientation. Answer $= -(-4\pi^2 a^2 b)$.

12.5 Use spherical polar coordinates (see Table 11.1) with range $0 < \theta < \pi/2$, $0 < \psi < 2\pi$. $EG - F^2 = a^4 \sin^2\theta$.

(a) $y^2 + z^2 = a^2\sin^2\theta\sin^2\psi + a^2\cos^2\theta$, $4\pi a^4/3$

(b) $(EG - F^2)^{1/2}(x^2 + y^2 + (z + a)^2)^{-1/2} = a\sin\theta/(2 + 2\cos\theta)^{1/2}$, $2\pi a(2 - \sqrt{2})$.

12.6 Truncated cone, $f(r, \theta) = (r\cos\theta, r\sin\theta, r)$, $0 < \theta < 2\pi$, $1 < r < 3$, $f_r \times f_\theta = (-r\cos\theta, -r\sin\theta, r)$, $\langle \mathbf{F}(f(r, \theta)), f_r \times f_\theta \rangle = 2r$, 16π.

12.7 $F_r \times F_\theta = (\sin\theta, -\cos\theta, r)$. $\langle \mathbf{G}(F(r, \theta)), F_r \times F_\theta \rangle = \cdots = -r\theta \,/\, (r^2 + \theta^2)^{3/2}$, $\iint_{\mathbf{S}} \mathbf{G} = \int_\pi^{2\pi} \{\int_\pi^\theta (-r\theta \,/\, (r^2 + \theta^2)^{3/2}) dr\} d\theta = \pi(3 - \sqrt{10})/\sqrt{2}$.

12.8 Method (b). Use polar coordinates on each coordinate plane and $z = 0$ on the $xy-$plane. $\iint_{P_{yz}(\mathbf{S})} (y^2 + z^2) dy\, dz + \iint_{P_{zx}(\mathbf{S})} \tan^{-1}(x/z) dz\, dx + \iint_{P_{xy}(\mathbf{S})} 0 e^{x^2+y^2} dx\, dy = \iint_{0<r<a, 0<\theta<\pi/2} r^2 r dr\, d\theta + \iint_{0<r<a, 0<\theta<\pi/2} \theta r dr\, dt = a^2\pi(2a^2 + \pi)/16$.

Chapter 13

13.1 (a) $\text{curl}(z\mathbf{i} - x\mathbf{k}) = (0, 2, 0)$, $(3\pi - 8)2a^2/3$.
(b) $\text{curl}(-y\mathbf{i} + x\mathbf{j}) = (0, 0, 2)$, πa^2.
(c) $\text{curl}(-z\mathbf{j} + y\mathbf{k}) = (2, 0, 0)$, $8a^2/3$.

13.2 $P_r \times P_\theta = \left(\frac{-r^2}{a}\sin\theta, \frac{-r^2}{a}\cos\theta, r\right)$, $\text{curl}(y, z, x) = (-1, -1, -1)$, $P_r \times P_\theta \cdot (-1, -1, -1) = \frac{r^2}{a}(\sin\theta + \cos\theta) - r$. $\int_\Gamma y\,dx + z\,dy + x\,dz = \int_0^b \left\{\int_0^{2\pi}(\frac{r^2}{a}(\sin\theta + \cos\theta) - r)d\theta\right\} dr = -\pi b^2$.

13.3 The mapping $\theta \longrightarrow (b + b\cos\theta, b - b\cos\theta, \sqrt{2}b\sin\theta)$, $0 \le \theta \le 2\pi$, parametrizes Γ a circle of radius $\sqrt{2}b$ containing the points $(2b, 0, 0)$, $(b, b, \sqrt{2}b)$, $(0, 2b, 0)$ and $(b, b, -\sqrt{2}b)$. Since $\text{curl}(y, z, x) = (-1, -1, -1)$ and $\mathbf{n} \cdot (-1, -1, -1) = -\sqrt{2}$, $\int_\Gamma y\,dx + z\,dy + x\,dz = -\sqrt{2}$(Area circle)$= -\sqrt{2}\,\pi(\sqrt{2}b)^2 = -2\sqrt{2}\pi b^2$.

13.4 Γ coils around S n times. $\int_\Gamma -y\,dx + x\,dy = \int_0^{2\pi}(b + a\cos nt)^2 dt = \pi(2b^2 + a^2)$. Area$=\int_0^1\int_0^{2\pi} r(b + a\cos nt)^2 dr\,dt = \pi(b^2 + \frac{a^2}{2})$.

13.5 If $\mathbf{G}$ is the vector field in Example 6.6 then $\text{curl}(\mathbf{G}) = F$. For $d > 0$ let $\Gamma_d = \{(x, y, z) : x^2 + y^2 = 1 - d^2\}$. Then $\Gamma_b \cup \Gamma_c$ suitably oriented is the boundary of $\mathbf{S}_{b,c}$ and $P(\theta) = (\sqrt{1 - d^2}\cos\theta, \sqrt{1 - d^2}\sin\theta, \sqrt{1 - d^2})$, $0 \le \theta \le 2\pi$, parametrizes Γ_d so that the sphere above the plane $z = d$ is on the left as Γ_d is rotated. Hence $\mathbf{G}(P(\theta)) \cdot P'(\theta) = (\sin\theta, -\cos\theta, 0) \cdot \sqrt{1 - d^2}(-\sin\theta, \cos\theta, 0) = -\sqrt{1 - d^2}$ and $\int_{\Gamma_d}\mathbf{G} = (-\sqrt{1 - d^2}) \cdot l(\Gamma_d) = -\sqrt{1 - d^2}\,2\pi\sqrt{1 - d^2} = -2\pi(1 - d^2)$. By Stokes' Theorem

$$\iint_{\mathbf{S}_{b,c}} \mathbf{F} = \int_{\Gamma_c} \mathbf{G} - \int_{\Gamma_b} \mathbf{G} = 2\pi(c^2 - b^2).$$

13.6 $P(r, \theta) = (r\cos\theta, r\sin\theta, b(1 - \frac{r\cos\theta}{a}))$, $0 < r < a$, $0 < \theta < 2\pi$ parametrises the portion S of the plane inside the cylinder. $P_r \times P_\theta = (br/a, 0, r)$, $\text{curl}(y - z, z - x, x - y) = (-2, -2, -2)$, $\iint_S \langle \text{curl}(y - z, z - x, x - y), P_r \times P_\theta\rangle dr\,d\theta = -2\pi a(a + b)$. Orientation inconsistent with positive answer. Choose opposite orientation. Answer $2\pi a(a + b)$.

Chapter 14

14.1 $P(r,\theta,z) = (r\cos\theta, a + r\sin\theta, z)$, $0 < r < a$, $0 < \theta < 2\pi$, $0 < z < (a^2 + r^2 + 2ar\sin\theta)/4a$, $|det(P')| = r$.

14.2 $f(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, $0 < \theta < \pi$, $0 < z < r$, $0 < r < 2a\sin\theta$, volume$= \int_0^{\pi}\left\{\int_0^{2a\sin\theta}\left\{\int_0^r r\,dz\right\}dr\right\}d\theta = 8a^3/3\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta = 32a^3/9$. Note that the parametrization in Exercise 14.1 leads to an unsuitable integral. Use Fig. 14.13.

14.3 Use $(r,\theta,z) \to (r\cos\theta, r\sin\theta, z)$, $0 < r < 1$, $0 < \theta < 2\pi$, $\theta < z < 2\pi$.

14.4 Paraboloid and cone intersect when $x^2 + y^2 = z^2 = 2 - z$, $z^2 + z - 2 = 0$, $z = 1$, $z = -2$. Above plane $z = 0$, hence $z = 1$. Region projects onto the disc $\{(x, y, 1) : x^2 + y^2 \le 1\}$ with paraboloid above and cone below. Parametrization $(r,\theta,z) \longrightarrow (r\cos\theta, r\sin\theta, z)$, $0 < r < 1$, $0 < \theta < 2\pi$, $r < z < 2 - r^2$. Volume $= \int_0^{2\pi}\left\{\int_0^1\left\{\int_r^{2-r^2} r\,dz\right\}dr\right\}d\theta = \frac{5\pi}{6}$.

14.5 (a) 8π;
(b) 8π;
(c) $\frac{81\pi}{4} - \frac{216}{5}$.

14.7 2/3.

14.8 $V = \int_0^1\{\int_0^1\{\int_0^{(1+x+y)^{1/2}} dx\}dy\}dx = 4(9\sqrt{3} - 8\sqrt{2} + 1)/15$. By symmetry the second volume is $8V$.

14.9 (a) Volume $= \int_0^{2a} A(z)dz = \int_0^{\sqrt{3}a}\pi(z^2/3)dz + \int_{\sqrt{3}a}^{2a}\pi(4a^2 - z^2)dz = (\pi a^3)/\sqrt{3} + \pi a^3(\frac{16}{3} - 3\sqrt{3}) = \frac{8\pi a^3}{3}(2 - \sqrt{3})$. (b) Use spherical polar coordinates $w : (r,\theta,\psi) \longrightarrow (r\sin\theta\cos\psi, r\sin\theta\sin\psi, r\cos\theta)$, $0 < \theta < \pi/6$, $0 < r < 2a$, and $0 < \psi < 2\pi$. Since $\det(w') = r^2\sin\theta$, Volume$= \int_0^{\pi/6}\sin\theta d\theta\cdot\int_0^{2a} r^2 dr\cdot\int_0^{2\pi} d\psi = \frac{8\pi a^3}{3}(2 - \sqrt{3})$. See also Example 15.3.

14.10 $\frac{7}{12}$.

14.11 $det(F') = u^2 v$, $\iiint\limits_V x\,dx\,dy\,dz = \int_0^1 u^3 du\int_0^1 v(1-v)dv\int_0^1 dw = 1/24$,
$\iiint\limits_V \frac{dx\,dy\,dz}{y+z} = \int_0^1\int_0^1\int_0^1 \frac{u^2 v}{uv} du\,dv\,dw = 1/2$.

Chapter 15

15.2 $\text{div}(\mathbf{F}) = y^2 + x^2$. Interior of $S = \{(r\cos\theta, r\sin\theta, z) : 0 < r < \sqrt{3}$, $\pi/2 < \theta < 3\pi/2, 0 < z < -r\cos\theta\}$, $\text{div}(\mathbf{F}) = r^2$, $18\sqrt{3}/5$.

15.3 $\int_0^a (\int_0^{b(1-\frac{x}{a})} (\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} xdz)dy)dx = a^2cb/24$.

15.4 Order of integration is important, first x and then y to get $(e-2)/2e$.

15.5 $\text{div}(x^2, -y^2, 3xz) = 5x - 2y$,

$$\iiint\limits_{\substack{x,y,x\geq 0\\ x^2+y^2+x^2\leq 4}} xdx\,dy\,dz = \iiint\limits_{\substack{x,y,x\geq 0\\ x^2+y^2+x^2\leq 4}} ydx\,dy\,dz = \pi.$$ Solution 3π.

15.8 Volume $= \iint\limits_D (ax^2 + by^2)dx\,dy$ where $D = \{(x, y) : (x/c)^2 + (y/d)^2 < 1\}$. If $P(r, \theta) = (rc\cos\theta, rd\sin\theta), 0 < r < 1, 0 < \theta < 2\pi$ then $dx\,dy = rcd \cdot dr\,d\theta$. Volume $= \int_0^1 \int_0^{2\pi} (ac^2r^2\cos^2\theta + bd^2r^2\sin^2\theta)rcd \cdot dr\,d\theta = cd\int_0^1 r^3dr \cdot \int_0^{2\pi}(ac^2\cos^2\theta + bd^2\sin^2\theta)d\theta = cd\pi(ac^2 + bd^2)/4$.

Chapter 16

16.1 $E = 1$, $F = 0$, $G = t^2 + b^2$, $l = 0$, $m = -b(t^2 + b^2)^{-1/2}$,
$K = -b^2 \big/ (t^2 + b^2)^2$.

16.2 Since the surface of a cone always lies on one side side of each tangent plane, $K \geq 0$, and since it contains a line, $K = 0$ and 0 is a principal curvature. $P(r, \theta) = (r\cos\theta, r\sin\theta, r)$, $\mathbf{n} = (-\cos\theta, -\sin\theta, 1)/\sqrt{2}$, $E = 2$, $F = 0$, $G = r^2$, $l = 0$, $m = 0$, $n = -r/\sqrt{2}$. Principal curvatures 0, $-1/r\sqrt{2}$.

16.3 Use (16.6), $E = 5$, $F = 4$, $G = 5$, $\mathbf{n} = (-2, -2, 1)/3$, $l = -2/3$, $m = 0$, $n = -2/3$. $\phi_u(1, 1) = (1, 0, 2)$, $\phi_v(1, 1) = (0, 1, 2)$, $v_1 = 2$, $v_2 = 1$, $k_p(\mathbf{v}) = 10/123$.

16.4 By (16.6), $\kappa_P(\mathbf{v}) = \frac{v_1^2 l + 2v_1v_2m + v_2^2 n}{v_1^2 E + 2v_1v_2F + v_2^2 G} = \alpha$ (constant) for all $(v_1, v_2) \Leftrightarrow v_1^2(l - \alpha E) + 2v_1v_2(m - \alpha F) + v_2^2(n - \alpha G) = 0$ for all $(v_1, v_2) \Leftrightarrow l = \alpha E,\ m = \alpha F$ and $n = \alpha G$.
$E = 1 + y^2$, $F = xy$, $G = 1 + x^2$ and $l = 0$, $m = (1 + x^2 + y^2)^{-1/2}$, $n = 0$. At an umbilic point $l = \alpha E$ implies $\alpha = 0$ and $m = \alpha F$ implies $\alpha \neq 0$. Hence there are no umbilics.

16.5 $K = 36uv \cdot (1 + 9u^4 + 9v^4)^{-2}$, elliptic points $uv > 0$, hyperbolic points when $uv < 0$.

16.6 $m = n = 0 \Rightarrow K \equiv 0$. This is the cylinder over the ellipse $x^2 + (y/2)^2 = 1$.

16.7 $u \to (av, -bv, 0) + u(a, b, v)$, $v \to (au, bu, 0) + v(a, -b, u)$.

16.8 $K = \frac{1}{a^2b^2c^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-2}$.

16.9 It suffices to note that the coefficient of λ in (16.9) is $En + Gl - 2Fm$.
$P(x, y) = (x, y, \log\cos y - \log\cos x)$, $E = \sec^2 x$, $F = -\tan x \tan y$, $G = \sec^2 y$, $l = \sec^2 x \big/ (1 + \tan^2 x + \tan^2 y)^{1/2}$, $m = 0$, $n = -\sec^2 y \big/ (1 + \tan^2 x + \tan^2 y)^{1/2}$.

Chapter 17

17.1 Use toroidal polar coordinates, $L(xP_\theta + yP_\psi) = \frac{x}{a}P_\theta + \frac{y\cos\theta}{b+a\cos\theta}P_\psi$.

17.2 $\phi(u,v) = (u,v,u^2-2v^2)$, $\phi_u = (1,0,2u)$, $\phi_v = (0,1,-4v)$, $\phi_{uu} = (0,0,2)$, $\phi_{uv} = (0,0,0)$, $\phi_{vv} = (0,0,-4)$, $E = 1+4u^2$, $F = -8uv$, $G = 1+16v^2$, $\mathbf{n} = (-2u,4v,1)/(1+4u^2+16v^2)^{1/2}$, $l = 2/(1+4u^2+16v^2)^{1/2}$, $m=0$, $n = -4/(1+4u^2+16v^2)^{1/2}$. $\kappa_P\left(\frac{v_1\phi_u+v_2\phi_v}{\|v_1\phi_u+v_2\phi_v\|}\right) = \frac{2v_1^2-4v_2^2}{(4u^2+16v^2+1)\|v_1\phi_u+v_2\phi_v\|^2}$. Let $P(t) = \phi(\sqrt{2}t,t) = (\sqrt{2}t,t,0)$. By the chain rule $P'(t) = \sqrt{2}\phi_u+\phi_v$, and $k_{P(t)}(P'(t)) = C(2(\sqrt{2})^2-4) = 0$. The straight line $t \longrightarrow P(t)$ lies in $\mathbf{S}$.

17.3 Suppose $\kappa \neq 0$ and $\mathbf{n}_1$ and $\mathbf{n}_2$, the normals to $\mathbf{S}_1$ and $\mathbf{S}_2$, are linearly independent. If $\mathbf{n}_1\cdot\mathbf{n}_2 = \cos\theta$, then $\|\mathbf{n}_2 - \langle\mathbf{n}_1,\mathbf{n}_2\rangle\mathbf{n}_1\|^2 = \sin^2\theta$ and $\sin\theta \neq 0$. Let P denote a unit speed parametrization of Γ with $P(0) = p$ and $P'(t) = T(t)$. If N is the normal to Γ in $\mathbb{R}^3$ then $T \perp \mathbf{n}_1$, $\mathbf{n}_2$ and $N \in \mathrm{span}(\mathbf{n}_1,\mathbf{n}_2)$. Then $D_\mathbf{T}\mathbf{T} = P''(t) = \kappa\mathbf{N}$ and $\lambda_i = -\langle D_\mathbf{T}\mathbf{n}_i,\mathbf{T}\rangle = \langle\mathbf{n}_i, D_\mathbf{T}\mathbf{T}\rangle = \langle\mathbf{n}_i,\kappa\mathbf{N}\rangle$. Hence

$$\kappa N = \langle\mathbf{n}_i,\kappa\mathbf{N}\rangle\mathbf{n}_i + \frac{\langle\mathbf{n}_2-\mathbf{n}_1\cos\theta,\kappa\mathbf{N}\rangle}{\sin\theta}\frac{(\mathbf{n}_2-\mathbf{n}_1\cos\theta)}{\sin\theta}$$

and $\kappa^2 = \lambda_1^2 + \frac{\lambda_2^2-2\lambda_1\lambda_2\cos\theta+\lambda_1^2\cos^2\theta}{\sin^2\theta}$, $\kappa^2\sin^2\theta = \lambda_1^2\sin^2\theta + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta + \lambda_1^2\cos^2\theta = \lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\cos\theta$, $\kappa = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$. $\mathbf{n}_1$ and $\mathbf{n}_2$ are not linearly independent $\Rightarrow \mathbf{n}_1 = \pm\mathbf{n}_2$ and $\lambda_1 = \pm\lambda_2$.

17.4 $P(1) = (1,1,2)$ and $P'(1)/\|P'(1)\| = (2,1,6)/\sqrt{41} = \mathbf{v}$, $S = g^{-1}(0)$ where $g(u,v,w) = u^2+v^2-w$, $\nabla g/\|\nabla g\| = \frac{(2u,2v,-1)}{(4u^2+4v^2+1)^{1/2}}$, $\frac{\nabla g}{\|\nabla g\|}$ coincides with $\mathbf{n}$ on S and is defined on an open set containing S. Calculate $D(\nabla g/\|\nabla g\|)$ and $-D_\mathbf{v}(\nabla g/\|\nabla g\|) = -D(\nabla g/\|\nabla g\|)\cdot\mathbf{v}$.

$$D\left(\frac{\nabla g}{\|\nabla g\|}\right)(1,1,2) = \frac{1}{27}\begin{pmatrix}10 & -8 & 0\\ -8 & 10 & 0\\ 4 & 4 & 0\end{pmatrix}, D_\mathbf{v}\left(\frac{\nabla g}{\|\nabla g\|}\right)(1,1,2) = \frac{1}{9\sqrt{41}}(4,-2,4),$$

$\kappa_\mathbf{v}(p) = \frac{1}{9\sqrt{41}}(4,-2,4)\cdot(2,1,6)/\sqrt{41} = 10/123$.

17.5 Use Euler's formula.

Chapter 18

18.1 Cylinder $= f^{-1}(0),\ f(x, y, z) = x^2 + y^2 - 1,\ \nabla f = (2x, 2y, 0)$. Hence $\mathbf{n}(\phi(t)) \parallel (2\cos(at + b), 2\sin(at + b), 0)$, $\phi''(t) = \big(a^2\cos(at + b), a^2\sin(at + b), 0\big)$.

18.2 Unit speed parametrization $\phi(t) = (x(t), y(t)) \Rightarrow x''x' + y''y' = 0$. Note that φ is also unit speed. $\mathbf{n}(\varphi(t)) \parallel (y'(t), -x'(t)\cos\theta, -x'(t)\sin\theta)$, while $\varphi''(t) = (x''(t), y''(t)\cos\theta, y''(t)\sin\theta) \Rightarrow \varphi''(t) = \alpha(t)\mathbf{n}(\varphi(t))$ where $\alpha(t) = x''(t)/y'(t)$ if $y'(t) \neq 0$ and $\alpha(t) = y''(t)/x'(t)$ if $x'(t) \neq 0$. Since φ is a unit speed geodesic (18.5) implies $\varphi''(t) = T'(t) = \kappa_n(t)\mathbf{n}(\varphi(t))$, and the normal curvature at $P(t, \theta_0)$ is $x''(t)/y'(t)$, if $y'(t) \neq 0$, and $y''(t)/x'(t)$, if $x'(t) \neq 0$.

18.3 If $P(t) = \mathbf{a} + t\mathbf{b}$ parametrizes the straight line then P has constant speed and, since $P''(t) = T'(t) = 0$, (18.4) implies the geodesic curvature is zero and, by Proposition 18.2, straight lines are geodesics. $(x_0, y_0, z_0) + t(a, b, c)$ lies in the surface $z = x^2 - y^2 \Leftrightarrow (x_0 + ta)^2 - (y_0 + tb)^2 = z_0 + tc$ for all $t \Leftrightarrow (2x_0a - 2y_0b - c)t + (a^2 - b^2)t^2 = 0$ for all t. Letting $a = 1,\ b = 1,\ 2x_0 - 2y_0 = c$ and $a = 1,\ b = -1,\ 2x_0 + 2y_0 = c$ gives two lines on the surface.

18.4 Since P is unit speed $T(t) = P'(t)$ and $\kappa_n(t) = k_{P(t)}(T(t))$. By (18.3), $\tau_g = 0 \Leftrightarrow \mathbf{n}' = -\kappa_n T \Leftrightarrow L_{P(t)}(T(t)) = \kappa_n(t)T(t) \Leftrightarrow T(t)$ is a principal curvature for all $t \Leftrightarrow \Gamma$ is a line of curvature. Γ a geodesic $\Longrightarrow \kappa_g = 0 \Longrightarrow T' = \kappa N = \kappa_n\mathbf{n}$. Hence $N = \pm\mathbf{n}$, $B = T \times N = T \times (\pm\mathbf{n}) = \mp\mathbf{n}_s$ and $-\tau N = B' = \mp(\mathbf{n}_s)' = \mp(\tau_g\mathbf{n}) = \mp\tau_g(\pm N) = -\tau_g N$ and $\tau = \tau_g$.

18.5 By Proposition 8.1, Γ lies in a plane. Using the proof in Example 8.2, we see that the non-zero curvature is constant. An application of the result in Example 8.4 completes the proof.

18.6 By (18.5), since the tangent space to $\mathbf{S}$ at $P(t)$ is 2 dimensional and $T(t) \perp N(t)$, we have $k_{P(t)}(T(t)) = 0 \Longleftrightarrow N(t)\|\mathbf{n_S}(t) \Longleftrightarrow N(P(t)) \in T_{\mathbf{S}}(P(t)) \Longleftrightarrow T_{\mathbf{S}}(P(t)) = \text{span}\,\{T(t), N(t)\} \Longleftrightarrow$ tangent plane= osculating plane.

18.7 Assume sphere has centre at the origin and radius r. Let P be a unit speed parametrization of the curve. By (16.5) the normal curvature is $\pm 1/r$ and, by (18.5), $\kappa_g(t)$ constant $\Longleftrightarrow \kappa(t)$ constant. By Example 8.2, $\kappa(t) \neq 0$ and $\tau(t)\langle P(t) - c, B(t)\rangle = -(\frac{1}{\kappa(t)})' = 0$. This implies $\tau(t) = 0$ and Exercise 18.5 implies the required result.

Index

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7,