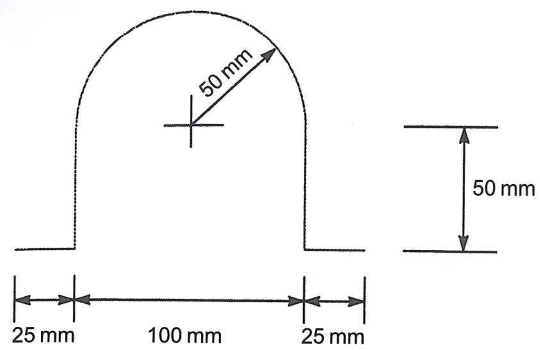


FIGURE P.10.15

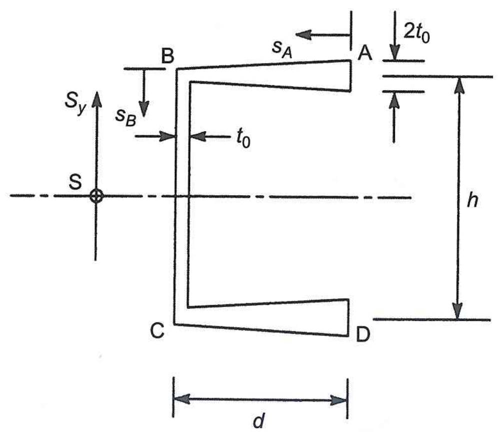


FIGURE P.10.16

P.10.17 Calculate the position of the shear centre of the thin-walled unsymmetrical channel section shown in Fig. P.10.17.

Ans. 23.1 mm from web BC, 76.3 mm from flange CD.

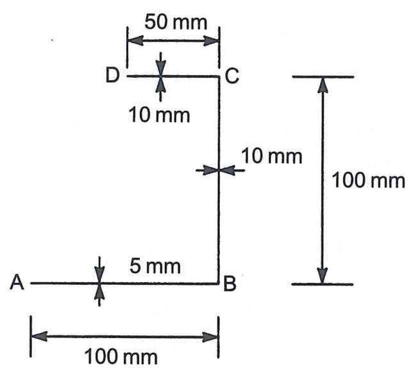


FIGURE P.10.17

P.10.18 The closed, thin-walled, hexagonal section shown in Fig. P.10.18 supports a shear load of 30 kN applied along one side. Determine the shear flow distribution round the section if the walls are of constant thickness throughout.

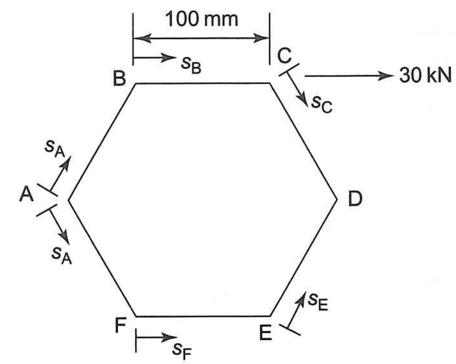


FIGURE P.10.18

Ans. $q_{AB} = 1.2s_A - 0.003\, s_A^2 + 50$
$q_{BC} = 0.6s_B - 0.006\, s_B^2 + 140$
$q_{CD} = -0.6s_C - 0.003\, s_C^2 + 140.$
Remainder of distribution follows by symmetry. All shear flows in N/mm.

P.10.19 A closed section, thin-walled beam has the shape of a quadrant of a circle and is subjected to a shear load S applied tangentially to its curved side as shown in Fig. P.10.19. If the walls are of constant thickness throughout determine the shear flow distribution round the section.

Ans. $q_{OA} = S(1.61\cos\theta - 0.81)/r \quad q_{AB} = S(0.57s^2 - 1.14rs - 0.33)/r.$

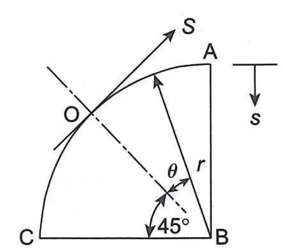


FIGURE P.10.19

P.10.20 Calculate the position of the shear centre of the beam section shown in Fig. P.10.19.

Ans. $0.61r$ from B on OB.

P.10.21 An overhead crane runs on tracks supported by a thin-walled beam whose closed cross section has the shape of an isosceles triangle (Fig. P.10.21). If the walls of the section are of constant thickness throughout determine the position of its shear centre.

Ans. 0.7 m from horizontal wall.

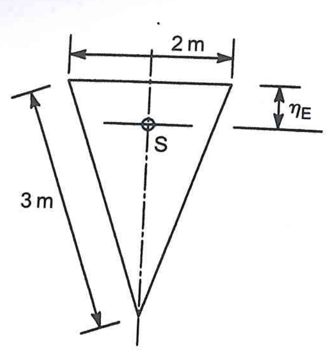


FIGURE P.10.21

P.10.22 A box girder has the singly symmetrical trapezoidal cross section shown in Fig. P.10.22. It supports a vertical shear load of 500 kN applied through its shear centre and in a direction perpendicular to its parallel sides. Calculate the shear flow distribution and the maximum shear stress in the section.

Ans. $q_{OA} = 0.25 s_A$

$q_{AB} = 0.21 s_B - 2.14 \times 10^{-4} s_B^2 + 250$

$q_{BC} = -0.17 s_C + 246$

$\tau_{max} = 30.2\ \text{N/mm}^2$.

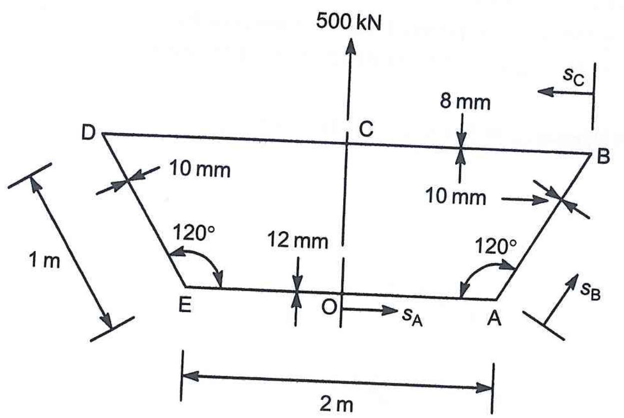


FIGURE P.10.22

CHAPTER

Torsion of Beams

11

Torsion in beams arises generally from the action of shear loads whose points of application do not coincide with the shear centre of the beam section. Examples of practical situations where this occurs are shown in Fig. 11.1 where, in Fig. 11.1(a), a concrete encased I-section steel beam supports an offset masonry wall and in Fig. 11.1(b) a floor slab, cast integrally with its supporting reinforced concrete beams, causes torsion of the beams as it deflects under load. Codes of Practice either imply or demand that torsional stresses and deflections be checked and provided for in design.

The solution of torsion problems is complex particularly in the case of beams of solid section and arbitrary shape for which exact solutions do not exist. Use is then made of empirical formulae which are conveniently expressed in terms of correction factors based on the geometry of a particular shape of cross section. The simplest case involving the torsion of solid section beams (as opposed to hollow cellular sections) is that of a circular section shaft or bar. Therefore, this case forms an instructive introduction to the more complex cases of the torsion of solid section, thin-walled open section and closed section beams.

11.1 Torsion of solid and hollow circular section bars

Initially, as in the cases of bending and shear, we shall examine the physical aspects of torsion.

Suppose that the circular section bar shown in Fig. 11.2(a) is cut at some point along its length and that the two parts of the bar are threaded onto a spindle along its axis. Now we draw a line ABC along the surface of the bar parallel to its axis and apply equal and opposite torques, *T*, at each end as shown in Fig. 11.2(b). The two parts of the bar will rotate relative to each other so that the line ABC becomes stepped. For this to occur there must be a relative slippage between the two internal surfaces in contact.

If, now, we glue the two parts of the bar together this relative slippage is prevented. The glue, therefore, produces an in-plane force which must, from a consideration of the equilibrium of either part of the bar, be equal to the applied torque *T*. This internal torque is distributed over each face of the cross section of the bar in the form of torsional shear stresses whose resultant must be a pure torque. It follows that the form of these internal shear stresses is that shown in Fig. 11.3 in which they act on a series of small elements positioned on an internal circle of radius *r*. Of course, there are an infinite number of elements on this circle and an infinite number of circles within the cross section.

Our discussion so far applies to all cross sections of the bar. The problem is to determine the distribution of shear stress and the actual twisting of the bar that the torque causes.

Figure 11.4(a) shows a circular section bar of length *L* subjected to equal and opposite torques, *T*, at each end. The torque at any section of the bar is therefore equal to *T* and is constant along its length. We shall assume that cross sections remain plane during twisting, that radii remain straight during twisting and that all normal cross sections equal distances apart suffer the same relative rotation.

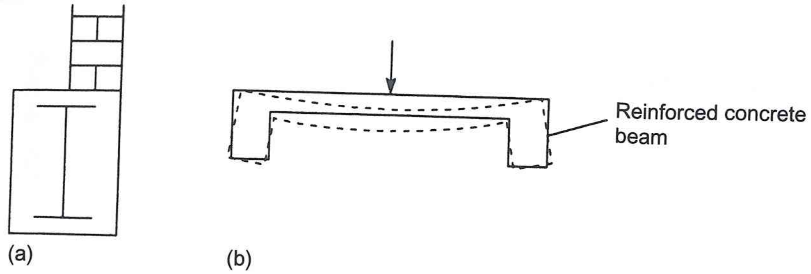


FIGURE 11.1

Causes of torsion in beams.

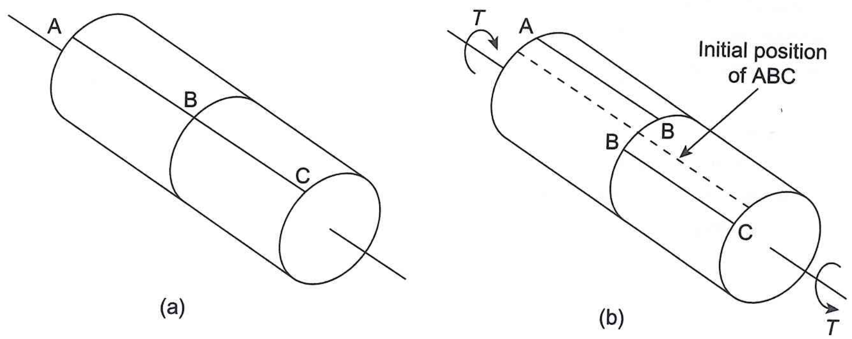


FIGURE 11.2

Torsion of a circular section bar.

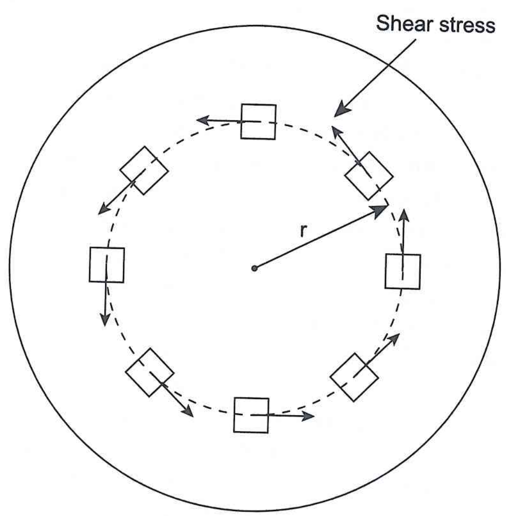


FIGURE 11.3

Shear stresses produced by a pure torque

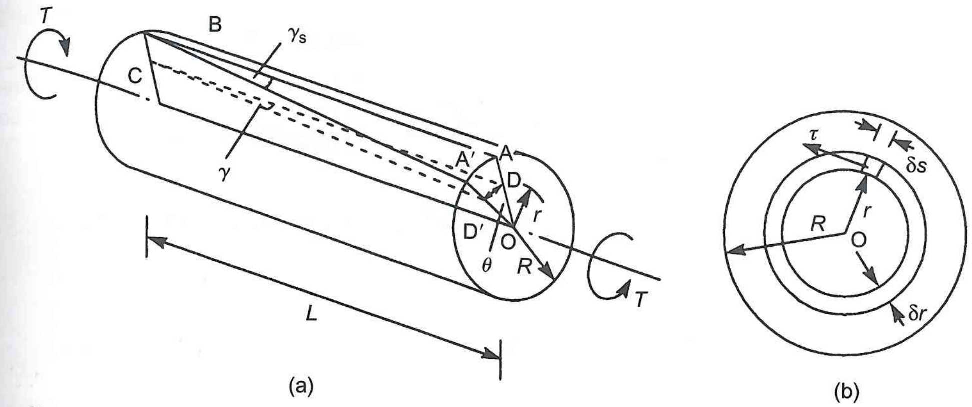


FIGURE 11.4

Torsion of a solid circular section bar.

Consider the generator AB on the surface of the bar and parallel to its longitudinal axis. Due to twisting, the end A is displaced to A′ so that the radius OA rotates through a small angle, θ, to OA′. The shear strain, γ_s, on the surface of the bar is then equal to the angle ABA′ in radians so that

$$\gamma_s = \frac{\mathrm{AA}'}{L} = \frac{R\theta}{L}$$

Similarly the shear strain, γ, at any radius r is given by the angle DCD′ so that

$$\gamma = \frac{\mathrm{DD}'}{L} = \frac{r\theta}{L}$$

The shear stress, τ, at the radius r is related to the shear strain γ by Eq. (7.9). Then

$$\gamma = \frac{\tau}{G} = \frac{r\theta}{L}$$

or, rearranging

$$\frac{\tau}{r} = G\frac{\theta}{L} \tag{11.1}$$

Consider now any cross section of the bar as shown in Fig. 11.4(b). The shear stress, τ, on an element δs of an annulus of radius r and width δr is tangential to the annulus, is in the plane of the cross section and is constant round the annulus since the cross section of the bar is perfectly symmetrical (see also Fig. 11.3). The shear force on the element δs of the annulus is then $\tau\ \delta s\ \delta r$ and its moment about the centre, O, of the section is $\tau\ \delta s\ \delta r r$. Summing the moments on all such elements of the annulus we obtain the torque, δT, on the annulus, i.e.

$$\delta T = \int_0^{2\pi r} \tau\ \delta r r\ \mathrm{d}s$$

which gives

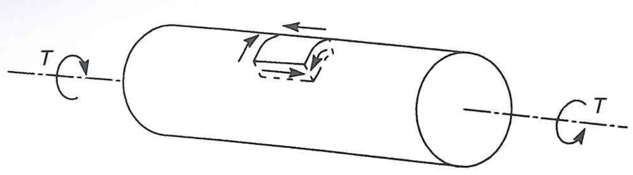


FIGURE 11.5

Shear and complementary shear stresses at the surface of a circular section bar subjected to torsion.

The total torque on the bar is now obtained by summing the torques from each annulus in the cross section. Thus

$$T = \int_0^R 2\pi r^2 \tau \, dr \tag{11.2}$$

Substituting for τ in Eq. (11.2) from Eq. (11.1) we have

$$T = \int_0^R 2\pi r^3 G \frac{\theta}{L} dr$$

which gives

$$T = \frac{\pi R^4}{2} G \frac{\theta}{L}$$

or

$$T = JG\frac{\theta}{L} \tag{11.3}$$

where $J = \pi R^4/2$ ($= \pi D^4/32$) is defined as the polar second moment of area of the cross section (see Eq. (9.42)). Combining Eqs (11.1) and (11.3) we have

$$\frac{T}{J} = \frac{\tau}{r} = G\frac{\theta}{L} \tag{11.4}$$

Note that for a given torque acting on a given bar the shear stress is a maximum at the outer surface of the bar. Note also that these shear stresses induce complementary shear stresses on planes parallel to the axis of the bar but not on the actual surface (Fig. 11.5).

Torsion of a circular section hollow bar

The preceding analysis may be applied directly to a hollow bar of circular section having outer and inner radii R_o and R_i, respectively. Equation (11.2) then becomes

$$T = \int_{R_i}^{R_o} 2\pi r^2 \tau \, dr$$

Substituting for τ from Eq. (11.1) we have

$$T = \int_{R_i}^{R_o} 2\pi r^3 G \frac{\theta}{L} dr$$

from which

$$T = \frac{\pi}{2}(R_o^4 - R_i^4) G \frac{\theta}{L}$$

The polar second moment of area, J, is then

$$J = \frac{\pi}{2}(R_o^4 - R_i^4) \tag{11.5}$$

EXAMPLE 11.1

A hollow shaft of outside diameter 220 mm and thickness 40 mm is required to transmit power at a speed of 80 rpm. If the maximum shear stress in the shaft is limited to 60 N/mm^2 determine the power transmitted and the angle of twist in a length of 10 m. Take $G = 80000$ N/mm^2.

(Note: Power P (watts) = Torque (Nm) × angular speed (rad/sec)).

From Eq. (11.5)

$$J = (\pi/2)(110^4 - 70^4) = 192.3 \times 10^6 \text{ mm}^4$$

From Eq. (11.4)

$$T = \frac{\tau_{max}}{r_o} J = \frac{60 \times 192.3 \times 10^6}{110} = 104.9 \times 10^6 \text{ Nmm}$$

so that

$$T = 104.9 \times 10^3 \text{ Nm}$$

Then

$$P = 104.9 \times 10^3 \times 2\pi \times 80/60 = 878808.2 \text{ watts}$$

or

$$P = 878.8 \text{ kW}$$

Again, from Eq. (11.4)

$$\theta = \frac{TL}{GJ} = \frac{104.9 \times 10^6 \times 10 \times 10^3}{80000 \times 192.3 \times 10^6} = 0.068 \text{ rad}$$

or

$$\theta = 3.9°$$

EXAMPLE 1.2

If a solid shaft was used to transmit the same power as the hollow shaft in Ex.1.1 with the same length and limiting stress what would be the percentage increase in weight of the material used.

For a shaft diameter of D mm

$$J = \frac{\pi D^4}{32}$$

Then, substituting for the torque from Eq. (11.4) in the expression for power

$$P = \frac{2\tau_{max} J}{D} \times \frac{2\pi \times 80}{60}$$

that is

$$878.8 \times 10^3 = \frac{2 \times 60 \times \pi D^4}{10^3 \times 32 \times D} \times \frac{2\pi \times 80}{60}$$

which gives $D = 207.3$ mm

Since weight is proportional to cross sectional area the percentage weight increase is given by

$$\% \text{ weight increase} = \frac{(\pi \times 207.3^2/4) - \pi(220^2 - 140^2)/4}{\pi(220^2 - 140^2)/4} \times 100$$

that is % weight increase = 49.0%.

EXAMPLE 11.3

Determine the angle of twist and the maximum shear stress in the tapered shaft shown in Fig. 11.6. The shear modulus of the material of the shaft is G.

Suppose that the diameter of the shaft at a distance x from the left-hand end is d. Then, from Eq. (11.4), the angle of twist, $\delta\theta$, over the length δx is given by

$$\delta\theta = \frac{T}{GJ}\delta x = \frac{T}{(G\pi d^4/32)}\delta x \qquad \text{(i)}$$

If the change in diameter over the length δx is δd then

$$\frac{\delta d}{\delta x} = \frac{D_2 - D_1}{L} \qquad \text{(Note that } \delta d \text{ is a decrease in diameter)}$$

Substituting for δx in Eq. (i)

$$\delta\theta = \frac{32TG}{\pi d^4}\frac{L}{D_2 - D_1}\delta d$$

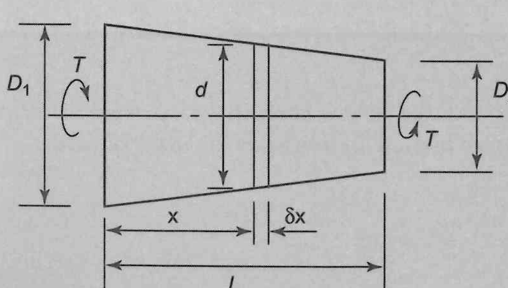


FIGURE 11.6

Tapered shaft of Ex 11.3.

The angle of twist over the complete length of the shaft is then given by

$$\theta = \frac{32TGL}{\pi(D_2 - D_1)}\int_{D_1}^{D_2} d^{-4}\,\mathrm{d}d = -\frac{32TGL}{3\pi(D_2 - D_1)}[d^{-3}]_{D1}^{D2}$$

which gives

$$\theta = \frac{-32TGL}{3\pi(D_2 - D_1)}[(1/D_2^3) - (1/D_1^3)]$$

which simplifies to

$$\theta = \frac{-32TGL}{3\pi D_1^3 D_2^3}(D_1^2 + D_1 D_2 + D_2^2) \qquad \text{(ii)}$$

From Eq. (11.4)

$$\tau = \frac{Td/2}{\pi d^4/32} = \frac{16T}{\pi d^3}$$

The maximum shear stress therefore occurs at the section where d is a minimum, that is where $d = D_2$. Then

$$\tau_{max} = \frac{16T}{\pi D_2^3}$$

Statically indeterminate circular section bars under torsion

In many instances bars subjected to torsion are supported in such a way that the support reactions are statically indeterminate. These reactions must be determined, however, before values of maximum stress and angle of twist can be obtained.

Figure 11.7(a) shows a bar of uniform circular cross section firmly supported at each end and subjected to a concentrated torque at a point B along its length. From equilibrium we have

$$T = T_A + T_C \qquad (11.6)$$

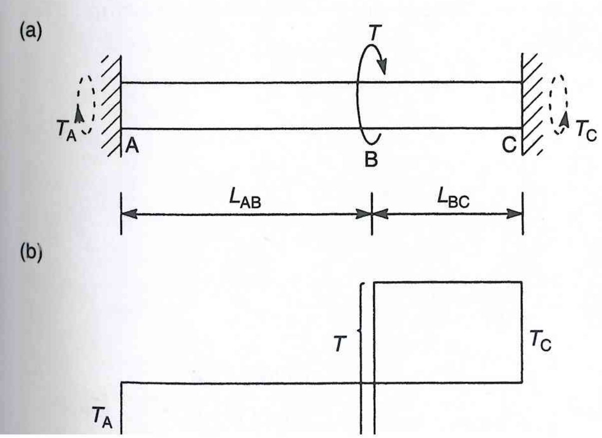


FIGURE 11.7

Torsion of a circular section bar with

A second equation is obtained by considering the compatibility of displacement at B of the two lengths AB and BC. Thus the angle of twist at B in AB must equal the angle of twist at B in BC, i.e.

$$\theta_{B(AB)} = \theta_{B(BC)}$$

or using Eq. (11.3)

$$\frac{T_A L_{AB}}{GJ} = \frac{T_C L_{BC}}{GJ}$$

whence

$$T_A = T_C \frac{L_{BC}}{L_{AB}}$$

Substituting in Eq. (11.6) for T_A we obtain

$$T = T_C\left(\frac{L_{BC}}{L_{AB}} + 1\right)$$

which gives

$$T_C = \frac{L_{AB}}{L_{AB} + L_{BC}} T \tag{11.7}$$

Hence

$$T_A = \frac{L_{BC}}{L_{AB} + L_{BC}} T \tag{11.8}$$

The distribution of torque along the length of the bar is shown in Fig. 11.7(b). Note that if $L_{AB} > L_{BC}$, T_C is the maximum torque in the bar.

EXAMPLE 11.4

A bar of circular cross section is 2.5 m long (Fig. 11.8). For 2 m of its length its diameter is 200 mm while for the remaining 0.5 m its diameter is 100 mm. If the bar is firmly supported at its ends and subjected to a torque of 50 kNm applied at its change of section, calculate the maximum stress in the bar and the angle of twist at the point of application of the torque. Take $G = 80000$ N/mm^2.

In this problem Eqs (11.7) and (11.8) cannot be used directly since the bar changes section at B. Thus from equilibrium

$$T = T_A + T_C \tag{i}$$

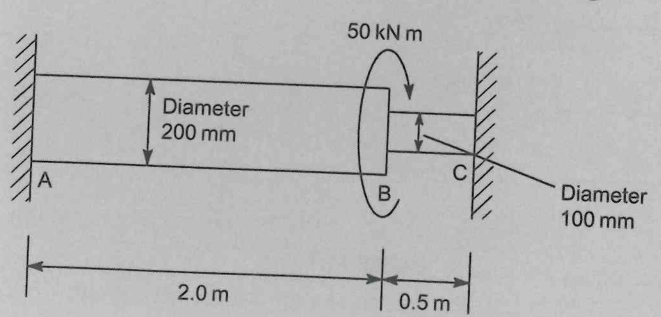


FIGURE 11.8
Bar of Ex. 11.4.

and from the compatibility of displacement at B in the lengths AB and BC

$$\theta_{B(AB)} = \theta_{B(BC)}$$

or using Eq. (11.3)

$$\frac{T_A L_{AB}}{GJ_{AB}} = \frac{T_C L_{BC}}{GJ_{BC}}$$

whence

$$T_A = \frac{L_{BC} J_{AB}}{L_{AB} J_{BC}} T_C \tag{ii}$$

Substituting in Eq. (i) we obtain

$$T = T_C\left(\frac{L_{BC}}{L_{AB}}\frac{J_{AB}}{J_{BC}} + 1\right)$$

or

$$50 = T_C\left[\frac{0.5}{2.0} \times \left(\frac{200 \times 10^{-3}}{100 \times 10^{-3}}\right)^4 + 1\right]$$

from which

$$T_C = 10 \text{ kN m}$$

Hence, from Eq. (i)

$$T_A = 40 \text{ kN m}$$

Although the maximum torque occurs in the length AB, the length BC has the smaller diameter. It can be seen from Eq. (11.4) that shear stress is directly proportional to torque and inversely proportional to diameter (or radius) cubed. Therefore, we conclude that in this case the maximum shear stress occurs in the length BC of the bar and is given by

$$\tau_{max} = \frac{10 \times 10^6 \times 100 \times 32}{2 \times \pi \times 100^4} = 50.9 \text{ N/mm}^2$$

Also the rotation at B is given by either

$$\theta_B = \frac{T_A L_{AB}}{GJ_{AB}} \text{ or } \theta_B = \frac{T_C L_{BC}}{GJ_{BC}}$$

Using the first of these expressions we have

$$\theta_B = \frac{40 \times 10^6 \times 2 \times 10^3 \times 32}{80\,000 \times \pi \times 200^4} = 0.0064 \text{ rad}$$

or

$$\theta_B = 0.37°$$

11.2 Strain energy due to torsion

It can be seen from Eq. (11.3) that for a bar of a given material, a given length, L, and radius, R, the angle of twist is directly proportional to the applied torque. Therefore a torque–angle of twist graph is linear and for a gradually applied torque takes the form shown in Fig. 11.9. The work done by a gradually applied torque, T, is equal to the area under the torque–angle of twist curve and is given by

$$\text{Work done} = \frac{1}{2}T\theta$$

The corresponding strain energy stored, U, is therefore also given by

$$U = \frac{1}{2}T\theta$$

Substituting for T and θ from Eq. (11.4) in terms of the maximum shear stress, τ_{max}, on the surface of the bar we have

$$U = \frac{1}{2}\frac{\tau_{max}J}{R} \times \frac{\tau_{max}L}{GR}$$

or

$$U = \frac{1}{4}\frac{\tau_{max}^2}{G}\pi R^2 L \quad \text{since } J = \frac{\pi R^4}{2}$$

Hence

$$U = \frac{\tau_{max}^2}{4G} \times \text{volume of bar} \qquad (11.9)$$

Alternatively, in terms of the applied torque T we have

$$U = \frac{1}{2}T\theta = \frac{T^2 L}{2GJ} \qquad (11.10)$$

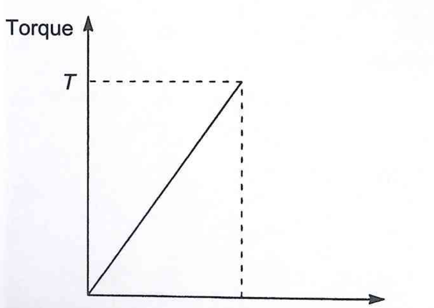


FIGURE 11.9

Torque–angle of twist relationship for a gradually applied

EXAMPLE 11.5

Determine the angle of twist at the free end of the shaft shown in Fig. 11.10. Take $G = 80000$ N/mm^2.

From Eq. (11.10) the total strain energy, U, in the shaft is

$$U = \frac{(4 \times 10^6)^2 \times 100 \times 32}{80000 \times \pi \times 50^4} + \frac{(4 \times 10^6)^2 \times 200 \times 32}{80000 \times \pi \times 100^4}$$

so that $U = 36669.3$ Nmm

The total strain energy in the shaft is equal to the work done by the applied torque. Therefore

$$4 \times 10^6\ \theta/2 = 36669.3$$

which gives

$$\theta = 0.018 \text{ rad} = 1.05°$$

Again, as in the case of trusses, strain energy can only be used directly when a shaft is subjected to a single applied torque. Further, it is only possible to obtain the angle of twist at the section where the torque is applied.

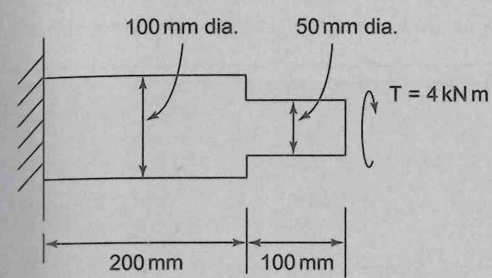


FIGURE 11.10

Shaft of Ex 11.5.

11.3 Plastic torsion of circular section bars

Equation (11.4) apply only if the shear stress–shear strain curve for the material of the bar in torsion is linear. Stresses greater than the yield shear stress, τ_Y, induce plasticity in the outer region of the bar and this extends radially inwards as the torque is increased. It is assumed, in the plastic analysis of a circular section bar subjected to torsion, that cross sections of the bar remain plane and that radii remain straight.

For a material, such as mild steel, which has a definite yield point the shear stress–shear strain curve may be idealized in a similar manner to that for direct stress (see Fig. 18.1) as shown in Fig. 11.11. Thus, after yield, the shear strain increases at a more or less constant value of shear stress. It follows that the shear stress in the plastic region of a mild steel bar is constant and equal to τ_Y. Figure 11.12 illustrates the various stages in the development of full plasticity in a mild steel bar of circular section. In Fig. 11.12(a) the maximum stress at the outer surface of the bar has reached the yield stress, τ_Y. Equations (11.4) still apply, therefore, so that at the outer surface of the bar

$$\frac{T_Y}{J} = \frac{\tau_Y}{R}$$

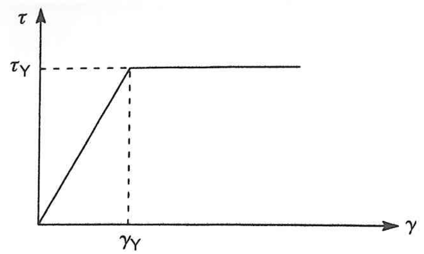


FIGURE 11.11

Idealized shear stress–shear strain curve for a mild steel bar.

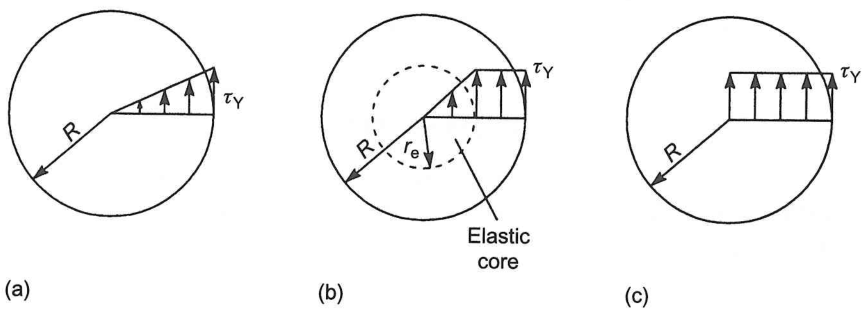


FIGURE 11.12

Plastic torsion of a circular section bar.

or

$$T_Y = \frac{\pi R^3}{2}\tau_Y \tag{11.11}$$

where T_Y is the torque producing yield. In Fig. 11.12(b) the torque has increased above the value T_Y so that the plastic region extends inwards to a radius r_e. Within r_e the material remains elastic and forms an *elastic core*. At this stage the total torque is the sum of the contributions from the elastic core and the plastic zone, i.e.

$$T = \frac{\tau_Y J_e}{r_e} + \int_{r_e}^{R} 2\pi r^2 \tau_Y \mathrm{d}r$$

where J_e is the polar second moment of area of the elastic core and the contribution from the plastic zone is derived in an identical manner to Eq. (11.2) but in which $\tau = \tau_Y =$ constant. Hence

$$T = \frac{\tau_Y \pi r_e^3}{2} + \frac{2}{3}\pi\tau_Y(R^3 - r_e^3)$$

which simplifies to

$$T = \frac{2\pi R^3}{3}\tau_Y\left(1 - \frac{r_e^3}{4R^3}\right) \tag{11.12}$$

Note that for a given value of torque, Eq. (11.12) fixes the radius of the elastic core of the section. In stage three (Fig. 11.12(c)) the cross section of the bar is completely plastic so that r_e in Eq. (11.12) is zero and the ultimate torque or fully plastic torque, T_P, is given by

Comparing Eqs (11.11) and (11.13) we see that

$$\frac{T_P}{T_Y} = \frac{4}{3} \tag{11.14}$$

so that only a one-third increase in torque is required after yielding to bring the bar to its ultimate load-carrying capacity.

Since we have assumed that radii remain straight during plastic torsion, the angle of twist of the bar must be equal to the angle of twist of the elastic core which may be obtained directly from Eq. (11.3) in which the torque is T_e the portion of the total torque carried by the elastic core. Thus for a bar of length L and shear modulus G,

$$\theta = \frac{T_e L}{GJ_e} = \frac{2T_e L}{\pi G r_e^4} \tag{11.15}$$

or, in terms of the shear stress, τ_Y, at the outer surface of the elastic core

$$\theta = \frac{\tau_Y L}{G r_e} \tag{11.16}$$

Either of Eq. (11.15) or (11.16) shows that θ is inversely proportional to the radius, r_e, of the elastic core. Clearly, when the bar becomes fully plastic, $r_e \rightarrow 0$ and θ becomes, theoretically, infinite. In practical terms this means that twisting continues with no increase in torque in the fully plastic state.

EXAMPLE 11.6

A solid circular section bar has a diameter of 100 mm and the material of the bar has a yield stress in shear of 150 N/mm^2. Determine the maximum torque the bar can transmit without yielding occurring and also the radius of the elastic core of the section if this torque is increased by 20%. What is the value of torque which would result in the section becoming fully plastic?

From Eq. (11.11)

$$T_Y = (\pi R^3/2)\tau_Y = (\pi \times 50^3/2) \times 150$$

which gives

$$T_Y = 29.5 \times 10^6 \text{Nmm}$$

Then, from Eq. (11.12)

$$1.2 \times 29.5 \times 10^6 = (2\pi \times 50^3/3) \times 150[1 - r_e^3/(4 \times 50^3)]$$

from which $r_e = 36.7$ mm

From Eq. (11.14) (or Eq. (11.13))

$$T_p = 4 \times 29.5 \times 10^6/3 = 39.3 \times 10^6 \text{ Nmm}$$

EXAMPLE 11.7

If the bar in Ex.11.6 is subjected to a torque of 32.5 kNm calculate the angle of twist in the bar over a length of 3 m. Take $G = 80000$ N/mm^2.

From Eq. (11.12)

$$32.5\times10^6=(2\pi\times50^3/3)\times150[1-r_e^3/(4\times50^3)]$$

from which $r_e=44.1$ mm

The torque resisted by the elastic core (and therefore the torque producing the twist) is then given by

$$T_e=\frac{\tau_Y J_e}{r_e}=\frac{\tau_Y\pi r_e^3}{2}=\frac{150\times\pi\times44.1^3}{2}$$

i.e. $T_e=20.2\times10^6$ Nmm

Then, from Eq. (11.15) (or Eq. (11.16))

$$\theta=\frac{2\times20.2\times10^6\times3\times10^3}{\pi\times80000\times44.1^4}=0.127\text{ rad}$$

or $\theta=7.3°$

EXAMPLE 11.8

A hollow circular section bar has external and internal radii, R_o and R_i, respectively and carries a torque, T. If the yield stress in shear of the material of the bar is τ_Y and the value of T is sufficient to cause the plastic core to penetrate to a radius, r_e, ($R_o > r_e > R_i$) derive an expression for r_e.

The total torque is, as for a solid bar, equal to the sum of the contributions from the elastic and plastic cores. Then

$$T=(\tau_Y J_e)/r_e+\int_{re}^{Ro}2\pi r^2\tau_Y\,\mathrm{d}r$$

so that

$$T=\tau_Y\pi(r_e^4-R_i^4)/2r_e+2\pi\tau_Y(R_o^3-r_e^3)/3$$

Simplifying

$$T=\frac{\pi\tau_Y}{2r_e}\left(\frac{4}{3}R_o^3r_e-\frac{1}{3}r_e^4-R_i^4\right)\qquad\text{(i)}$$

from which, for given values of T, τ_Y, R_o and R_i, r_e can be determined.

Note that Eq. (i) reduces to Eq. (11.12) for the case of a solid bar for which $R_i=0$.

11.4 Torsion of a thin-walled closed section beam

Although the analysis of torsion problems is generally complex and in some instances relies on empirical methods for a solution, the torsion of a thin-walled beam of arbitrary closed section is relatively straightforward.

Figure 11.13(a) shows a thin-walled closed section beam subjected to a torque, T. The thickness, t,

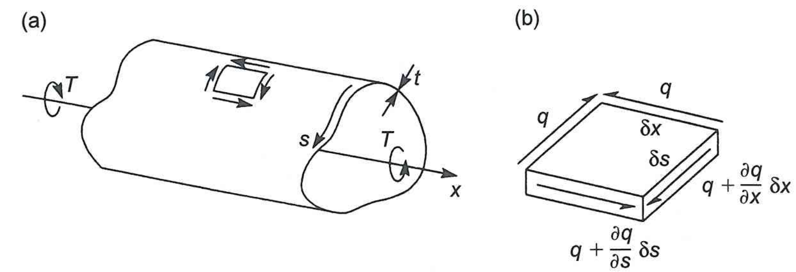


FIGURE 11.13

Torsion of a thin-walled closed section beam.

stress system in the walls of the beam which consists solely of shear stresses if the applied loading comprises only a pure torque. In some cases structural or loading discontinuities or the method of support produce a system of direct stresses in the walls of the beam even though the loading consists of torsion only. These effects, known as axial constraint effects, are considered in more advanced texts.

The shear stress system on an element of the beam wall may be represented in terms of the shear flow, q, (see Section 10.4) as shown in Fig. 11.13(b). Again we are assuming that the variation of t over the side δs of the element may be neglected. For equilibrium of the element in the x direction we have

$$\left(q+\frac{\partial q}{\partial s}\delta s\right)\delta x-q\,\delta x=0$$

which gives

$$\frac{\partial q}{\partial s}=0\qquad(11.17)$$

Considering equilibrium in the s direction

$$\left(q+\frac{\partial q}{\partial x}\delta x\right)\delta s-q\,\delta s=0$$

from which

$$\frac{\partial q}{\partial x}=0\qquad(11.18)$$

Equations (11.17) and (11.18) may only be satisfied simultaneously by a constant value of q. We deduce, therefore, that the application of a pure torque to a thin-walled closed section beam results in the development of a constant shear flow in the beam wall. However, the shear stress, τ, may vary round the cross section since we allow the wall thickness, t, to be a function of s.

The relationship between the applied torque and this constant shear flow may be derived by considering the torsional equilibrium of the section shown in Fig. 11.14. The torque produced by the shear flow acting on the element, δs, of the beam wall is $q\,\delta s\,p$. Hence

$$T=\oint pq\,\mathrm{d}s$$

or, since q = constant

$$T=q\oint p\,\mathrm{d}s\qquad(11.19)$$

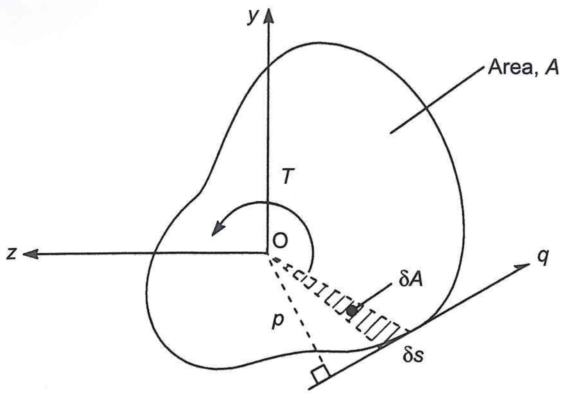


FIGURE 11.14

Torque–shear flow relationship in a thin-walled closed section beam.

We have seen in Section 10.5 that $\oint p\,\mathrm{d}s = 2A$ where A is the area enclosed by the midline of the beam wall. Hence

$$T = 2Aq \tag{11.20}$$

The theory of the torsion of thin-walled closed section beams is known as the *Bredt-Batho theory* and Eq. (11.20) is often referred to as the *Bredt-Batho formula.*

It follows from Eq. (11.20) that

$$\tau = \frac{q}{t} = \frac{T}{2At} \tag{11.21}$$

and that the maximum shear stress in a beam subjected to torsion will occur at the section where the torque is a maximum and at the point in that section where the thickness is a minimum. Thus

$$\tau_{\max} = \frac{T_{\max}}{2At_{\min}} \tag{11.22}$$

In Section 10.5 we derived an expression (Eq. (10.28)) for the rate of twist, $\mathrm{d}\theta/\mathrm{d}x$, in a shear-loaded thin-walled closed section beam. Equation (10.28) also applies to the case of a closed section beam under torsion in which the shear flow is constant if it is assumed that, as in the case of the shear-loaded beam, cross sections remain undistorted after loading. Thus, rewriting Eq. (10.28) for the case $q_s = q =$ constant, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{q}{2A}\oint\frac{\mathrm{d}s}{Gt} \tag{11.23}$$

Substituting for q from Eq. (11.20) we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2}\oint\frac{\mathrm{d}s}{Gt} \tag{11.24}$$

or, if G, the shear modulus, is constant round the section

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2G}\oint\frac{\mathrm{d}s}{t} \tag{11.25}$$

EXAMPLE 11.9

A thin-walled circular section beam has a diameter of 200 mm and is 2 m long; it is firmly restrained against rotation at each end. A concentrated torque of 30 kN m is applied to the beam at its mid-span point. If the maximum shear stress in the beam is limited to 200 N/mm^2 and the maximum angle of twist to 2°, calculate the minimum thickness of the beam walls. Take $G = 25000$ N/mm^2.

The minimum thickness of the beam corresponding to the maximum allowable shear stress of 200 N/mm^2 is obtained directly using Eq. (11.22) in which $T_{\max} = 15$ kNm. Thus

$$t_{\min} = \frac{15\times10^6\times4}{2\times\pi\times200^2\times200} = 1.2\text{ mm}$$

The rate of twist along the beam is given by Eq. (11.25) in which

$$\oint\frac{\mathrm{d}s}{t} = \frac{\pi\times200}{t_{\min}}$$

Hence

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{T}{4A^2G}\times\frac{\pi\times200}{t_{\min}} \tag{i}$$

Taking the origin for x at one of the fixed ends and integrating Eq. (i) for half the length of the beam we obtain

$$\theta = \frac{T}{4A^2G}\times\frac{200\pi}{t_{\min}}x + C_1$$

where C_1 is a constant of integration. At the fixed end where $x = 0$, $\theta = 0$ so that $C_1 = 0$. Hence

$$\theta = \frac{T}{4A^2G}\times\frac{200\pi}{t_{\min}}x$$

The maximum angle of twist occurs at the mid-span of the beam where $x = 1$ m. Hence

$$t_{\min} = \frac{15\times10^6\times200\times\pi\times1\times10^3\times180}{4\times(\pi\times200^2/4)^2\times25\,000\times2\times\pi} = 2.7\text{mm}$$

The minimum allowable thickness that satisfies both conditions is therefore 2.7 mm.

11.5 Torsion of solid section beams

Generally, by solid section beams, we mean beam sections in which the walls do not form a closed loop system. Examples of such sections are shown in Fig. 11.15. An obvious exception is the hollow circular section bar which is, however, a special case of the solid circular section bar. The prediction of stress distributions and angles of twist produced by the torsion of such sections is complex and relies on the St. Venant warping function or Prandtl stress function methods of solution. Both of these methods are based on the theory of elasticity which may be found in advanced texts devoted solely to this topic.

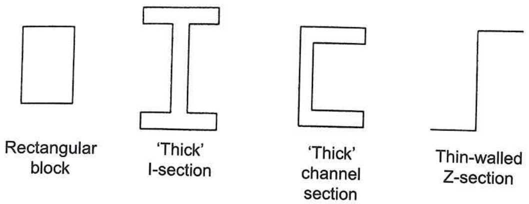


FIGURE 11.15

Examples of solid beam sections.

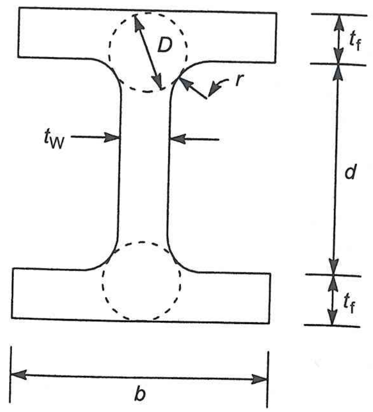


FIGURE 11.16

Torsion constant for a 'thick' I-section beam.

In all torsion problems, however, it is found that the torque, T, and the rate of twist, $d\theta/dx$, are related by the equation

$$T = GJ\frac{d\theta}{dx} \tag{11.26}$$

where G is the shear modulus and J is the *torsion constant*. For a circular section bar J is the polar second moment of area of the section (see Eq. (11.3)) while for a thin-walled closed section beam J, from Eq. (11.25), is seen to be equal to $4A^2/\oint(ds/t)$. It is J, in fact, that distinguishes one torsion problem from another.

For 'thick' sections of the type shown in Fig. 11.15 J is obtained empirically in terms of the dimensions of the particular section. For example, the torsion constant of the 'thick' I-section shown in Fig. 11.16 is given by

$$J = 2J_1 + J_2 + 2\alpha D^4$$

where

$$J_1 = \frac{bt_f^3}{3}\left[1 - 0.63\frac{t_f}{b}\left(1 - \frac{t_f^4}{12b^4}\right)\right]$$

$$J_2 = \frac{1}{3}dt_w^3$$

$$\alpha = \frac{t_1}{t_2}\left(0.15 + 0.1\frac{r}{t_f}\right)$$

in which $t_1 = t_f$ and $t_2 = t_w$ if $t_f < t_w$, or $t_1 = t_w$ and $t_2 = t_f$ if $t_f > t_w$.

It can be seen from the above that J_1 and J_2, which are the torsion constants of the flanges and web, respectively, are each equal to one-third of the product of their length and their thickness cubed multiplied, in the case of the flanges, by an empirical constant. The torsion constant for the complete section is then

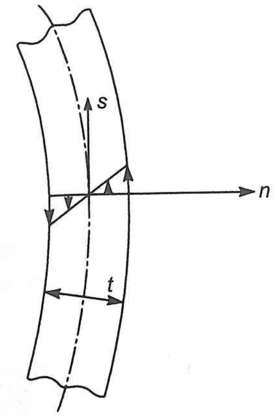


FIGURE 11.17

Shear stress distribution due to torsion in a thin-walled open section beam.

the sum of the torsion constants of the components plus a contribution from the material at the web/flange junction. If the section were thin-walled, $t_f \ll b$ and D^4 would be negligibly small, in which case

$$J \simeq 2\frac{bt_f^3}{3} + \frac{dt_w^3}{3}$$

Generally, for thin-walled sections the torsion constant J may be written as

$$J = \frac{1}{3}\sum st^3 \tag{11.27}$$

in which s is the length and t the thickness of each component in the cross section or if t varies with s

$$J = \frac{1}{3}\int_{\text{section}} t^3 ds \tag{11.28}$$

The shear stress distribution in a thin-walled open section beam (Fig. 11.17) may be shown to be related to the rate of twist by the expression

$$\tau = 2Gn\frac{d\theta}{dx} \tag{11.29}$$

where n is the distance to any point in the section wall measured normally from its midline. The distribution is therefore linear across the thickness as shown in Fig. 11.17 and is zero at the midline of the wall. An alternative expression for shear stress distribution is obtained, in terms of the applied torque, by substituting for $d\theta/dx$ in Eq. (11.29) from Eq. (11.26). Thus

$$\tau = 2n\frac{T}{J} \tag{11.30}$$

It is clear from either of Eqs. (11.29) or (11.30) that the maximum value of shear stress occurs at the outer surfaces of the wall when $n = \pm t/2$. Hence

$$\tau_{max} = \pm Gt\frac{d\theta}{dx} = \pm\frac{Tt}{J} \tag{11.31}$$

The positive and negative signs in Eq. (11.31) indicate the direction of the shear stress in relation to the assumed direction for s.

The behaviour of closed and open section beams under torsional loads is similar in that they twist and develop internal shear stress systems. However, the manner in which each resists torsion is different. It is clear from the preceding discussion that a pure torque applied to a beam section produces a closed, continuous shear stress system since the resultant of any other shear stress system would generally be a shear force unless, of course, the system were self-equilibrating. In a closed section beam this closed loop system of shear stresses is allowed to develop in a continuous path round the cross section, whereas in an open section beam it can only develop within the thickness of the walls;

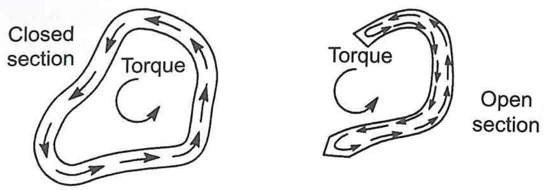


FIGURE 11.18

Shear stress development in closed and open section beams subjected to torsion.

in which torsion is resisted by closed and open section beams and the reason for the comparatively low torsional stiffness of thin-walled open sections. Clearly the development of a closed loop system of shear stresses in an open section is restricted by the thinness of the walls.

EXAMPLE 11.10

The thin-walled section shown in Fig. 11.19 is symmetrical about a horizontal axis through O. The thickness t_0 of the centre web CD is constant, while the thickness of the other walls varies linearly from t_0 at points C and D to zero at the open ends A, F, G and H. Determine the torsion constant J for the section and also the maximum shear stress produced by a torque T.

Since the thickness of the section varies round its profile except for the central web, we use both Eqs (11.27) and (11.28) to determine the torsion constant. Thus,

$$J = \frac{2at_0^3}{3} + 2 \times \frac{1}{3}\int_0^a \left(\frac{s_A t_0}{a}\right)^3 ds_A + 2 \times \frac{1}{3}\int_0^{3a} \left(\frac{s_B t_0}{3a}\right)^3 ds_B$$

which gives

$$J = \frac{4at_0^3}{3}$$

The maximum shear stress is now obtained using Eq. (11.31), i.e.

$$\tau_{\max} = \pm\frac{Tt_0}{J} = \pm\frac{3Tt_0}{4at_0^3} = \pm\frac{3T}{4at_0^2}$$

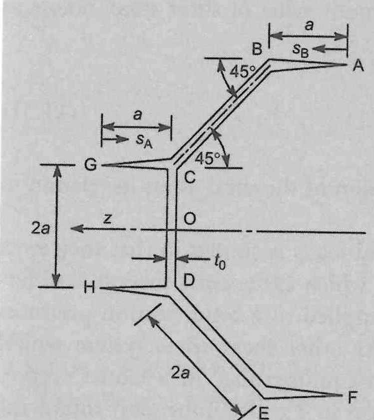


FIGURE 11.19

Beam section of Ex. 11.10.

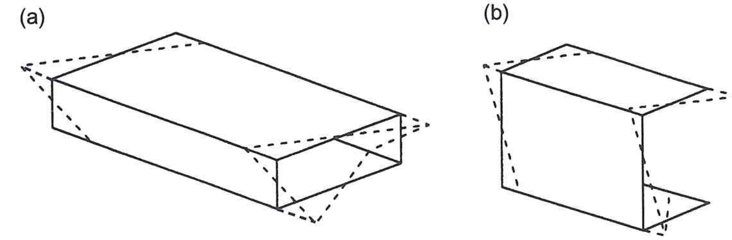


FIGURE 11.20

Warping of beam sections due to torsion.

11.6 Warping of cross sections under torsion

Although we have assumed that the shapes of closed and open beam sections remain undistorted during torsion, they do not remain plane. Thus, for example, the cross section of a rectangular section box beam, although remaining rectangular when twisted, warps out of its plane as shown in Fig. 11.20(a), as does the channel section of Fig. 11.20(b). The calculation of warping displacements is covered in more advanced texts and is clearly of importance if a beam is, say, built into a rigid foundation at one end. In such a situation the warping is suppressed and direct tensile and compressive stresses are induced which must be investigated in design particularly if a beam is of concrete where even low tensile stresses can cause severe cracking.

Some beam sections do not warp under torsion; these include solid (and hollow) circular section bars and square box sections of constant thickness.

PROBLEMS

P.11.1 The solid bar of circular cross section shown in Fig. P.11.1 is subjected to a torque of 1 kN m at its free end and a torque of 3 kN m at its change of section. Calculate the maximum shear stress in the bar and the angle of twist at its free end. $G = 70000$ N/mm^2.

Ans. 40.7 N/mm^2, 0.6°.

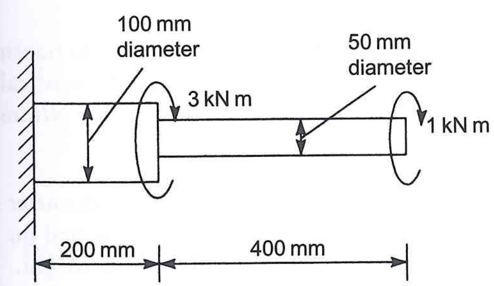


FIGURE P.11.1

P.11.2 A hollow circular section shaft 2 m long is firmly supported at each end and has an outside diameter of 80 mm. The shaft is subjected to a torque of 12 kN m applied at a point 1.5 m from one end. If the shear stress in the shaft is limited to 150 N/mm^2 and the angle of twist to 1.5°, calculate the maximum allowable internal diameter. The shear modulus $G = 80000$ N/mm^2.

P.11.3 A bar ABCD of circular cross section having a diameter of 50 mm is firmly supported at each end and carries two concentrated torques at B and C as shown in Fig. P.11.3. Calculate the maximum shear stress in the bar and the maximum angle of twist. Take $G = 70000$ N/mm^2.

Ans. 66.2 N/mm^2 in CD, 2.3° at B.

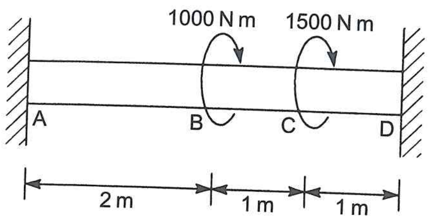


FIGURE P.11.3

P.11.4 A bar ABCD has a circular cross section of 75 mm diameter over half its length and 50 mm diameter over the remaining half of its length. A torque of 1 kN m is applied at C midway between B and D as shown in Fig. P.11.4. Sketch the distribution of torque along the length of the bar and calculate the maximum shear stress and the maximum angle of twist in the bar j $G = 70\ 000$ N/mm^2.

Ans. $\tau_{max} = 23.2$ N/mm^2 in CD, 0.38° at C.

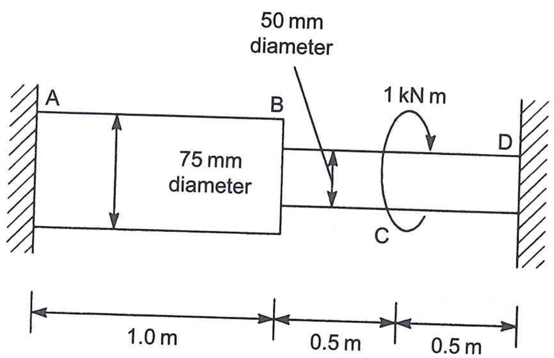


FIGURE P.11.4

P.11.5 A solid shaft has a circular cross section of diameter 150 mm and is required to transmit power at 90 rpm. If the maximum shear stress in the shaft is limited to 85 N/mm^2 calculate the power transmitted and the angle twist in a length of 5 m. Take $G = 80000$ N/mm^2.

Ans. 531 kW, 4.1°.

P.11.6 If the solid shaft of P.11.5 is replaced by a hollow shaft of the same external diameter and having walls 30 mm thick calculate the percentage reduction in power transmitted for the same limiting value of shear stress. Calculate also the percentage reduction in weight.

Ans. 12.8%, 36%.

P.11.7 The bar shown in Fig. P.11.7 carries a single torque of 10 kNm applied mid-way along its length. Use strain energy to calculate the angle of twist under the applied torque and hence the angle of twist at its free end. Take $G = 75000$ N/mm^2.

Ans. 2.9° at both points.

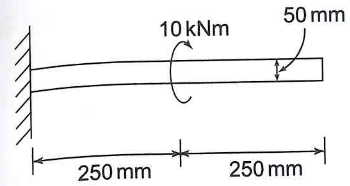


FIGURE P.11.7

P.11.8 Determine the minimum diameter of a solid bar which is required to transmit a torque of 40 kNm if the yield stress of the material of the bar is 120 N/mm^2.

Ans. 119.2 mm.

P.11.9 If the torque on the bar of P.11.8 is increased to 45 kNm calculate the diameter of the elastic core and the angle of twist of the bar over a length of 5 m.

Take $G = 80000$ N/mm^2.

Ans. 101.4 mm, 8.5°.

P.11.10 A hollow section bar has an outside diameter of 120 mm and an inside diameter of 60 mm. If the shear stress at yield in the material of the bar is 100 N/mm^2 calculate the maximum torque the bar can transmit without yielding occurring. If this torque is increased by 20% determine the outer radius of the elastic core of the section and the angle of twist over a length of 5 m. Take $G = 80000$ N/mm^2.

Ans. 42.8 mm (by trial and error), 8.3°.

P.11.11 A thin-walled rectangular section box girder carries a uniformly distributed torque loading of 1 kN m/mm over the outer half of its length as shown in Fig. P.11.11. Calculate the maximum shear stress in the walls of the box girder and also the distribution of angle of twist along its length; illustrate your answer with a sketch. Take $G = 70\ 000$ N/mm^2.

Ans. 133.3 N/mm^2. In AB, $\theta = 3.81 \times 10^{-6}x$ rad.

In BC, $\theta = 1.905 \times 10^{-9}(4000x - x^2/2) - 0.00381$ rad.

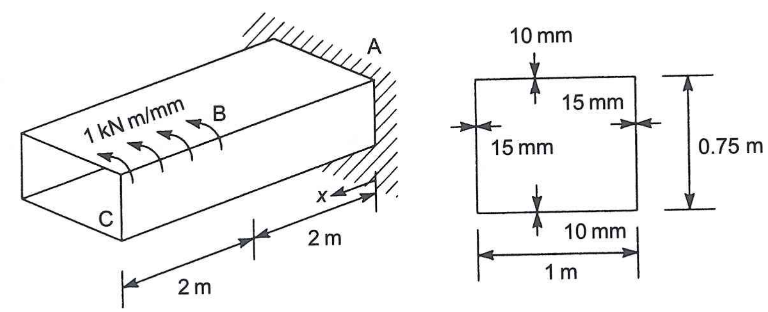


FIGURE P.11.11

P.11.12 The closed section thin-walled beam shown in Fig. P.11.12 is subjected to a torque of 4.5 kNm. For the curved wall 12 the thickness is 2 mm and the shear modulus is 22000 N/mm^2. For the walls 23, 34 and 41 the corresponding figures are 1.6 mm and 27500 N/mm^2. (Note: Gt = constant). Calculate the rate of twist of the beam section.

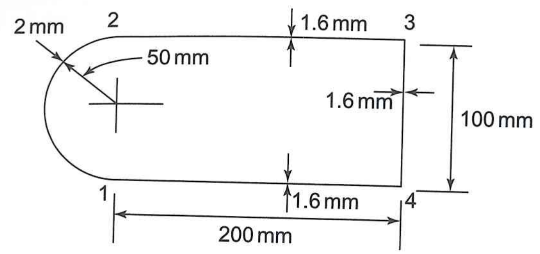


FIGURE P.11.12

P.11.13 The thin-walled box section beam ABCD shown in Fig. P.11.13 is attached at each end to supports which allow rotation of the ends of the beam in the longitudinal vertical plane of symmetry but prevent rotation of the ends in vertical planes perpendicular to the longitudinal axis of the beam. The beam is subjected to a uniform torque loading of 20 Nm/mm over the portion BC of its span. Calculate the maximum shear stress in the cross section of the beam and the distribution of angle of twist along its length; $G = 70\ 000$ N/mm^2.

Ans. 71.4 N/mm^2, $\theta_B = \theta_C = 0.36°$, θ at mid-span $= 0.72°$.

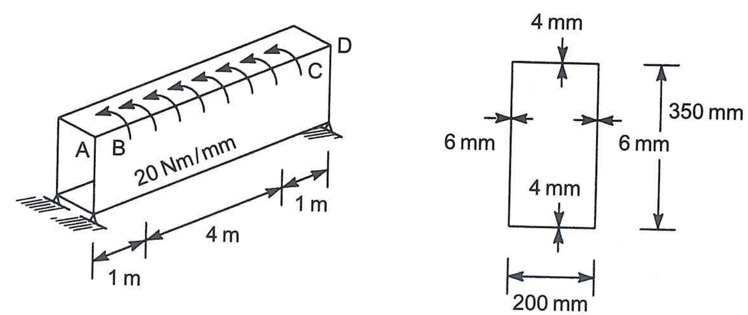


FIGURE P.11.13

P.11.14 Figure P.11.14 shows a thin-walled cantilever box-beam having a constant width of 50 mm and a depth which decreases linearly from 200 mm at the built-in end to 150 mm at the free end. If the beam is subjected to a torque of 1 kN m at its free end, plot the angle of twist of the beam at 500 mm intervals along its length and determine the maximum shear stress in the beam section. Take $G = 25\ 000$ N/mm^2.

Ans. $\tau_{max} = 33.3$ N/mm^2.

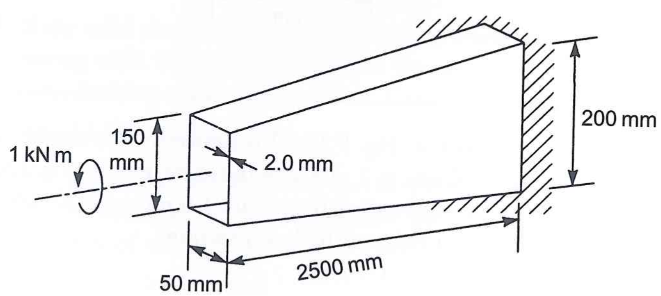


FIGURE P.11.14

P.11.15 The cold-formed section shown in Fig. P.11.15 is subjected to a torque of 50 Nm. Calculate the maximum shear stress in the section and its rate of twist. $G = 25\ 000$ N/mm^2.

Ans. $\tau_{max} = 220.6$ N/mm^2, $d\theta/dx = 0.0044$ rad/mm.

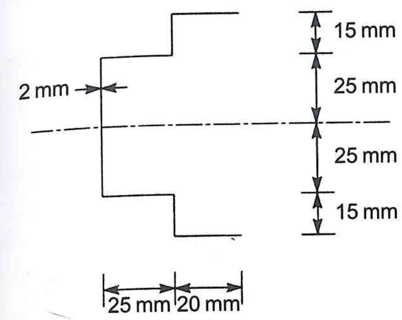


FIGURE P.11.15

P.11.16 The thin-walled angle section shown in Fig. P.11.16 supports shear loads that produce both shear and torsional effects. Determine the maximum shear stress in the cross section of the angle, stating clearly the point at which it acts.

Ans. 18.0 N/mm^2 on the inside of flange BC at 16.5 mm from point B.

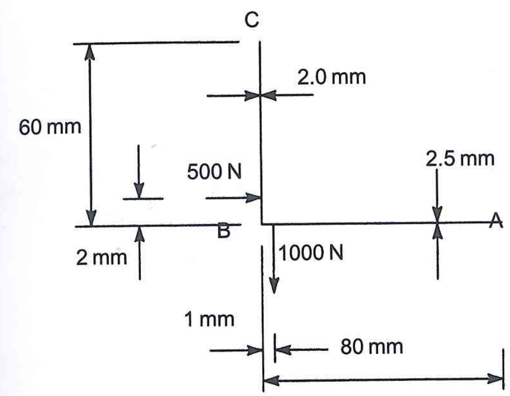


FIGURE P.11.16

P.11.17 Figure P.11.17 shows the cross section of a thin-walled inwardly lipped channel. The lips are of constant thickness while the flanges increase linearly in thickness from 1.27 mm, where

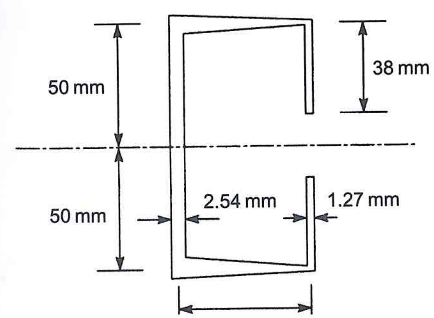


FIGURE P.11.17

they meet the lips, to 2.54 mm at their junctions with the web. The web has a constant thickness of 2.54 mm and the shear modulus G is 26 700 N/mm^2. Calculate the maximum shear stress in the section and also its rate of twist if it is subjected to a torque of 100 Nm.

Ans. $\tau_{max} = \pm 297.4$ N/mm^2, $d\theta/dx = 0.0044$ rad/mm.

P.11.18 The thin-walled section shown in Fig. P.11.18 is subjected to a unit torque. Calculate the maximum shear stress in the section.

Ans. $\pm 0.42/rt^2$.

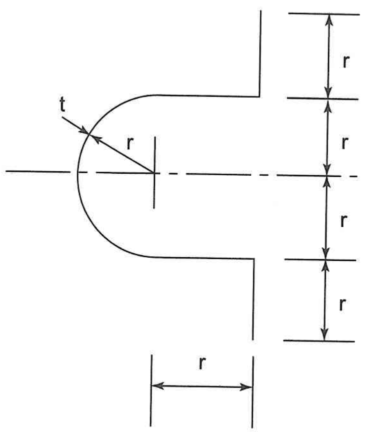


FIGURE P.11.18

P.11.19 Determine the maximum shear stress in the beam section shown in Fig. P.11.19 stating clearly the point at which it occurs. Calculate also the rate of twist of the section if the shear modulus $G = 25000$ N/mm^2.

Ans. 70.2 N/mm^2 on the underside of 24 at 2 or on the upper surface of 32 at 2. 9.0×10^{-4} rad/mm.

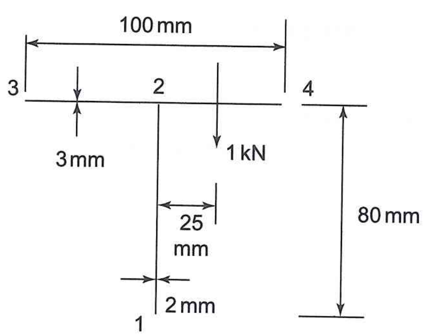


FIGURE P.11.19

CHAPTER

Composite Beams

12

Frequently in civil engineering construction beams are fabricated from comparatively inexpensive materials of low strength which are reinforced by small amounts of high-strength material, such as steel. In this way a timber beam of rectangular section may have steel plates bolted to its sides or to its top and bottom surfaces. Again, concrete beams are reinforced in their weak tension zones and also, if necessary, in their compression zones, by steel-reinforcing bars. Other instances arise where steel beams support concrete floor slabs in which the strength of the concrete may be allowed for in the design of the beams. The design of reinforced concrete beams, and concrete and steel beams is covered by Codes of Practice and relies, as in the case of steel beams, on ultimate load analysis. The design of steel-reinforced timber beams is not covered by a code, and we shall therefore limit the analysis of this type of beam to an elastic approach.

12.1 Steel-reinforced timber beams

The timber joist of breadth b and depth d shown in Fig. 12.1 is reinforced by two steel plates bolted to its sides, each plate being of thickness t and depth d. Let us suppose that the beam is bent to a radius R at this section by a positive bending moment, M. Clearly, since the steel plates are firmly attached to the sides of the timber joist, both are bent to the same radius, R. Then, from Eq. (9.7), the bending moment, M_t, carried by the timber joist is

$$M_t = \frac{E_t I_t}{R} \tag{12.1}$$

where E_t is Young's modulus for the timber and I_t is the second moment of area of the timber section about the centroidal axis, Gz. Similarly for the steel plates

$$M_s = \frac{E_s I_s}{R} \tag{12.2}$$

in which I_s is the combined second moment of area about Gz of the two plates. The total bending moment is then

$$M = M_t + M_s = \frac{1}{R}(E_t I_t + E_s I_s)$$

from which

$$\frac{1}{R} = \frac{M}{E_t I_t + E_s I_s} \tag{12.3}$$

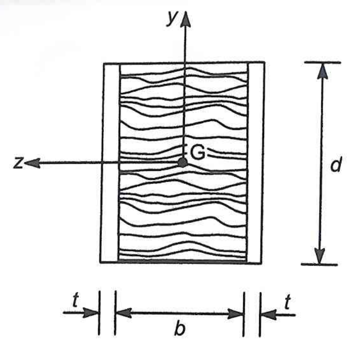


FIGURE 12.1

Steel-reinforced timber beam.

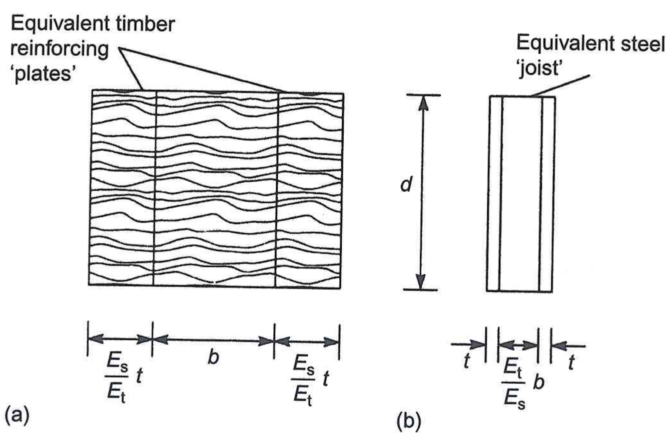


FIGURE 12.2

Equivalent beam sections.

From a comparison of Eqs (12.3) and (9.7) we see that the composite beam behaves as a homogeneous beam of bending stiffness EI where

$$EI = E_t I_t + E_s I_s$$

or

$$EI = E_t\left(I_t + \frac{E_s}{E_t} I_s\right) \tag{12.4}$$

The composite beam may therefore be treated wholly as a timber beam having a total second moment of area

$$I_t + \frac{E_s}{E_t} I_s$$

This is equivalent to replacing the steel-reinforcing plates by timber 'plates' each having a thickness $(E_s/E_t)t$ as shown in Fig. 12.2(a). Alternatively, the beam may be transformed into a wholly steel beam by writing Eq. (12.4) as

$$EI = E_s\left(\frac{E_t}{E_s} I_t + I_s\right)$$

so that the second moment of area of the equivalent steel beam is

$$\frac{E_t}{E_s} I_t + I_s$$

which is equivalent to replacing the timber joist by a steel 'joist' of breadth $(E_t/E_s)b$ (Fig. 12.2(b)). Note that the transformed sections of Fig. 12.2 apply only to the case of bending about the horizontal axis, Gz. Note also that the depth, d, of the beam is unchanged by either transformation.

The direct stress due to bending in the timber joist is obtained using Eq. (9.9), i.e.

$$\sigma_t = -\frac{M_t y}{I_t} \tag{12.5}$$

From Eqs (12.1) and (12.3)

$$M_t = \frac{E_t I_t}{E_t I_t + E_s I_s} M$$

or

$$M_t = \frac{M}{1 + \frac{E_s I_s}{E_t I_t}} \tag{12.6}$$

Substituting in Eq. (12.5) from Eq. (12.6) we have

$$\sigma_t = -\frac{My}{I_t + \frac{E_s}{E_t} I_s} \tag{12.7}$$

Equation (12.7) could in fact have been deduced directly from Eq. (9.9) since $I_t + (E_s/E_t)I_s$ is the second moment of area of the equivalent timber beam of Fig. 12.2(a). Similarly, by considering the equivalent steel beam of Fig. 12.2(b), we obtain the direct stress distribution in the steel, i.e.

$$\sigma_s = -\frac{My}{I_s + \frac{E_t}{E_s} I_t} \tag{12.8}$$

EXAMPLE 12.1

A beam is formed by connecting two timber joists each 100 mm × 400 mm with a steel plate 12 mm × 300 mm placed symmetrically between them (Fig. 12.3). If the beam is subjected to a bending moment of 50 kN m, determine the maximum stresses in the steel and in the timber. The ratio of Young's modulus for steel to that of timber is 12 : 1.

The second moments of area of the timber and steel about the centroidal axis, Gz, are

$$I_t = 2 \times 100 \times \frac{400^3}{12} = 1067 \times 10^6 \text{ mm}^4$$

and

$$I_s = 12 \times \frac{300^3}{12} = 27 \times 10^6 \text{ mm}^4$$

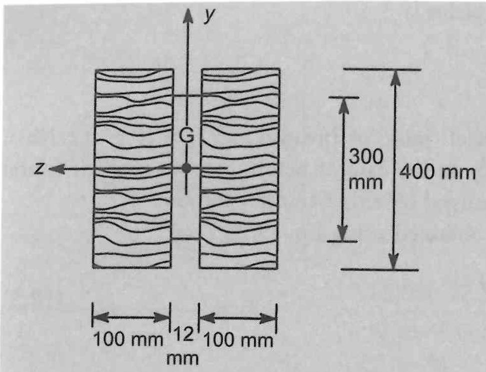


FIGURE 12.3

Steel-reinforced timber beam of Ex. 12.1.

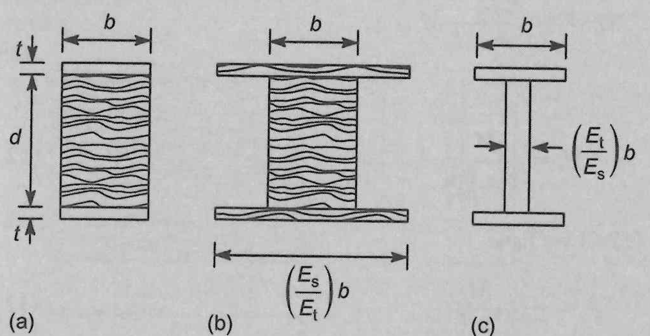


FIGURE 12.4

Reinforced timber beam with steel plates attached to its top and bottom surfaces.

respectively. Therefore, from Eq. (12.7) we have

$$\sigma_t = \pm \frac{50 \times 10^6 \times 200}{1067 \times 10^6 + 12 \times 27 \times 10^6} = \pm 7.2 \text{ N/mm}^2$$

and from Eq. (12.8)

$$\sigma_s = \pm \frac{50 \times 10^6 \times 150}{27 \times 10^6 + 1067 \times 10^6/12} = \pm 64.7 \text{ N/mm}^2$$

Consider now the steel-reinforced timber beam of Fig. 12.4(a) in which the steel plates are attached to the top and bottom surfaces of the timber. The section may be transformed into an equivalent timber beam (Fig. 12.4(b)) or steel beam (Fig. 12.4(c)) by the methods used for the beam of Fig. 12.1. The direct stress distributions are then obtained from Eqs (12.7) and (12.8). There is, however, one important difference between the beam of Fig. 12.1 and that of Fig. 12.4(a). In the latter case, when the beam is subjected to shear loads, the connection between the timber and steel must resist horizontal complementary shear stresses as shown in Fig. 12.5. Generally, it is sufficiently accurate to assume that the timber joist resists all the vertical shear and then calculate an average value of shear stress, τ_{av}, i.e.

$$\tau_{av} = \frac{S_y}{bd}$$

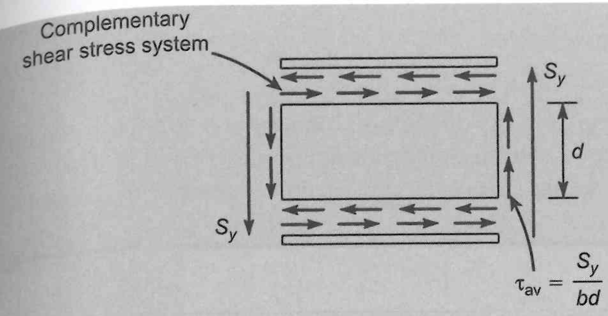


FIGURE 12.5

Shear stresses between steel plates and timber beam (side view of a length of beam).

so that, based on this approximation, the horizontal complementary shear stress is S_y/bd and the shear force per unit length resisted by the timber/steel connection is S_y/d.

EXAMPLE 12.2

A timber joist 100 mm × 200 mm is reinforced on its top and bottom surfaces by steel plates 15 mm thick × 100 mm wide. The composite beam is simply supported over a span of 4 m and carries a uniformly distributed load of 10 kN/m. Determine the maximum direct stress in the timber and in the steel and also the shear force per unit length transmitted by the timber/steel connection. Take $E_s/E_t = 15$.

The second moments of area of the timber and steel about a horizontal axis through the centroid of the beam are

$$I_t = \frac{100 \times 200^3}{12} = 66.7 \times 10^6 \text{ mm}^4$$

and

$$I_s = 2 \times 15 \times 100 \times 107.5^2 = 34.7 \times 10^6 \text{mm}^4$$

respectively. Note that the second moment of area of a steel plate about an axis through its own centroid is negligibly small. The maximum bending moment in the beam occurs at mid-span and is

$$M_{max} = \frac{10 \times 4^4}{8} = 20 \text{ kN m}$$

From Eq. (12.7)

$$\sigma_{t,max} = \pm \frac{20 \times 10^6 \times 100}{66.7 \times 10^6 + 15 \times 34.7 \times 10^6} = \pm 3.4 \text{ N/mm}^2$$

and from Eq. (12.8)

$$\sigma_{s,max} = \pm \frac{20 \times 10^6 \times 115}{34.7 \times 10^6 + 66.7 \times 10^6/15} = \pm 58.8 \text{ N/mm}^2$$

The maximum shear force in the beam occurs at the supports and is equal to 10 × 4/2 = 20 kN. The average shear stress in the timber joist is then

$$\tau_{av} = \frac{20 \times 10^3}{100 \times 200} = 1\ \text{N/mm}^2$$

It follows that the shear force per unit length in the timber/steel connection is $1 \times 100 = 100$ N/mm or 100 kN/m. Note that this value is an approximation for design purposes since, as we saw in Chapter 10, the distribution of shear stress through the depth of a beam of rectangular section is not uniform.

12.2 Reinforced concrete beams

As we have noted in Chapter 8, concrete is a brittle material which is weak in tension. It follows that a beam comprised solely of concrete would have very little bending strength since the concrete in the tension zone of the beam would crack at very low values of load. Concrete beams are therefore reinforced in their tension zones (and sometimes in their compression zones) by steel bars embedded in the concrete. Generally, whether the beam is precast or forms part of a slab/beam structure, the bars are positioned in a mould (usually fabricated from timber and called formwork) into which the concrete is poured. On setting, the concrete shrinks and grips the steel bars; the adhesion or *bond* between the bars and the concrete transmits bending and shear loads from the concrete to the steel.

In the design of reinforced concrete beams the elastic method has been superseded by the ultimate load method. We shall, however, for completeness, consider both methods.

Elastic theory

Consider the concrete beam section shown in Fig. 12.6(a). The beam is subjected to a bending moment, M, and is reinforced in its tension zone by a number of steel bars of total cross-sectional area A_s. The centroid of the reinforcement is at a depth d_1 from the upper surface of the beam; d_1 is known as the *effective depth* of the beam. The bending moment, M, produces compression in the concrete above the neutral axis whose position is at some, as yet unknown, depth, n, below the upper surface of the beam. Below the neutral axis the concrete is in tension and is assumed to crack so that its contribution to the bending strength of the beam is negligible. All tensile forces are therefore resisted by the reinforcing steel.

The reinforced concrete beam section may be conveniently analysed by the method employed in Section 12.1 for steel-reinforced beams. The steel reinforcement is, therefore, transformed into an equivalent area, mA_s, of concrete in which m, the *modular ratio*, is given by

$$m = \frac{E_s}{E_c}$$

where E_s and E_c are Young's moduli for steel and concrete, respectively. The transformed section is shown in Fig. 12.6(b). Taking moments of areas about the neutral axis we have

$$bn\frac{n}{2} = mA_s(d_1 - n)$$

which, when rearranged, gives a quadratic equation in n, i.e.

$$\frac{bn^2}{2} + mA_s n - mA_s d_1 = 0 \qquad (12.9)$$

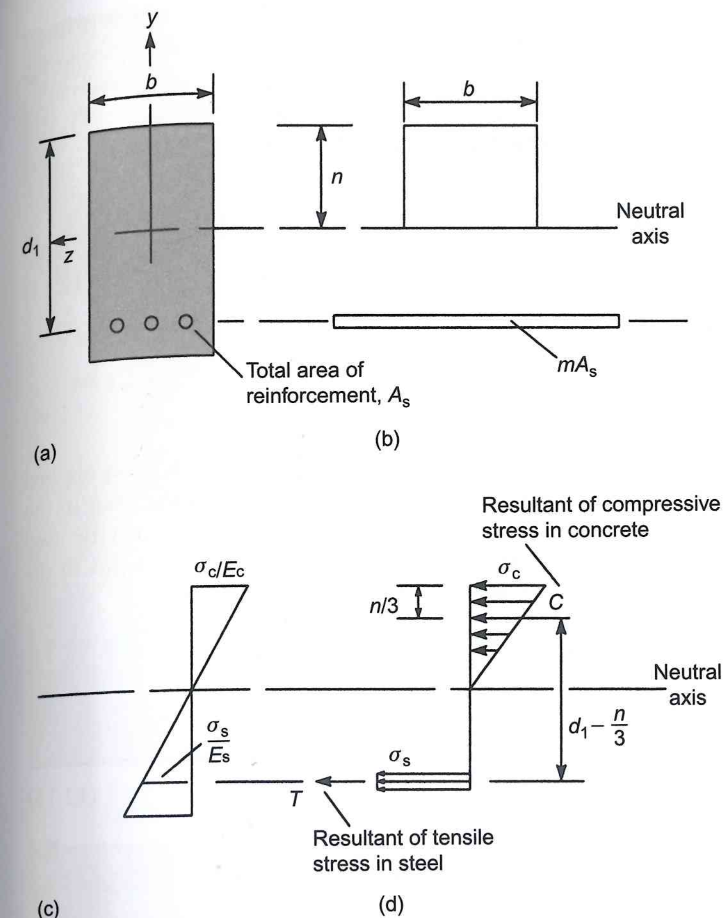


FIGURE 12.6

Reinforced concrete beam.

solving gives

$$n = \frac{mA_s}{b}\left(\sqrt{1 + \frac{2bd_1}{mA_s}} - 1\right) \qquad (12.10)$$

Note that the negative solution of Eq. (12.9) has no practical significance and is therefore ignored. The second moment of area, I_c, of the transformed section is

$$I_c = \frac{bn^3}{3} + mA_s(d_1 - n)^2 \qquad (12.11)$$

and the maximum stress, σ_c, induced in the concrete is

$$\sigma_c = -\frac{Mn}{I_c} \qquad (12.12)$$

The stress, σ_s, in the steel may be deduced from the strain diagram (Fig. 12.6(c)) which is linear throughout the depth of the beam since the beam section is assumed to remain plane during bending. Then

$$\frac{\sigma_s/E_s}{d_1 - n} = -\frac{\sigma_c/E_c}{n} \quad \text{(note: strains are of opposite sign)}$$

from which

$$\sigma_s = -\sigma_c \frac{E_s}{E_c}\left(\frac{d_1 - n}{n}\right) = -\sigma_c m\left(\frac{d_1 - n}{n}\right) \tag{12.13}$$

Substituting for σ_c from Eq. (12.12) we obtain

$$\sigma_s = \frac{mM}{I_c}(d_1 - n) \tag{12.14}$$

Frequently, instead of determining stresses in a given beam section subjected to a given applied bending moment, we wish to calculate the moment of resistance of a beam when either the stress in the concrete or the steel reaches a maximum allowable value. Equations (12.12) and (12.14) may be used to solve this type of problem but an alternative and more direct method considers moments due to the resultant loads in the concrete and steel. From the stress diagram of Fig. 12.6(d)

$$M = C\left(d_1 - \frac{n}{3}\right)$$

so that

$$M = \frac{\sigma_c}{2} bn\left(d_1 - \frac{n}{3}\right) \tag{12.15}$$

Alternatively, taking moments about the centroid of the concrete stress diagram

$$M = T\left(d_1 - \frac{n}{3}\right)$$

or

$$M = \sigma_s A_s\left(d_1 - \frac{n}{3}\right) \tag{12.16}$$

Equation (12.16) may also be used in conjunction with Eq. (12.13) to 'design' the area of reinforcing steel in a beam section subjected to a given bending moment so that the stresses in the concrete and steel attain their maximum allowable values simultaneously. Such a section is known as a *critical* or *economic* section. The position of the neutral axis is obtained directly from Eq. (12.13) in which σ_s, σ_c, m and d_1 are known. The required area of steel is then determined from Eq. (12.16).

EXAMPLE 12.3

A rectangular section reinforced concrete beam has a breadth of 200 mm and is 350 mm deep to the centroid of the steel reinforcement which consists of two steel bars each having a diameter of

20 mm. If the beam is subjected to a bending moment of 30 kN m, calculate the stress in the concrete and in the steel. The modular ratio m is 15.

The area A_s of the steel reinforcement is given by

$$A_s = 2 \times \frac{\pi}{4} \times 20^2 = 628.3 \text{ mm}^2$$

The position of the neutral axis is obtained from Eq. (12.10) and is

$$n = \frac{15 \times 628.3}{200}\left(\sqrt{1 + \frac{2 \times 200 \times 350}{15 \times 628.3}} - 1\right) = 140.5 \text{ mm}$$

Now using Eq. (12.11)

$$I_c = \frac{200 \times 140.5^3}{3} + 15 \times 628.3(350 - 140.5)^2 = 598.5 \times 10^6 \text{ mm}^4$$

The maximum stress in the concrete follows from Eq. (12.12), i.e.

$$\sigma_c = -\frac{30 \times 10^6 \times 140.5}{598.5 \times 10^6} = -7.0 \text{ N/mm}^2\text{(compression)}$$

and from Eq. (12.14)

$$\sigma_s = \frac{15 \times 30 \times 10^6}{598.5 \times 10^6}(350 - 140.5) = 157.5 \text{ N/mm}^2\text{(tension)}$$

EXAMPLE 12.4

A reinforced concrete beam has a rectangular section of breadth 250 mm and a depth of 400 mm to the steel reinforcement, which consists of three 20 mm diameter bars. If the maximum allowable stresses in the concrete and steel are 7.0 N/mm² and 140 N/mm², respectively, determine the moment of resistance of the beam. The modular ratio $m = 15$.

The area, A_s, of steel reinforcement is

$$A_s = 3 \times \frac{\pi}{4} \times 20^2 = 942.5 \text{ mm}^2$$

From Eq. (12.10)

$$n = \frac{15 \times 942.5}{250}\left(\sqrt{1 + \frac{2 \times 250 \times 400}{15 \times 942.5}} - 1\right) = 163.5 \text{ mm}$$

The maximum bending moment that can be applied such that the permissible stress in the concrete is not exceeded is given by Eq. (12.15). Thus

$$M = \frac{7}{2} \times 250 \times 163.5\left(400 - \frac{163.5}{3}\right) \times 10^{-6} = 49.4 \text{ kNm}$$

Similarly, from Eq. (12.16) the stress in the steel limits the applied moment to

$$M = 140 \times 942.5\left(400 - \frac{163.5}{3}\right) \times 10^{-6} = 45.6 \text{ kNm}$$

The steel is therefore the limiting material and the moment of resistance of the beam is 45.6 kN m.

EXAMPLE 12.5

A rectangular section reinforced concrete beam is required to support a bending moment of 40 kN m and is to have dimensions of breadth 250 mm and effective depth 400 mm. The maximum allowable stresses in the steel and concrete are 120 N/mm^2 and 6.5 N/mm^2, respectively; the modular ratio is 15. Determine the required area of reinforcement such that the limiting stresses in the steel and concrete are attained simultaneously.

Using Eq. (12.13) we have

$$120 = 6.5 \times 15\left(\frac{400}{n} - 1\right)$$

from which $n = 179.3$ mm.

The required area of steel is now obtained from Eq. (12.16); hence

$$A_s = \frac{M}{\sigma_s(d_1 - n/3)}$$

i.e.

$$A_s = \frac{40 \times 10^6}{120(400 - 179.3/3)} = 979.7 \text{ mm}^2$$

It may be seen from Ex. 12.4 that for a beam of given cross-sectional dimensions, increases in the area of steel reinforcement do not result in increases in the moment of resistance after a certain value has been attained. When this stage is reached the concrete becomes the limiting material, so that additional steel reinforcement only serves to reduce the stress in the steel. However, the moment of resistance of a beam of a given cross section may be increased above the value corresponding to the limiting concrete stress by the addition of steel in the compression zone of the beam.

Figure 12.7(a) shows a concrete beam reinforced in both its tension and compression zones. The centroid of the compression steel of area A_{sc} is at a depth d_2 below the upper surface of the beam, while the tension steel of area A_{st} is at a depth d_1. The section may again be transformed into an equivalent concrete section as shown in Fig. 12.7(b).

However, when determining the second moment of area of the transformed section it must be remembered that the area of concrete in the compression zone is reduced due to the presence of the steel. Thus taking moments of areas about the neutral axis we have

$$\frac{bn^2}{2} - A_{sc}(n - d_2) + mA_{sc}(n - d_2) = mA_{st}(d_1 - n)$$

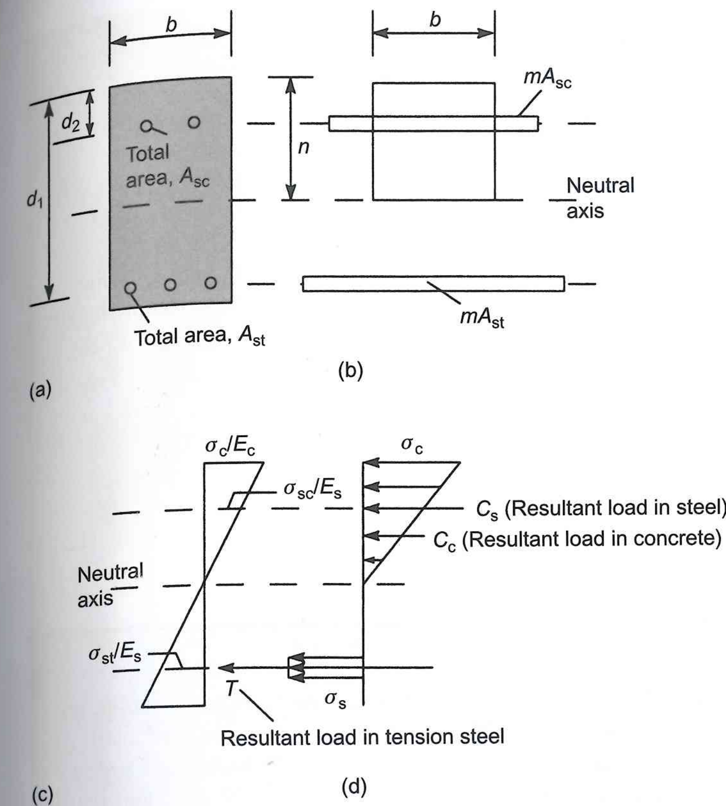


FIGURE 12.7

Reinforced concrete beam with steel in tension and compression zones.

or, rearranging

$$\frac{bn^2}{2} + (m-1)A_{sc}(n - d_2) = mA_{st}(d_1 - n) \tag{12.17}$$

It can be seen from Eq. (12.17) that multiplying A_{sc} by $(m - 1)$ in the transformation process rather than m automatically allows for the reduction in the area of concrete caused by the presence of the compression steel. Thus the second moment of area of the transformed section is

$$I_c = \frac{bn^3}{3} + (m-1)A_{sc}(n-d_2)^2 + mA_{st}(d_1-n)^2 \tag{12.18}$$

The maximum stress in the concrete is then

$$\sigma_c = -\frac{Mn}{I_c} \quad \text{(see Eq. (12.12))}$$

The stress in the tension steel and in the compression steel are obtained from the strain diagram of Fig. 12.7(c). Hence

$$\frac{\sigma_{sc}/E_s}{n - d_2} = \frac{\sigma_c/E_c}{n} \quad \text{(both strains have the same sign)} \tag{12.19}$$

so that

$$\sigma_{sc} = \frac{m(n - d_2)}{n}\sigma_c = -\frac{mM(n - d_2)}{I_c} \qquad (12.20)$$

and

$$\sigma_{st} = \frac{mM}{I_c}(d_1 - n) \text{ as before} \qquad (12.21)$$

An alternative expression for the moment of resistance of the beam is derived by taking moments of the resultant steel and concrete loads about the compressive reinforcement. Therefore from the stress diagram of Fig. 12.7(d)

$$M = T(d_1 - d_2) - C_c\left(\frac{n}{3} - d_2\right)$$

whence

$$M = \sigma_{st}A_{st}(d_1 - d_2) - \frac{\sigma_c}{2}bn\left(\frac{n}{3} - d_2\right) \qquad (12.22)$$

EXAMPLE 12.6

A rectangular section concrete beam is 180 mm wide and has a depth of 360 mm to its tensile reinforcement. It is subjected to a bending moment of 45 kN m and carries additional steel reinforcement in its compression zone at a depth of 40 mm from the upper surface of the beam. Determine the necessary areas of reinforcement if the stress in the concrete is limited to 8.5 N/mm² and that in the steel to 140 N/mm². The modular ratio $E_s/E_c = 15$.

Assuming that the stress in the tensile reinforcement and that in the concrete attain their limiting values we can determine the position of the neutral axis using Eq. (12.13). Thus

$$140 = 8.5 \times 15\left(\frac{360}{n} - 1\right)$$

from which

$$n = 171.6 \text{ mm}$$

Substituting this value of n in Eq. (12.22) we have

$$45 \times 10^6 = 140A_{st}(360 - 40) + \frac{8.5}{2} \times 180 \times 171.6\left(\frac{171.6}{3} - 40\right)$$

which gives

$$A_{st} = 954 \text{ mm}^2$$

We can now use Eq. (12.17) to determine A_{sc} or, alternatively, we could equate the load in the tensile steel to the combined compressive load in the concrete and compression steel. Substituting for n and A_{st} in Eq. (12.17) we have

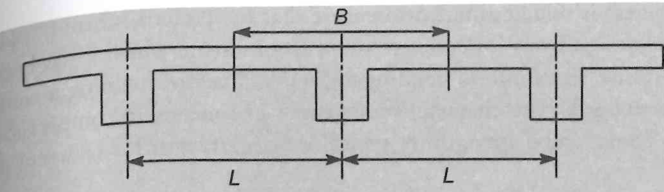


FIGURE 12.8

Slab-reinforced concrete beam arrangement.

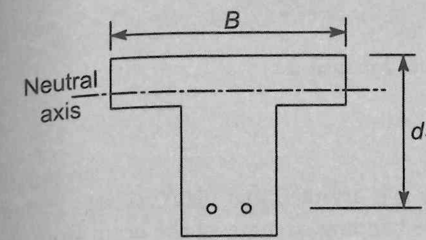


FIGURE 12.9

Analysis of a reinforced concrete T-beam.

$$\frac{180 \times 171.6^2}{2} + (15 - 1)A_{sc}(171.6 - 40) = 15 \times 954(360 - 171.6)$$

from which

$$A_{sc} = 24.9 \text{ mm}^2$$

The stress in the compression steel may be obtained from Eq. (12.20), i.e.

$$\sigma_{sc} = -15\frac{(171.6 - 40)}{171.6} \times 8.5 = -97.8 \text{ N/mm}^2 \quad \text{(compression)}$$

In many practical situations reinforced concrete beams are cast integrally with floor slabs, as shown in Fig. 12.8. Clearly, the floor slab contributes to the overall strength of the structure so that the part of the slab adjacent to a beam may be regarded as forming part of the beam. The result is a T-beam whose flange, or the major portion of it, is in compression. The assumed width, B, of the flange cannot be greater than L, the distance between the beam centres; in most instances B is specified in Codes of Practice.

It is usual to assume in the analysis of T-beams that the neutral axis lies within the flange or coincides with its under surface. In either case the beam behaves as a rectangular section concrete beam of width B and effective depth d_1 (Fig. 12.9). Therefore, the previous analysis of rectangular section beams still applies.

Ultimate load theory

We have previously noted in this chapter and also in Chapter 8 that the modern design of reinforced concrete structures relies on ultimate load theory. The calculated moment of resistance of a beam section is therefore based on the failure strength of concrete in compression and the yield strength of the steel reinforcement in tension modified by suitable factors of safety. Typical values are 1.5 for concrete (based on its 28-day cube strength) and 1.15 for steel. However, failure of the concrete in compression could occur suddenly in a reinforced concrete beam, whereas failure of the steel by yielding would be gradual. It is therefore preferable that failure occurs in the reinforcement rather than in the concrete.

Thus, in design, the capacity of the concrete is underestimated to ensure that the preferred form of failure occurs. A further factor affecting the design stress for concrete stems from tests in which it has been found that concrete subjected to compressive stress due to bending always fails before attaining a compressive stress equal to the 28-day cube strength. The characteristic strength of concrete in compression is therefore taken as two-thirds of the 28-day cube strength. A typical design strength for concrete in compression is then

$$\frac{\sigma_{cu}}{1.5} \times 0.67 = 0.45\sigma_{cu}$$

where σ_{cu} is the 28-day cube strength. The corresponding figure for steel is

$$\frac{\sigma_Y}{1.15} = 0.87\sigma_Y$$

In the ultimate load analysis of reinforced concrete beams it is assumed that plane sections remain plane during bending and that there is no contribution to the bending strength of the beam from the concrete in tension. From the first of these assumptions we deduce that the strain varies linearly through the depth of the beam as shown in Fig. 12.10(b). However, the stress diagram in the concrete is not linear but has the rectangular–parabolic shape shown in Fig. 12.10(c). Design charts in Codes of Practice are based on this stress distribution, but for direct calculation purposes a reasonably accurate approximation can be made in which the rectangular–parabolic stress distribution of Fig. 12.10(c) is replaced by an equivalent rectangular distribution as shown in Fig. 12.11(b) in which the compressive stress in the concrete is assumed to extend down to the mid-effective depth of the section at the maximum condition, i.e. at the ultimate moment of resistance, M_u, of the section.

M_u is then given by

$$M_u = C\tfrac{3}{4}d_1 = 0.40\sigma_{cu}b\tfrac{1}{2}d_1\tfrac{3}{4}d_1$$

which gives

$$M_u = 0.15\sigma_{cu}b(d_1)^2 \tag{12.23}$$

or

$$M_u = T\tfrac{3}{4}d_1 = 0.87\sigma_Y A_s\tfrac{3}{4}d_1$$

from which

$$M_u = 0.65\sigma_Y A_s d_1 \tag{12.24}$$

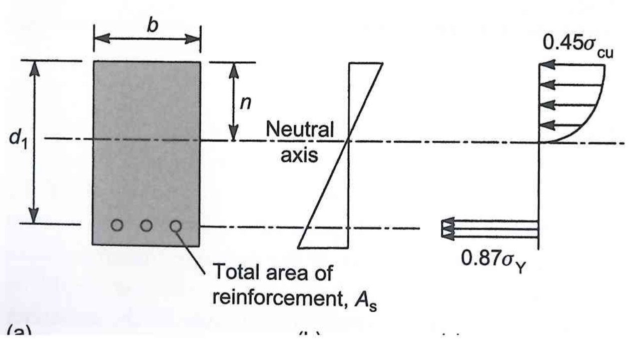


FIGURE 12.10

Stress and strain distributions in a

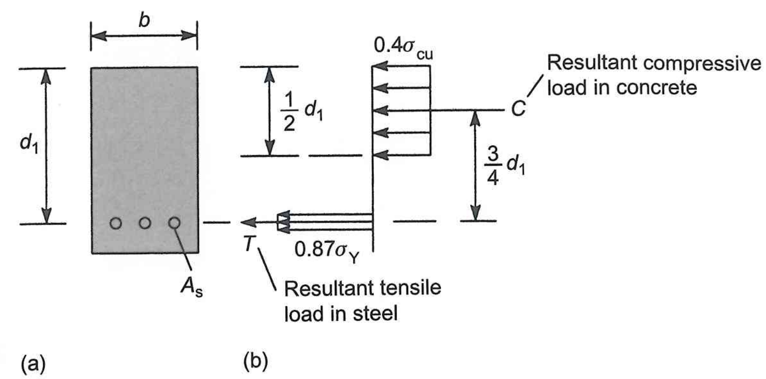


FIGURE 12.11

Approximation of stress distribution in concrete.

whichever is the lesser. For applied bending moments less than M_u a rectangular stress block may be assumed for the concrete in which the stress is $0.4\sigma_{cu}$ but in which the depth of the neutral axis must be calculated. For beam sections in which the applied bending moment is greater than M_u, compressive reinforcement is required.

EXAMPLE 12.7

A reinforced concrete beam having an effective depth of 600 mm and a breadth of 250 mm is subjected to a bending moment of 350 kN m. If the 28-day cube strength of the concrete is 30 N/mm^2 and the yield stress in tension of steel is 400 N/mm^2, determine the required area of reinforcement.

First it is necessary to check whether or not the applied moment exceeds the ultimate moment of resistance provided by the concrete. Hence, using Eq. (12.23)

$$M_u = 0.15 \times 30 \times 250 \times 600^2 \times 10^{-6} = 405 \text{ kN m}$$

Since this is greater than the applied moment, the beam section does not require compression reinforcement.

We now assume the stress distribution shown in Fig. 12.12 in which the neutral axis of the section is at a depth n below the upper surface of the section. Thus, taking moments about the tensile reinforcement we have

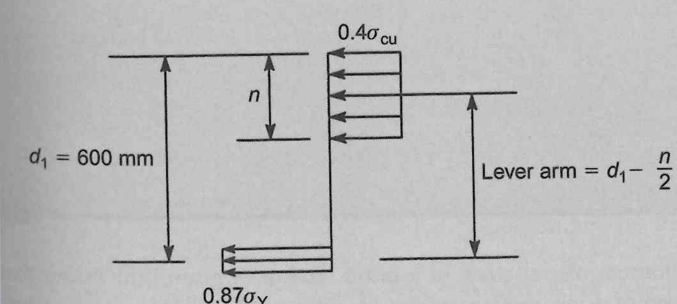


FIGURE 12.12

Stress distribution in beam of Ex. 12.7.

$$350 \times 10^6 = 0.4 \times 30 \times 250n\left(600 - \frac{n}{2}\right)$$

from which

$$n = 243.3 \text{ mm}$$

The lever arm is therefore equal to 600 − 243.3/2 = 478.4 mm. Now taking moments about the centroid of the concrete we have

$$0.87 \times 400 \times A_s \times 478.4 = 350 \times 10^6$$

which gives

$$A_s = 2102.3 \text{ mm}^2$$

EXAMPLE 12.8

A reinforced concrete beam of breadth 250 mm is required to have an effective depth as small as possible. Design the beam and reinforcement to support a bending moment of 350 kN m assuming that $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 400$ N/mm^2.

In this example the effective depth of the beam will be as small as possible when the applied moment is equal to the ultimate moment of resistance of the beam. Then, using Eq. (12.23)

$$350 \times 10^6 = 0.15 \times 30 \times 250 \times d_1^2$$

which gives

$$d_1 = 557.8 \text{ mm}$$

This is not a practical dimension since it would be extremely difficult to position the reinforcement to such accuracy. We therefore assume $d_1 = 558$ mm. Since the section is stressed to the limit, we see from Fig. 12.11(b) that the lever arm is

$$\frac{3}{4}d_1 = \frac{3}{4} \times 558 = 418.5 \text{ mm}$$

Hence, from Eq. (12.24)

$$350 \times 10^6 = 0.87 \times 400A_s \times 418.5$$

from which

$$A_s = 2403.2 \text{ mm}^2$$

A comparison of Exs 12.7 and 12.8 shows that the reduction in effective depth is only made possible by an increase in the area of steel reinforcement.

We have noted that the ultimate moment of resistance of a beam section of given dimensions can only be increased by the addition of compression reinforcement. However, although the design stress for tension reinforcement is 0.87σ...

avoid the possibility of the reinforcement buckling between the binders or stirrups. The method of designing a beam section to include compression reinforcement is simply treated as an extension of the singly reinforced case and is best illustrated by an example.

EXAMPLE 12.9

A reinforced concrete beam has a breadth of 300 mm and an effective depth to the tension reinforcement of 618 mm. Compression reinforcement, if required, will be placed at a depth of 60 mm. If $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 410$ N/mm^2, design the steel reinforcement if the beam is to support a bending moment of 650 kN m.

The ultimate moment of resistance provided by the concrete is obtained using Eq. (12.23) and is

$$M_u = 0.15 \times 30 \times 300 \times 618^2 \times 10^{-6} = 515.6 \text{ kN m}$$

This is less than the applied moment so that compression reinforcement is required to resist the excess moment of 650−515·6 = 134·4 kN m. If A_{sc} is the area of compression reinforcement

$$134.4 \times 10^6 = \text{lever arm} \times 0.72 \times 410A_{sc}$$

i.e.

$$134.4 \times 10^6 = (618 - 60) \times 0.72 \times 410A_{sc}$$

which gives

$$A_{sc} = 815.9 \text{ mm}^2$$

The tension reinforcement, a_{st}, is required to resist the moment of 515.6 kN m (as though the beam were singly reinforced) plus the excess moment of 134.4 kN m. Hence

$$A_{st} = \frac{515.6 \times 10^6}{0.75 \times 618 \times 0.87 \times 410} + \frac{134.4 \times 10^6}{(618 - 60) \times 0.87 \times 410}$$

from which

$$A_{st} = 3793.8 \text{ mm}^2$$

The ultimate load analysis of reinforced concrete T-beams is simplified in a similar manner to the elastic analysis by assuming that the neutral axis does not lie below the lower surface of the flange. The ultimate moment of a T-beam therefore corresponds to a neutral axis position coincident with the lower surface of the flange as shown in Fig. 12.13(a). M_u is then the lesser of the two values given by

$$M_u = 0.4\sigma_{cu}Bh_f\left(d_1 - \frac{h_f}{2}\right) \tag{12.25}$$

or

$$M_u = 0.87\sigma_Y A_s\left(d_1 - \frac{h_f}{2}\right) \tag{12.26}$$

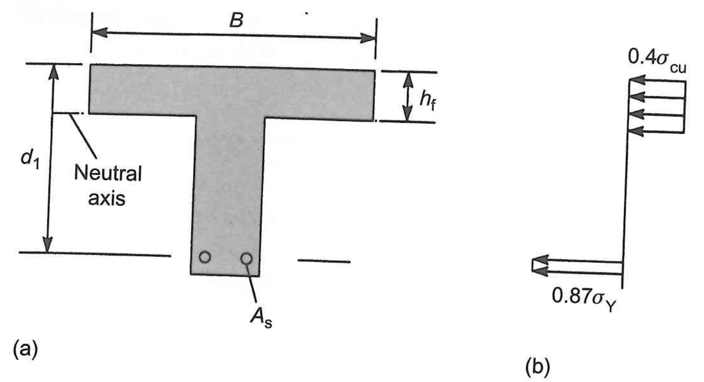


FIGURE 12.13

Ultimate load analysis of a reinforced concrete T-beam.

For T-beams subjected to bending moments less than m_u, the neutral axis lies within the flange and must be found before, say, the amount of tension reinforcement can be determined. Compression reinforcement is rarely required in T-beams due to the comparatively large areas of concrete in compression.

EXAMPLE 12.10

A reinforced concrete T-beam has a flange width of 1200 mm and an effective depth of 618 mm; the thickness of the flange is 150 mm. Determine the required area of reinforcement if the beam is required to resist a bending moment of 500 kN m. Take $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 410$ N/mm^2.

M_u for this beam section may be determined using Eq. (12.25), i.e.

$$M_u = 0.4 \times 30 \times 1200 \times 150\left(618 - \frac{150}{2}\right) \times 10^{-6} = 1173 \text{ kN m}$$

Since this is greater than the applied moment, we deduce that the neutral axis lies within the flange. Then from Fig. 12.14

$$500 \times 10^6 = 0.4 \times 30 \times 1200n\left(618 - \frac{n}{2}\right)$$

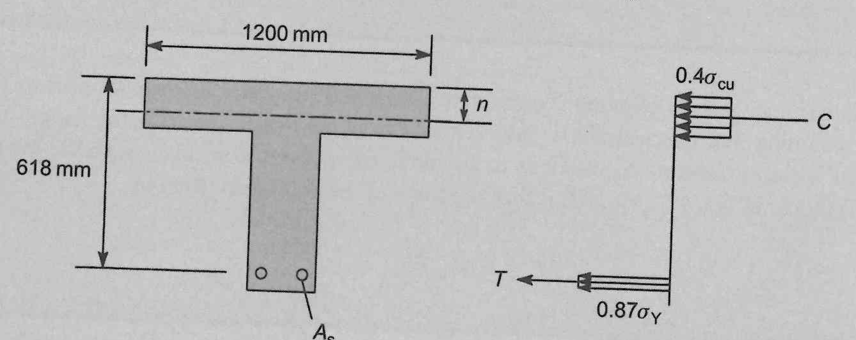


FIGURE 12.14

Reinforced concrete T-beam of Ex. 12.10.

the solution of which gives

$$n = 59 \text{ mm}$$

Now taking moments about the centroid of the compression concrete we have

$$500 \times 10^6 = 0.87 \times 410 \times A_s\left(618 - \frac{59}{2}\right)$$

which gives

$$A_s = 2381.9 \text{ mm}^2$$

EXAMPLE 12.11

A concrete floor slab whose partial cross section is shown in Fig. 12.15(a) is required to carry a uniformly distributed load of 100 kN/m^2. The beams supporting the slab are themselves simply supported over a span of 5 m. If $\sigma_{cu} = 25$ N/mm^2 and $\sigma_Y = 400$ N/mm^2 determine the required depth of the slab and the area of steel reinforcement.

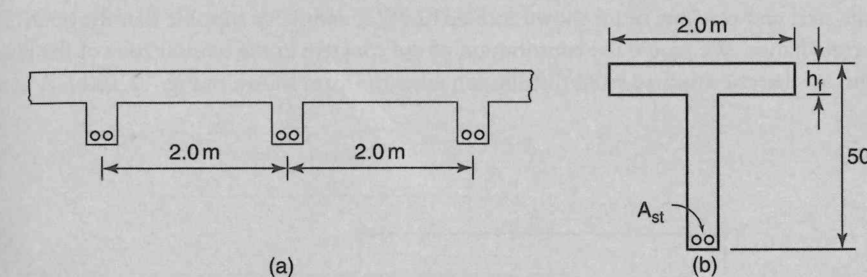


FIGURE 12.15

Beam/slab arrangement of Ex. 12.11.

The beam/slab arrangement may be designed as a T-beam having the cross section shown in Fig. 12.15(b). The maximum bending moment occurs at the mid-span of the beam and is given by

$$M_{max} = \frac{100 \times 2 \times 5^2}{8} = 625 \text{ kN m} \quad \text{(see Ex. 3.8)}$$

Then, assuming that the neutral axis coincides with the base of the slab, from Eq.(12.25)

$$625 \times 10^6 = 0.4 \times 25 \times 2 \times 10^3 h_f(500 - h_f/2)$$

which simplifies to the quadratic equation

$$h_f^2 - 1000h_f + 62500 = 0$$

Solving

$$h_f = 67 \text{ mm}$$

Then, from Eq.(12.26)

$$625 \times 10^6 = 0.87 \times 400A_s(500 - 67/2)$$

from which

$$A_s = 3849.9 \text{ mm}^2$$

12.3 Steel and concrete beams

In many instances concrete slabs are supported on steel beams, the two being joined together by shear connectors to form a composite structure. We therefore have a similar situation to that of the reinforced concrete T-beam in which the flange of the beam is concrete but the leg is a standard steel section.

Ultimate load theory is used to analyse steel and concrete beams with stress limits identical to those applying in the ultimate load analysis of reinforced concrete beams; plane sections are also assumed to remain plane.

Consider the steel and concrete beam shown in Fig. 12.16(a) and let us suppose that the neutral axis lies within the concrete flange. We ignore the contribution of the concrete in the tension zone of the beam to its bending strength, so that the assumed stress distribution takes the form shown in Fig. 12.16(b). A convenient

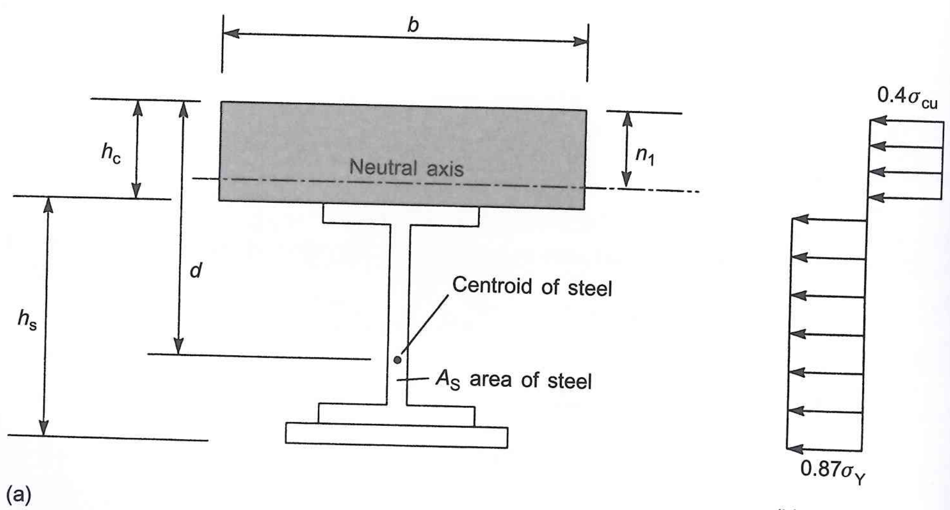


FIGURE 12.16

Ultimate load analysis of a steel and concrete beam, neutral axis

method of designing the cross section to resist a bending moment, M, is to assume the lever arm to be $(h_c + h_s)/2$ and then to determine the area of steel from the moment equation

$$M = 0.87\sigma_Y A_s \frac{(h_c + h_s)}{2} \tag{12.27}$$

The available compressive force in the concrete slab, $0.4\,\sigma_{cu}bh_c$, is then checked to ensure that it exceeds the tensile force, $0.87\sigma_Y A_s$, in the steel. If it does not, the neutral axis of the section lies within the steel and A_s given by Eq. (12.27) will be too small. If the neutral axis lies within the concrete slab the moment of resistance of the beam is determined by first calculating the position of the neutral axis. Thus, since the compressive force in the concrete is equal to the tensile force in the steel

$$0.4\sigma_{cu}bn_1 = 0.87\sigma_Y A_s \tag{12.28}$$

Then, from Fig. 12.16

$$M_u = 0.87\sigma_Y A_s\left(d - \frac{n_1}{2}\right) \tag{12.29}$$

If the neutral axis lies within the steel, the stress distribution shown in Fig. 12.17(b) is assumed in which the compressive stress in the steel above the neutral axis is the resultant of the tensile stress and twice the compressive stress. Thus, if the area of steel in compression is A_{sc}, we have, equating compressive and tensile forces

$$0.4\sigma_{cu}bh_c + 2 \times (0.87\sigma_Y)A_{sc} = 0.87\sigma_Y A_s \tag{12.30}$$

which gives A_{sc} and hence h_{sc}. Now taking moments

$$M_u = 0.87\sigma_Y A_s\left(d - \frac{h_c}{2}\right) - 2 \times (0.87\sigma_Y)A_{sc}\left(h_{sc} - \frac{h_c}{2}\right) \tag{12.31}$$

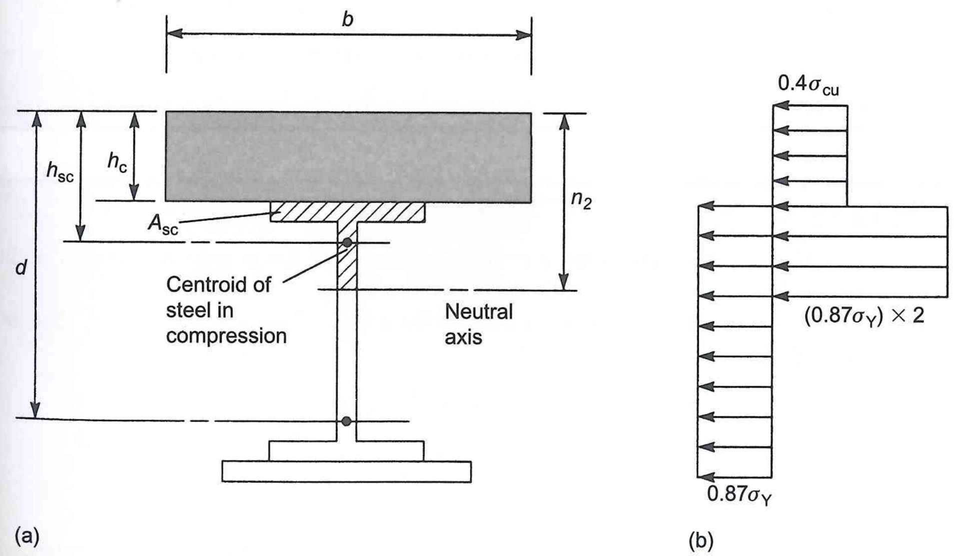


FIGURE 12.17

EXAMPLE 12.12

A concrete slab 150 mm thick is 1.8 m wide and is to be supported by a steel beam. The total depth of the steel/concrete composite beam is limited to 562 mm. Find a suitable beam section if the composite beam is required to resist a bending moment of 709 kN m. Take $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 350$ N/mm^2.

Using Eq. (12.27)

$$A_s = \frac{2 \times 709 \times 10^6}{0.87 \times 350 \times 562} = 8286 \text{ mm}^2$$

The tensile force in the steel is then

$$0.87 \times 350 \times 8286 \times 10^{-3} = 2523 \text{ kN}$$

and the compressive force in the concrete is

$$0.4 \times 1.8 \times 10^3 \times 150 \times 30 \times 10^{-3} = 3240 \text{ kN}$$

The neutral axis therefore lies within the concrete slab so that the area of steel in tension is, in fact, equal to A_s. From Steel Tables we see that a Universal Beam of nominal size 406 mm × 152 mm × 67 kg/m has an actual overall depth of 412 mm and a cross-sectional area of 8530 mm^2. The position of the neutral axis of the composite beam incorporating this beam section is obtained from Eq. (12.28); hence

$$0.4 \times 30 \times 1800 n_1 = 0.87 \times 350 \times 8530$$

which gives

$$n_1 = 120 \text{ mm}$$

Substituting for n_1 in Eq. (12.29) we obtain the moment of resistance of the composite beam

$$M_u = 0.87 \times 350 \times 8530(356 - 60) \times 10^{-6} = 769 \text{ kN m}$$

Since this is greater than the applied moment we deduce that the beam section is satisfactory.

EXAMPLE 12.13

If the concrete in the steel/concrete composite beam of Ex.12.12 has a reduced strength of 20 N/mm^2 determine whether or not the composite beam section is still satisfactory.

The cross sectional area of the steel beam chosen from Steel Tables is 8530 mm^2. The tensile force in the steel is then

$$0.87 \times 350 \times 8530 \times 10^{-3} = 2597.4 \text{ kN}$$

The compressive force in the concrete is

$$0.4 \times 1.8 \times 10^3 \times 150 \times 20 \times 10^{-3} = 2160 \text{ kN}$$

Since this is less than the tensile force in the steel the neutral axis of the beam section lies within the steel. Then, from Eq.(12.30)

from which

$$A_{sc} = 718.2 \text{ mm}^2$$

From Steel Tables, the Universal Beam has a flange width of 153 mm and a flange thickness of 16 mm. Therefore, by inspection, the neutral axis lies within the flange of the steel beam. Then

$$158 h_f = 718.2$$

so that

$$h_f = 4.7 \text{ mm}$$

where h_f is the depth of the flange in compression. Then

$$h_{sc} = 4.7 + 150 = 154.7 \text{ mm}$$

From Eq.(12.31)

$$M_u = 0.87 \times 350 \times 8530\,(356 - 150/2) - 2 \times 0.87 \times 350 \times 718.2\,(154.7 - 150/2)$$

which gives

$$M_u = 695 \text{ kNm}$$

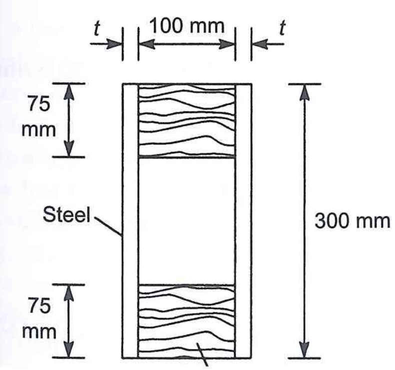
This is less than the applied bending moment so that the beam section is no longer satisfactory.

PROBLEMS

P.12.1 A timber beam 200 mm wide by 300 mm deep is reinforced on its top and bottom surfaces by steel plates each 12 mm thick by 200 mm wide. If the allowable stress in the timber is 8 N/mm^2 and that in the steel is 110 N/mm^2, find the allowable bending moment. The ratio of the modulus of elasticity of steel to that of timber is 20.

Ans. 94.7 kN m.

P.12.2 A simply supported beam of span 3.5 m carries a uniformly distributed load of 46.5 kN/m. The beam has the box section shown in Fig. P.12.2. Determine the required thickness of the



steel plates if the allowable stresses are 124 N/mm^2 for the steel and 8 N/mm^2 for the timber. The modular ratio of steel to timber is 20.

Ans. 17 mm.

P.12.3 A timber beam 150 mm wide by 300 mm deep is reinforced by a steel plate 150 mm wide and 12 mm thick which is securely attached to its lower surface. Determine the percentage increase in the moment of resistance of the beam produced by the steel-reinforcing plate. The allowable stress in the timber is 12 N/mm^2 and in the steel, 155 N/mm^2. The modular ratio is 20.

Ans. 176%.

P.12.4 A singly reinforced rectangular concrete beam of effective span 4.5 m is required to carry a uniformly distributed load of 16.8 kN/m. The overall depth, D, is to be twice the breadth and the centre of the steel is to be at $0.1D$ from the underside of the beam. Using elastic theory find the dimensions of the beam and the area of steel reinforcement required if the stresses are limited to 8 N/mm^2 in the concrete and 140 N/mm^2 in the steel. Take $m = 15$.

Ans. $D = 406.7$ mm, $A_s = 980.6$ mm^2.

P.12.5 A reinforced concrete beam is of rectangular section 300 mm wide by 775 mm deep. It has five 25 mm diameter bars as tensile reinforcement in one layer with 25 mm cover and three 25 mm diameter bars as compression reinforcement, also in one layer with 25 mm cover. Find the moment of resistance of the section using elastic theory if the allowable stresses are 7.5 N/mm^2 and 125 N/mm^2 in the concrete and steel, respectively. The modular ratio is 16.

Ans. 214.5 kN m.

P.12.6 A reinforced concrete T-beam is required to carry a uniformly distributed load of 42 kN/m on a simply supported span of 6 m. The slab is 125 mm thick, the rib is 250 mm wide and the effective depth to the tensile reinforcement is 550 mm. The working stresses are 8.5 N/mm^2 in the concrete and 140 N/mm^2 in the steel; the modular ratio is 15. Making a reasonable assumption as to the position of the neutral axis find the area of steel reinforcement required and the breadth of the compression flange.

Ans. 2655.7 mm^2, 700 mm (neutral axis coincides with base of slab).

P.12.7 Repeat P.12.4 using ultimate load theory assuming $\sigma_{cu} = 24$ N/mm^2 and $\sigma_Y = 280$ N/mm^2.

Ans. $D = 307.8$ mm, $A_s = 843$ mm^2.

P.12.8 Repeat P.12.5 using ultimate load theory and take $\sigma_{cu} = 22.5$ N/mm^2, $\sigma_Y = 250$ N/mm^2.

Ans. 222.5 kNm.

P.12.9 Repeat P.12.6 using ultimate load theory. Assume $\sigma_{cu} = 25.5$ N/mm^2 and $\sigma_Y = 280$ N/mm^2.

Ans. 1592 mm^2, 304 mm (neutral axis coincides with base of slab).

P.12.10 A concrete slab 175 mm thick and 2 m wide is supported by, and firmly connected to, a 457 mm × 152 mm × 74 kg/m Universal Beam whose actual depth is 461.3 mm and whose cross-sectional area is 9490 mm^2. If $\sigma_{cu} = 30$ N/mm^2 and $\sigma_Y = 350$ N/mm^2, find the moment of resistance of the resultant steel and concrete beam.

Ans. 919.5 kNm.

P.12.11 If the concrete in the composite beam in P.12.10 has a reduced strength of 15 N/mm^2 determine its resulting moment of resistance.

CHAPTER

Deflection of Beams

13

In Chapters 9, 10 and 11 we investigated the *strength* of beams in terms of the stresses produced by the action of bending, shear and torsion, respectively. An associated problem is the determination of the deflections of beams caused by different loads for, in addition to strength, a beam must possess sufficient *stiffness* so that excessive deflections do not have an adverse effect on adjacent structural members. In many cases, maximum allowable deflections are specified by Codes of Practice in terms of the dimensions of the beam, particularly the span; typical values are quoted in Section 8.7. We also saw in Section 8.7 that beams may be designed using either elastic or plastic analysis. However, since beam deflections must always occur within the elastic limit of the material of a beam they are determined using elastic theory.

There are several different methods of obtaining deflections in beams, the choice depending upon the type of problem being solved. For example, the double integration method gives the complete shape of a beam whereas the moment-area method can only be used to determine the deflection at a particular beam section. The latter method, however, is also useful in the analysis of statically indeterminate beams.

Generally beam deflections are caused primarily by the bending action of applied loads. In some instances, however, where a beam's cross-sectional dimensions are not small compared with its length, deflections due to shear become significant and must be calculated. We shall consider beam deflections due to shear in addition to those produced by bending. We shall also include deflections due to unsymmetrical bending.

13.1 Differential equation of symmetrical bending

In Chapter 9 we developed an expression relating the curvature, $1/R$, of a beam to the applied bending moment, M, and flexural rigidity, EI, i.e.

$$\frac{1}{R} = \frac{M}{EI} \quad \text{(Eq. (9.11))}$$

For a beam of a given material and cross section, EI is constant so that the curvature is directly proportional to the bending moment. We have also shown that bending moments produced by shear loads vary along the length of a beam, which implies that the curvature of the beam also varies along its length; Eq. (9.11) therefore gives the curvature at a particular section of a beam.

Consider a beam having a vertical plane of symmetry and loaded such that at a section of the beam the deflection of the neutral plane, referred to arbitrary axes Oxy, *is* v and the slope of the tangent to the neutral plane at this section is dv/dx (Fig. 13.1). Also, if the applied loads produce a positive, i.e. sagging, bending moment at this section, then the upper surface of the beam is concave and the centre of curvature lies above the beam as shown. For the system of axes shown in Fig. 13.1, the sign convention usually adopted in mathematical theory gives a positive value for this curvature, i.e.

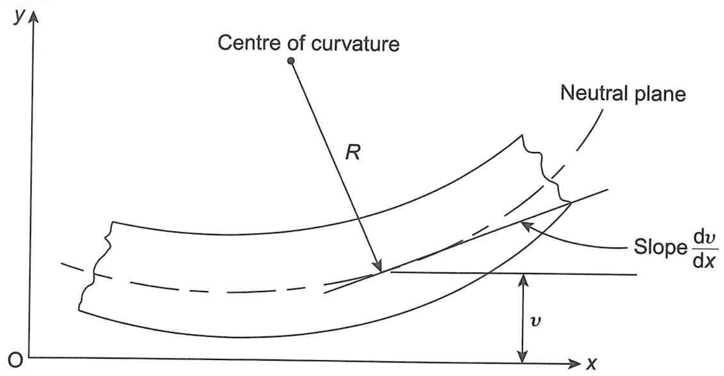


FIGURE 13.1

Deflection and curvature of a beam due to bending.

$$\frac{1}{R} = \frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}} \tag{13.1}$$

For small deflections dv/dx is small so that $(dv/dx)^2$ is negligibly small compared with unity. Equation (13.1) then reduces to

$$\frac{1}{R} = \frac{d^2v}{dx^2} \tag{13.2}$$

whence, from Eq. (9.11)

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{13.3}$$

Double integration of Eq. (13.3) then yields the equation of the deflection curve of the neutral plane of the beam.

In the majority of problems concerned with beam deflections the bending moment varies along the length of a beam and therefore M in Eq. (13.3) must be expressed as a function of x before integration can commence. Alternatively, it may be convenient in cases where the load is a known function of x to use the relationships of Eq. (3.8). Thus

$$\frac{d^3v}{dx^3} = -\frac{S}{EI} \tag{13.4}$$

$$\frac{d^4v}{dx^4} = -\frac{w}{EI} \tag{13.5}$$

We shall now illustrate the use of Eqs (13.3), (13.4) and (13.5) by considering some standard cases of beam deflection.

EXAMPLE 13.1

Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 13.2(a); the flexural rigidity of the cantilever is EI.

The load W causes the cantilever to deflect such that its neutral plane takes up the curved shape shown in Fig. 13.2(b); the deflection at any section X is then v while that at its free end is v_{tip}. The axis system is chosen so that the origin coincides with the built-in end where the deflection is clearly zero.

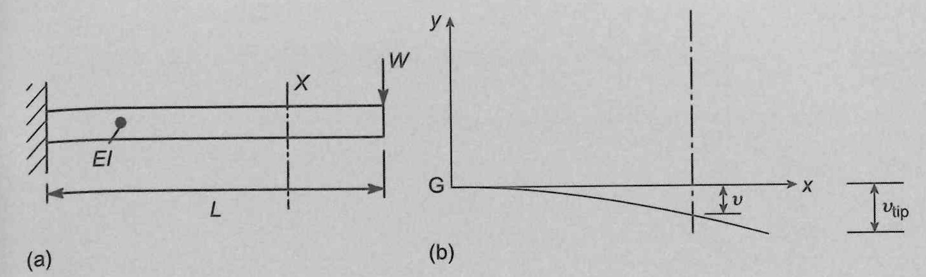


FIGURE 13.2

Deflection of a cantilever beam carrying a concentrated load at its free end (Ex. 13.1).

The bending moment, M, at the section X is, from Fig. 13.2(a)

$$M = -W(L-x) \quad \text{(i.e. hogging)} \tag{i}$$

Substituting for M in Eq. (13.3) we obtain

$$\frac{d^2v}{dx^2} = -\frac{W}{EI}(L-x)$$

or in more convenient form

$$EI\frac{d^2v}{dx^2} = -W(L-x) \tag{ii}$$

Integrating Eq. (ii) with respect to x gives

$$EI\frac{dv}{dx} = -W\left(Lx - \frac{x^2}{2}\right) + C_1$$

where C_1 is a constant of integration which is obtained from the boundary condition that $dv/dx = 0$ at the built-in end where $x = 0$. Hence $C_1 = 0$ and

$$EI\frac{dv}{dx} = -W\left(Lx - \frac{x^2}{2}\right) \tag{iii}$$

Integrating Eq. (iii) we obtain

$$EIv = -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

in which C_2 is again a constant of integration. At the built-in end $v = 0$ when $x = 0$ so that $C_2 = 0$. Hence the equation of the deflection curve of the cantilever is

$$v = -\frac{W}{6EI}(3Lx^2 - x^3) \tag{iv}$$

The deflection, v_{tip}, at the free end is obtained by setting $x = L$ in Eq. (iv). Thus

$$v_{tip} = -\frac{WL^3}{3EI} \tag{v}$$

EXAMPLE 13.2

Determine the deflection curve and the deflection of the free end of the cantilever shown in Fig. 13.3(a).

The bending moment, M, at any section X is given by

$$M = -\frac{w}{2}(L-x)^2 \tag{i}$$

Substituting for M in Eq. (13.3) and rearranging we have

$$EI\frac{d^2v}{dx^2} = -\frac{w}{2}(L-x)^2 = -\frac{w}{2}(L^2 - 2Lx + x^2) \tag{ii}$$

Integration of Eq. (ii) yields

$$EI\frac{dv}{dx} = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + C_1$$

When $x = 0$ at the built-in end, $dv/dx = 0$ so that $C_1 = 0$ and

$$EI\frac{dv}{dx} = -\frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) \tag{iii}$$

Integrating Eq. (iii) we have

$$EIv = -\frac{w}{2}\left(L^2\frac{x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + C_2$$

and since $v = 0$ when $x = 0$, $C_2 = 0$. The deflection curve of the beam therefore has the equation

$$v = -\frac{w}{24EI}(6L^2x^2 - 4Lx^3 + x^4) \tag{iv}$$

and the deflection at the free end where $x = L$ is

$$v_{tip} = -\frac{wL^4}{8EI} \tag{v}$$

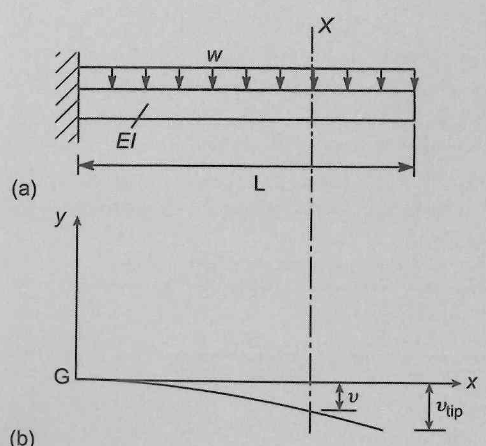


FIGURE 13.3

Deflection of a cantilever beam carrying a uniformly distributed load.

which is again negative and downwards. The applied loading in this case may be easily expressed in mathematical form so that a solution can be obtained using Eq. (13.5), i.e.

$$\frac{d^4v}{dx^4} = -\frac{w}{EI} \tag{vi}$$

in which w = constant. Integrating Eq. (vi) we obtain

$$EI\frac{d^3v}{dx^3} = -wx + C_1$$

We note from Eq. (13.4) that

$$\frac{d^3v}{dx^3} = -\frac{S}{EI} \quad \text{(i.e. } -S = -wx + C_1\text{)}$$

When $x = 0$, $S = -wL$ so that

$$C_1 = wL$$

Alternatively we could have determined C_1 from the boundary condition that when $x = L$, $S = 0$.

Hence

$$EI\frac{d^3v}{dx^3} = -w(x - L) \tag{vii}$$

Integrating Eq. (vii) gives

$$EI\frac{d^2v}{dx^2} = -w\left(\frac{x^2}{2} - Lx\right) + C_2$$

From Eq. (13.3) we see that

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

and when $x = 0$, $M = -wL^2/2$ (or when $x = L$, $M = 0$) so that

$$C_2 = -\frac{wL^2}{2}$$

and

$$EI\frac{d^2v}{dx^2} = -\frac{w}{2}(x^2 - 2Lx + L^2)$$

which is identical to Eq. (ii). The solution then proceeds as before.

EXAMPLE 13.3

The cantilever beam shown in Fig. 13.4(a) carries a uniformly distributed load over part of its span. Calculate the deflection of the free end.

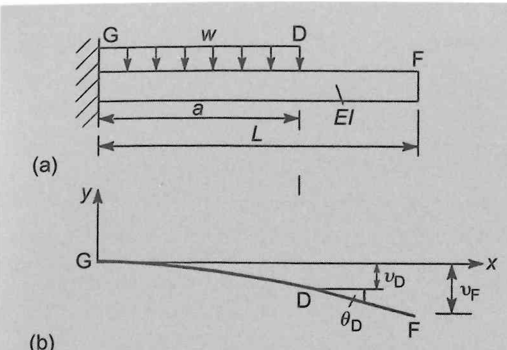


FIGURE 13.4

Cantilever beam of Ex. 13.3.

If we assume that the cantilever is weightless then the bending moment at all sections between D and F is zero. It follows that the length DF of the beam remains straight. The deflection at D can be deduced from Eq. (v) of Ex. 13.2 and is

$$v_D = -\frac{wa^4}{8EI}$$

Similarly the slope of the cantilever at D is found by substituting $x = a$ and $L = a$ in Eq. (iii) of Ex. 13.2; thus

$$\left(\frac{dv}{dx}\right)_D = \theta_D = -\frac{wa^3}{6EI}$$

The deflection, v_F, at the free end of the cantilever is then given by

$$v_F = -\frac{wa^4}{8EI} - (L-a)\frac{wa^3}{6EI}$$

which simplifies to

$$v_F = -\frac{wa^3}{24EI}(4L - a)$$

EXAMPLE 13.4

Determine the deflection curve and the mid-span deflection of the simply supported beam shown in Fig. 13.5(a).

The support reactions are each $wL/2$ and the bending moment, M, at any section X, a distance x from the left-hand support is

$$M = \frac{wL}{2}x - \frac{wx^2}{2} \tag{i}$$

Substituting for M in Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = \frac{w}{2}(Lx - x^2) \tag{ii}$$

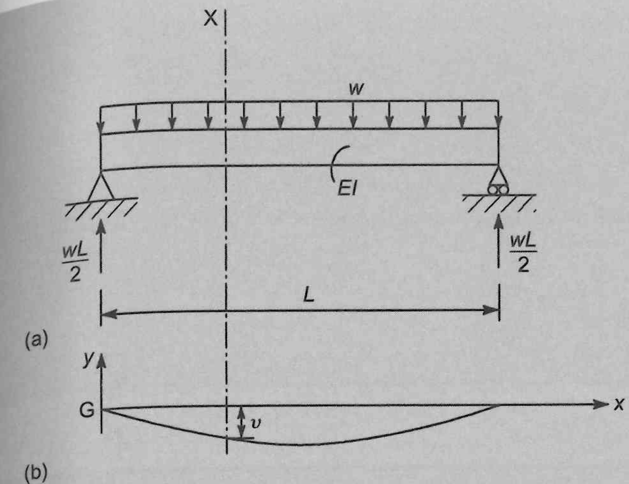


FIGURE 13.5

Deflection of a simply supported beam carrying a uniformly distributed load (Ex. 13.4).

Integrating we have

$$EI\frac{dv}{dx} = \frac{w}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1$$

From symmetry it is clear that at the mid-span section the gradient $dv/dx = 0$. Hence

$$0 = \frac{w}{2}\left(\frac{L^3}{8} - \frac{L^3}{24}\right) + C_1$$

whence

$$C_1 = -\frac{wL^3}{24}$$

Therefore

$$EI\frac{dv}{dx} = \frac{w}{24}(6Lx^2 - 4x^3 - L^3) \tag{iii}$$

Integrating again gives

$$EIv = \frac{w}{24}(2Lx^3 - x^4 - L^3x) + C_2$$

Since $v = 0$ when $x = 0$ (or since $v = 0$ when $x = L$) it follows that $C_2 = 0$ and the deflected shape of the beam has the equation

$$v = \frac{w}{24EI}(2Lx^3 - x^4 - L^3x) \tag{iv}$$

The maximum deflection occurs at mid-span where $x = L/2$ and is

$$v_{\text{mid-span}} = -\frac{5wL^4}{384EI} \tag{v}$$

So far the constants of integration were determined immediately they arose. However, in some cases a relevant boundary condition, say a value of gradient, is not obtainable. The method is then to carry the unknown constant through the succeeding integration and use known values of deflection at two sections of the beam. Thus in the previous example Eq. (ii) is integrated twice to obtain

$$EIv = \frac{w}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_1x + C_2$$

The relevant boundary conditions are $v = 0$ at $x = 0$ and $x = L$. The first of these gives $C_2 = 0$ while from the second we have $C_1 = -wL^3/24$. Thus the equation of the deflected shape of the beam is

$$v = \frac{w}{24EI}(2Lx^3 - x^4 - L^3x)$$

as before.

EXAMPLE 13.5

Figure 13.6(a) shows a simply supported beam carrying a concentrated load W at mid-span. Determine the deflection curve of the beam and the maximum deflection.

The support reactions are each $W/2$ and the bending moment M at a section X a distance x ($\leq L/2$) from the left-hand support is

$$M = \frac{w}{2}x \tag{i}$$

From Eq. (13.3) we have

$$EI\frac{d^2v}{dx^2} = \frac{W}{2}x \tag{ii}$$

Integrating we obtain

$$EI\frac{dv}{dx} = \frac{W}{2}\frac{x^2}{2} + C_1$$

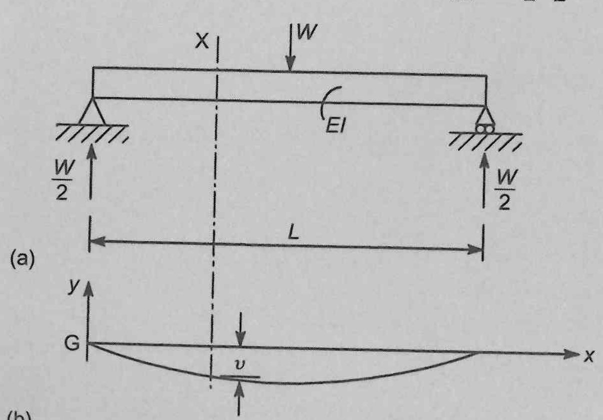


FIGURE 13.6

Deflection of a simply supported beam carrying a concentrated load at mid-span (Ex. 13.5).

From symmetry the slope of the beam is zero at mid-span where $x = L/2$. Thus $C_1 = -WL^2/16$ and

$$EI\frac{dv}{dx} = \frac{W}{16}(4x^2 - L^2) \tag{iii}$$

Integrating Eq. (iii) we have

$$EIv = \frac{W}{16}\left(\frac{4x^3}{3} - L^2x\right) + C_2$$

and when $x = 0$, $v = 0$ so that $C_2 = 0$. The equation of the deflection curve is, therefore

$$v = \frac{W}{48EI}(4x^3 - 3L^2x) \tag{iv}$$

The maximum deflection occurs at mid-span and is

$$v_{\text{mid-span}} = -\frac{WL^3}{48EI} \tag{v}$$

Note that in this problem we could not use the boundary condition that $v = 0$ at $x = L$ to determine C_2 since Eq. (i) applies only for $0 \leq x \leq L/2$; it follows that Eqs (iii) and (iv) for slope and deflection apply only for $0 \leq x \leq L/2$ although the deflection curve is clearly symmetrical about mid-span.

EXAMPLE 13.6

The simply supported beam shown in Fig. 13.7(a) carries a concentrated load W at a distance a from the left-hand support. Determine the deflected shape of the beam, the deflection under the load and the maximum deflection.

Considering the moment and force equilibrium of the beam we have

$$R_A = \frac{W}{L}(L - a) \qquad R_B = \frac{Wa}{L}$$

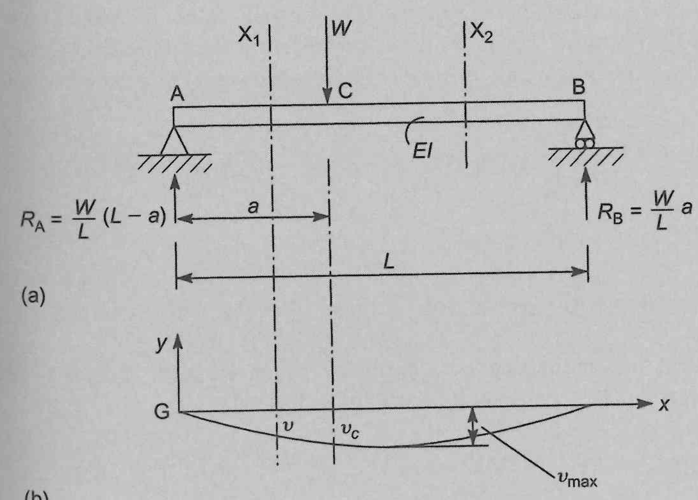


FIGURE 13.7

Deflection of a simply supported beam carrying a concentrated load not at mid-span (Ex. 13.6).

At a section X_1, a distance x from the left-hand support where $x \leq a$, the bending moment is

$$M = R_A x \quad \text{(i)}$$

At the section X_2, where $x \geq a$

$$M = R_A x - W(x - a) \quad \text{(ii)}$$

Substituting both expressions for M in turn in Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = R_A x \quad (x \leq a) \quad \text{(iii)}$$

and

$$EI\frac{d^2v}{dx^2} = R_A x - W(x - a) \quad (x \geq a) \quad \text{(iv)}$$

Integrating Eqs (iii) and (iv) we obtain

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} + C_1 \quad (x \leq a) \quad \text{(v)}$$

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} - W\left(\frac{x^2}{2} - ax\right) + C_1' \quad (x \geq a) \quad \text{(vi)}$$

and

$$EIv = R_A\frac{x^3}{6} + C_1 x + C_2 \quad (x \leq a) \quad \text{(vii)}$$

$$EIv = R_A\frac{x^3}{6} - W\left(\frac{x^3}{6} - \frac{ax^2}{2}\right) + C_1' x + C_2' x \quad (x \geq a) \quad \text{(viii)}$$

in which C_1, C_1', C_2, C_2' are arbitrary constants. In using the boundary conditions to determine these constants, it must be remembered that Eqs (v) and (vii) apply only for $0 \leq x \leq a$ and Eqs (vi) and (viii) apply only for $a \leq x \leq L$. At the left-hand support $v = 0$ when $x = 0$, therefore, from Eq. (vii), $C_2 = 0$. It is not possible to determine C_1, C_1' and C_2' directly since the application of further known boundary conditions does not isolate any of these constants. However, since $v = 0$ when $x = L$ we have, from Eq. (viii)

$$0 = R_A\frac{L^3}{6} - W\left(\frac{L^3}{6} - \frac{aL^2}{2}\right) + C_1' L + C_2'$$

which, after substituting $R_A = W(L - a)/L$, simplifies to

$$0 = \frac{WaL^2}{3} + C_1' L + C_2' \quad \text{(ix)}$$

Additional equations are obtained by considering the continuity which exists at the point of application of the load; at this section Eqs (v)–(viii) apply. Thus, from Eqs (v) and (vi)

$$R_A\frac{a^2}{2} + C_1 = R_A\frac{a^2}{2} - W\left(\frac{a^2}{2} - a^2\right) + C_1'$$

which gives

$$C_1 = \frac{Wa^2}{2} + C_1' \quad \text{(x)}$$

Now equating values of deflection at $x = a$ we have, from Eqs (vii) and (viii)

$$R_A\frac{a^3}{6} + C_1 a = R_A\frac{a^3}{6} - W\left(\frac{a^3}{6} - \frac{a^3}{2}\right) + C_1' a + C_2'$$

which yields

$$C_1 a = \frac{Wa^3}{3} + C_1' a + C_2' \quad \text{(xi)}$$

Solution of the simultaneous Eqs (ix), (x) and (xi) gives

$$C_1 = -\frac{Wa}{6L}(a - 2L)(a - L)$$

$$C_1' = -\frac{Wa}{6L}(a^2 + 2L^2)$$

$$C_2' = \frac{Wa^3}{6}$$

Equations (v)–(vii) then become respectively

$$EI\frac{dv}{dx} = -\frac{W(a - L)}{6L}\left[3x^2 + a(a - 2L)\right] \quad (x \leq a) \quad \text{(xii)}$$

$$EI\frac{dv}{dx} = -\frac{Wa}{6L}(3x^2 - 6Lx + a^2 + 2L^2) \quad (x \geq a) \quad \text{(xiii)}$$

$$EIv = -\frac{W(a - L)}{6L}\left[x^3 + a(a - 2L)x\right] \quad (x \leq a) \quad \text{(xiv)}$$

$$EIv = -\frac{Wa}{6L}\left[x^2 - 3Lx^2 + (a^2 + 2L^2)x - a^2 L\right] \quad (x \geq a) \quad \text{(xv)}$$

The deflection of the beam under the load is obtained by putting $x = a$ into either of Eq. (xiv) or (xv). Thus

$$v_C = -\frac{Wa^2(a-L)^2}{3EIL} \quad \text{(xvi)}$$

This is not, however, the maximum deflection of the beam. This will occur, if $a < L/2$, at some section between C and B. Its position may be found by equating dv/dx in Eq. (xiii) to zero. Hence

$$0 = 3x^2 - 6Lx + a^2 + 2L^2 \quad \text{(xvii)}$$

The solution of Eq. (xvii) is then substituted in Eq. (v) and the maximum deflection follows. For a central concentrated load $a = L/2$ and

$$v_C = -\frac{WL^3}{48EI}$$

EXAMPLE 13.7

Determine the deflection curve of the beam AB shown in Fig. 13.8 when it carries a distributed load that varies linearly in intensity from zero at the left-hand support to w_0 at the right-hand support.

To find the support reactions we first take moments about B. Thus

$$R_A L = \frac{1}{2} w_0 L \frac{L}{3}$$

which gives

$$R_A = \frac{w_0 L}{6}$$

Resolution of vertical forces then gives

$$R_B = \frac{w_0 L}{3}$$

The bending moment, M, at any section X, a distance x from A is

$$M = R_A x - \frac{1}{2}\left(w_0 \frac{x}{L}\right) x \frac{x}{3}$$

or

$$M = \frac{w_0}{6L}(L^2 x - x^3) \qquad \text{(i)}$$

Substituting for M in Eq. (13.3) we obtain

$$EI\frac{d^2 v}{dx^2} = \frac{w_0}{6L}(L^2 x - x^3) \qquad \text{(ii)}$$

which, when integrated, becomes

$$EI\frac{dv}{dx} = \frac{w_0}{6L}\left(L^2\frac{x^2}{2} - \frac{x^4}{4}\right) + C_1 \qquad \text{(iii)}$$

Integrating Eq. (iii) we have

$$EIv = \frac{w_0}{6L}\left(L^2\frac{x^3}{6} - 20\right) + x + C_2 \qquad \text{(iv)}$$

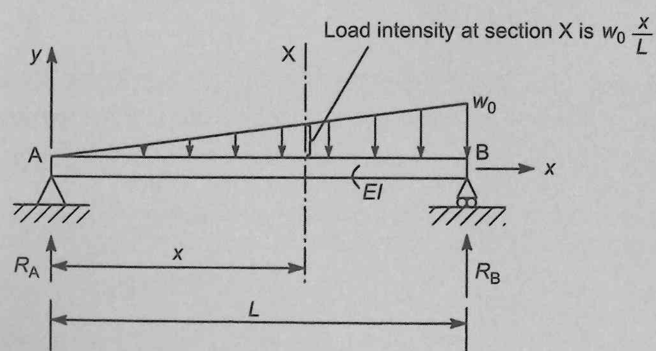


FIGURE 13.8

Deflection of a simply supported beam carrying a triangularly distributed load.

The deflection $v = 0$ at $x = 0$ and $x = L$. From the first of these conditions we obtain $C_2 = 0$, while from the second

$$0 = \frac{w_0}{6L}\left(\frac{L^5}{6} - \frac{L^5}{20}\right) + C_1 L$$

which gives

$$C_1 = -\frac{7 w_0 L^4}{360}$$

The deflection curve then has the equation

$$v = -\frac{w_0}{360EIL}(3x^5 - 10L^2 x^3 + 7L^4 x) \qquad \text{(v)}$$

An alternative method of solution is to use Eq. (13.5) and express the applied load in mathematical form. Thus

$$EI\frac{d^4 v}{dx^4} = -w = -w_0\frac{x}{L} \qquad \text{(vi)}$$

Integrating we obtain

$$EI\frac{d^3 v}{dx^3} = -w_0\frac{x^2}{2L} + C_3$$

When $x = 0$ we see from Eq. (13.4) that

$$EI\frac{d^3 v}{dx^3} = R_A = \frac{w_0 L}{6}$$

Hence

$$C_3 = \frac{w_0 L}{6}$$

and

$$EI\frac{d^3 v}{dx^3} = -w_0\frac{x^2}{2L} + \frac{w_0 L}{6} \qquad \text{(vii)}$$

Integrating Eq. (vii) we have

$$EI\frac{d^2 v}{dx^2} = -\frac{w_0 x^3}{6L} + \frac{w_0 L}{6}x + C_4$$

Since the bending moment is zero at the supports we have

$$EI\frac{d^2 v}{dx^2} = 0 \quad \text{when } x = 0$$

Hence $C_4 = 0$ and

$$EI\frac{d^2 v}{dx^2} = -\frac{w_0}{6L}(x^3 - L^2 x)$$

as before.

13.2 Singularity functions

A comparison of Exs 13.5 and 13.6 shows that the double integration method becomes extremely lengthy when even relatively small complications such as the lack of symmetry due to an offset load are introduced. Again the addition of a second concentrated load on the beam of Ex. 13.6 would result in a total of six equations for slope and deflection producing six arbitrary constants. Clearly the computation involved in determining these constants would be tedious, even though a simply supported beam carrying two concentrated loads is a comparatively simple practical case. An alternative approach is to introduce so-called *singularity* or *half-range* functions. Such functions were first applied to beam deflection problems by Macauley in 1919 and hence the method is frequently known as *Macauley's method.*

We now introduce a quantity $[x-a]$ and define it to be zero if $(x-a)<0$, i.e. $x<a$, and to be simply $(x-a)$ if $x>a$. The quantity $[x-a]$ is known as a singularity or half-range function and is defined to have a value only when the argument is positive in which case the square brackets behave in an identical manner to ordinary parentheses. Thus in Ex. 13.6 the bending moment at a section of the beam furthest from the origin for x may be written as

$$M = R_A x - W[x-a]$$

This expression applies to both the regions AC and CB since $W[x-a]$ disappears for $x<a$. Equations (iii) and (iv) in Ex. 13.6 then become the single equation

$$EI\frac{d^2 v}{dx^2} = R_A x - W[x-a]$$

which on integration yields

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} - \frac{W}{2}[x-a]^2 + C_1$$

and

$$EIv = R_A\frac{x^3}{6} - \frac{W}{6}[x-a]^3 + C_1 x + C_2$$

Note that the square brackets *must be retained* during the integration. The arbitrary constants C_1 and C_2 are found using the boundary conditions that $v=0$ when $x=0$ and $x=L$. From the first of these and remembering that $[x-a]^3$ is zero for $x<a$, we have $C_2=0$. From the second we have

$$0 = R_A\frac{L^3}{6} - \frac{W}{6}[L-a]^3 + C_1 L$$

in which $R_A = W(L-a)/L$.

Substituting for R_A gives

$$C_1 = -\frac{Wa(L-a)}{6L}(2L-a)$$

Then

$$EIv = -\frac{W}{6L}\{-(L-a)x^3 + L[x-a]^3 + a(L-a)(2L-a)x\}$$

The deflection of the beam under the load is then

$$v_C = -\frac{Wa^2(L-a)^2}{3EIL}$$

as before.

EXAMPLE 13.8

Determine the position and magnitude of the maximum upward and downward deflections of the beam shown in Fig. 13.9.

A consideration of the overall equilibrium of the beam gives the support reactions; thus

$$R_A = \frac{3}{4}W \text{ (upward)} \quad R_F = \frac{3}{4}W \text{ (downward)}$$

Using the method of singularity functions and taking the origin of axes at the left-hand support, we write down an expression for the bending moment, M, at any section X between D and F, *the region of the beam furthest from the origin.* Thus

$$M = R_A x - W[x-a] - W[x-2a] + 2W[x-3a] \quad \text{(i)}$$

Substituting for M in Eq. (13.3) we have

$$EI\frac{d^2 v}{dx^2} = \frac{3}{4}Wx - W[x-a] - W[x-2a] + 2W[x-3a] \quad \text{(ii)}$$

Integrating Eq. (ii) and retaining the square brackets we obtain

$$EI\frac{dv}{dx} = \frac{3}{8}Wx^2 - \frac{W}{2}[x-a]^2 - \frac{W}{2}[x-2a]^2 + W[x-3a]^2 + C_1 \quad \text{(iii)}$$

and

$$EIv = \frac{1}{8}Wx^3 - \frac{W}{6}[x-a]^3 - \frac{W}{6}[x-2a]^3 + \frac{W}{3}[x-3a]^3 + C_1 x + C_2 \quad \text{(iv)}$$

in which C_1 and C_2 are arbitrary constants. When $x=0$ (at A), $v=0$ and hence $C_2=0$. Note that the second, third and fourth terms on the right-hand side of Eq. (iv) disappear for $x<a$. Also $v=0$ at $x=4a$ (F) so that, from Eq. (iv), we have

$$0 = \frac{W}{8}64a^3 - \frac{W}{6}27a^3 - \frac{W}{6}8a^3 + \frac{W}{3}a^3 + 4aC_1$$

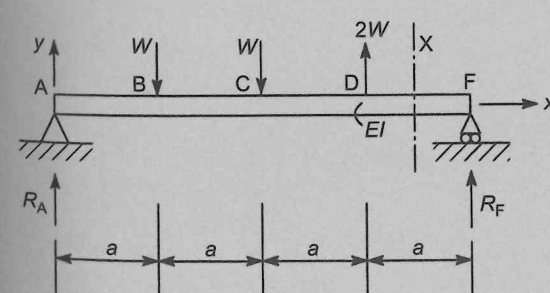


FIGURE 13.9

Macauley's method for the deflection of a simply supported beam (Ex. 13.8).

which gives

$$C_1 = \frac{5}{8} Wa^2$$

Equations (iii) and (iv) now become

$$EI\frac{dv}{dx} = \frac{3}{8} Wx^2 - \frac{W}{2}[x-a]^2 - \frac{W}{2}[x-2a]^2 + W[x-3a]^2 - \frac{5}{8} Wa^2 \qquad \text{(v)}$$

and

$$EIv = \frac{1}{8} Wx^3 - \frac{W}{6}[x-a]^3 - \frac{W}{6}[x-2a]^3 + \frac{W}{3}[x-3a]^3 - \frac{5}{8} Wa^2x \qquad \text{(vi)}$$

respectively.

To determine the maximum upward and downward deflections we need to know in which bays $dv/dx = 0$ and thereby which terms in Eq. (v) disappear when the exact positions are being located. One method is to select a bay and determine the sign of the slope of the beam at the extremities of the bay. A change of sign will indicate that the slope is zero within the bay.

By inspection of Fig. 13.9 it seems likely that the maximum downward deflection will occur in BC. At B, using Eq. (v)

$$EI\frac{dv}{dx} = \frac{3}{8} Wa^2 - \frac{5}{8} Wa^2$$

which is clearly negative. At C

$$EI\frac{dv}{dx} = \frac{3}{8} W4x^2 - \frac{W}{2} a^2 - \frac{5}{8} Wa^2$$

which is positive. Therefore, the maximum downward deflection does occur in BC and its exact position is located by equating dv/dx to zero for any section in BC. Thus, from Eq. (v)

$$0 = \frac{3}{8} Wx^2 - \frac{W}{2}[x-a]^2 - \frac{5}{8} Wa^2$$

or, simplifying,

$$0 = x^2 - 8ax + 9a^2 \qquad \text{(vii)}$$

Solution of Eq. (vii) gives

$$x = 1.35a$$

so that the maximum downward deflection is, from Eq. (vi)

$$EIv = \frac{1}{8} W(1.35a)^3 - \frac{W}{6}(0.35a)^3 - \frac{5}{8} Wa^2(1.35a)$$

i.e.

$$v_{max}\,(\text{downward}) = -\frac{0.54Wa^3}{EI}$$

In a similar manner it can be shown that the maximum upward deflection lies between D and F at $x = 3.42a$ and that its magnitude is

$$v_{max}\,(\text{upward}) = \frac{0.04Wa^3}{EI}$$

An alternative method of determining the position of maximum deflection is to select a possible bay, set $dv/dx = 0$ for that bay and solve the resulting equation in x. If the solution gives a value of x that lies within the bay, then the selection is correct, otherwise the procedure must be repeated for a second and possibly a third and a fourth bay. This method is quicker than the former if the correct bay is selected initially; if not, the equation corresponding to each selected bay must be completely solved, a procedure clearly longer than determining the sign of the slope at the extremities of the bay.

EXAMPLE 13.9

Determine the position and magnitude of the maximum deflection in the beam of Fig. 13.10.

Following the method of Ex. 13.8 we determine the support reactions and find the bending moment, M, at any section X in the bay furthest from the origin of the axes. Thus

$$M = R_A x - w\frac{L}{4}\left[x - \frac{5L}{8}\right] \qquad \text{(i)}$$

Examining Eq. (i) we see that the singularity function $[x - 5L/8]$ does not become zero until $x \le 5L/8$ although Eq. (i) is only valid for $x \ge 3L/4$. To obviate this difficulty we extend the distributed load to the support D while simultaneously restoring the status quo by applying an upward distributed load of the same intensity and length as the additional load (Fig. 13.11).

At the section X, a distance x from A, the bending moment is now given by

$$M = R_A x - \frac{w}{2}\left[x - \frac{L}{2}\right]^2 + \frac{w}{2}\left[x - \frac{3L}{4}\right]^2 \qquad \text{(ii)}$$

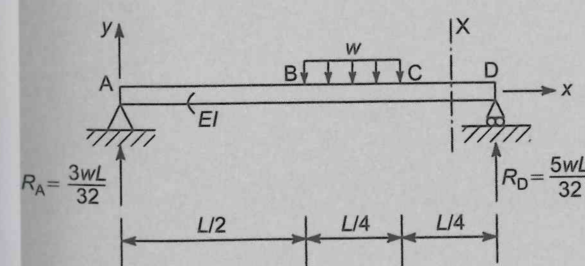


FIGURE 13.10

Deflection of a beam carrying a part span uniformly distributed load (Ex. 13.9).

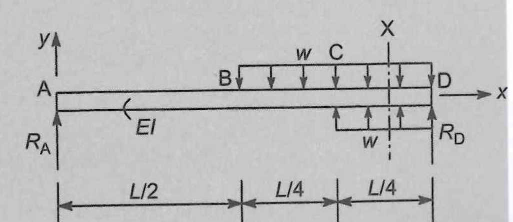


FIGURE 13.11

Method of solution for a part span uniformly distributed load.

Equation (ii) is now valid for all sections of the beam if the singularity functions are discarded as they become zero. Substituting Eq. (ii) into Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = \frac{3}{32}wLx - \frac{w}{2}\left[x-\frac{L}{2}\right]^2 + \frac{w}{2}\left[x-\frac{3L}{4}\right]^2 \qquad \text{(iii)}$$

Integrating Eq. (iii) gives

$$EI\frac{dv}{dx} = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x-\frac{L}{2}\right]^3 + \frac{w}{6}\left[x-\frac{3L}{4}\right]^3 + C_1 \qquad \text{(iv)}$$

$$EIv = \frac{wLx^3}{64} - \frac{w}{24}\left[x-\frac{L}{2}\right]^4 + \frac{w}{24}\left[x-\frac{3L}{4}\right]^4 + C_1x + C_2 \qquad \text{(v)}$$

where C_1 and C_2 are arbitrary constants. The required boundary conditions are $v = 0$ when $x = 0$ and $x = L$. From the first of these we obtain $C_2 = 0$ while the second gives

$$0 = \frac{wL^4}{64} - \frac{w}{24}\left(\frac{L}{2}\right)^4 + \frac{w}{24}\left(\frac{L}{4}\right)^4 + C_1L$$

from which

$$C_1 = -\frac{27wL^3}{2048}$$

Equations (iv) and (v) then become

$$EI\frac{dv}{dx} = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x-\frac{L}{2}\right]^3 + \frac{w}{6}\left[x-\frac{3L}{4}\right]^3 - \frac{27wL^3}{2048} \qquad \text{(vi)}$$

and

$$EIv = \frac{wLx^3}{64} - \frac{w}{24}\left[x-\frac{L}{2}\right]^4 + \frac{w}{24}\left[x-\frac{3L}{4}\right]^4 - \frac{27wL^3}{2048}x \qquad \text{(vii)}$$

In this problem, the maximum deflection clearly occurs in the region BC of the beam. Thus equating the slope to zero for BC we have

$$0 = \frac{3}{64}wLx^2 - \frac{w}{6}\left[x-\frac{L}{2}\right]^3 - \frac{27wL^3}{2048}$$

which simplifies to

$$x^3 - 1.78Lx^2 + 0.75xL^2 - 0.046L^3 = 0 \qquad \text{(viii)}$$

Solving Eq. (viii) by trial and error, we see that the slope is zero at $x \simeq 0.6L$. Hence from Eq. (vii) the maximum deflection is

$$v_{max} = -\frac{4.53 \times 10^{-3}wL^4}{EI}$$

EXAMPLE 13.10

Determine the deflected shape of the beam shown in Fig. 13.12.

In this problem an external moment M_0 is applied to the beam at B. The support reactions are found in the normal way and are

$$R_A = -\frac{M_0}{L}\ \text{(downwards)} \qquad R_C = \frac{M_0}{L}\ \text{(upwards)}$$

The bending moment at any section X between B and C is then given by

$$M = R_Ax + M_0 \qquad \text{(i)}$$

Equation (i) is valid only for the region BC and clearly does not contain a singularity function which would cause M_0 to vanish for $x \leq b$. We overcome this difficulty by writing

$$M = R_Ax + M_0[x-b]^0 \quad (\text{Note:}[x-b]^0 = 1) \qquad \text{(ii)}$$

Equation (ii) has the same value as Eq. (i) but is now applicable to all sections of the beam since $[x-b]^0$ disappears when $x \leq$ b. Substituting for M from Eq. (ii) in Eq. (13.3) we obtain

$$EI\frac{d^2v}{dx^2} = R_Ax + M_0[x-b]^0 \qquad \text{(iii)}$$

Integration of Eq. (iii) yields

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} + M_0[x-b] + C_1 \qquad \text{(iv)}$$

and

$$EIv = R_A\frac{x^3}{6} + \frac{M_0}{2}[x-b]^2 + C_1x + C_2 \qquad \text{(v)}$$

where C_1 and C_2 are arbitrary constants. The boundary conditions are $v = 0$ when $x = 0$ and $x = L$. From the first of these we have $C_2 = 0$ while the second gives

$$0 = -\frac{M_0}{L}\frac{L^3}{6} + \frac{M_0}{2}[L-b]^2 + C_1L$$

from which

$$C_1 = -\frac{M_0}{6L}(2L^2 - 6Lb + 3b^2)$$

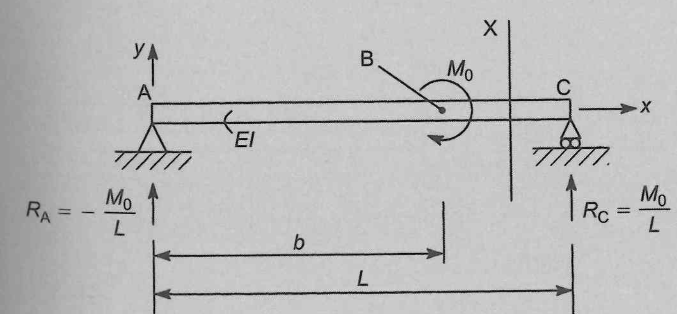


FIGURE 13.12

Deflection of a simply supported beam carrying a point moment (Ex. 13.10).

The equation of the deflection curve of the beam is then

$$v = \frac{M_0}{6EIL}\{x^3 + 3L[x-b]^2 - (2L^2 - 6Lb + 3b^2)x\} \qquad \text{(vi)}$$

EXAMPLE 13.11

Determine the vertical deflection of the point D in the beam ABCD shown in Fig. 13.13 in terms of its flexural rigidity EI; state clearly its direction.

The support reactions R_A and R_C are obtained in the usual way and are −3.75 kN and 18.75 kN respectively. Note that R_A is a downward reaction.

The distributed load is now extended to the end D of the beam as shown in Fig. 13.14 and the status quo restored by applying an equal and upward distributed load between C and D.

The bending moment at the section X in the bay CD a distance x from A is then given by

$$M = R_A x - \frac{5}{2}[x-1]^2 + \frac{5}{2}[x-2]^2 + R_C[x-2] \qquad \text{(i)}$$

Substituting for M in Eq.(13.3) we obtain

$$EI\frac{d^2v}{dx^2} = -3.75x - \frac{5}{2}[x-1]^2 + \frac{5}{2}[x-2]^2 + 18.75[x-2] \qquad \text{(ii)}$$

Then

$$EI\frac{dv}{dx} = -3.75\frac{x^2}{2} - \frac{5}{6}[x-1]^3 + \frac{5}{6}[x-2]^3 + \frac{18.75}{2}[x-2]^2 + C_1 \qquad \text{(iii)}$$

and

$$EIv = -3.75\frac{x^3}{6} - \frac{5}{24}[x-1]^4 + \frac{5}{24}[x-2]^4 + \frac{18.75}{6}[x-2]^3 + C_1x + C_2 \qquad \text{(iv)}$$

The boundary conditions are that $v = 0$ when $x = 0$ and $x = 2$ m. From the first of these $C_2 = 0$ and from the second

$$0 = \frac{-3.75\times 2^3}{6} - \frac{5}{24}[2-1]^4 + 2C_1$$

which gives $C_1 = 2.6$

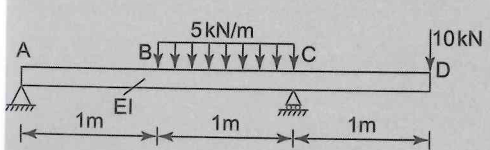


FIGURE 13.13

Beam of Ex. 13.11

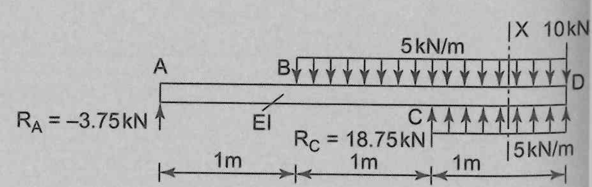


FIGURE 13.14

Solution of Ex. 13.11

Eq. (iv) therefore becomes

$$EIv = \frac{-3.75x^3}{6} - \frac{5}{24}[x-1]^4 + \frac{5}{24}[x-2]^4 + \frac{18.75}{6}[x-2]^3 + 2.6x \qquad \text{(v)}$$

Then, at D where $x = 3$ m

$$EIv_D = -9.08$$

or

$$v_D = \frac{-9.08}{EI} \quad \text{(ie downwards)}$$

Note that the 10 kN load does not enter directly into the moment equation. It could be included by adding an imaginary extension of the beam past D which would result in an additional term $-10[x-3]$ in the expression for bending moment, Eq. (i). However it is clear that this term would always disappear when considering any section of the beam between A and D so that such an approach is unnecessary."

13.3 Moment-area method for symmetrical bending

The double integration method and the method of singularity functions are used when the complete deflection curve of a beam is required. However, if only the deflection of a particular point is required, the moment-area method is generally more suitable.

Consider the curvature—moment equation (Eq. (13.3)), i.e.

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Integration of this equation between any two sections, say A and B, of a beam gives

$$\int_A^B \frac{d^2v}{dx^2}\,dx = \int_A^B \frac{M}{EI}\,dx \qquad (13.6)$$

or

$$\left[\frac{dv}{dx}\right]_A^B = \int_A^B \frac{M}{EI}\,dx$$

which gives

$$\left(\frac{dv}{dx}\right)_B - \left(\frac{dv}{dx}\right)_A = \int_A^B \frac{M}{EI}\,dx \qquad (13.7)$$

In qualitative terms Eq. (13.7) states that the change of slope between two sections A and B of a beam is numerically equal to the area of the M/EI diagram between those sections.

We now return to Eq. (13.3) and multiply both sides by x thereby retaining the equality. Thus

$$\frac{d^2v}{dx^2}x = \frac{M}{EI}x \qquad (13.8)$$

Integrating Eq. (13.8) between any two sections A and B of a beam we have

$$\int_A^B \frac{d^2v}{dx^2} x\, dx = \int_A^B \frac{M}{EI} x\, dx \tag{13.9}$$

The left-hand side of Eq. (13.9) may be integrated by parts and gives

$$\left[x\frac{dv}{dx}\right]_A^B - \int_A^B \frac{dv}{dx} dx = \int_A^B \frac{M}{EI} x\, dx$$

or

$$\left[x\frac{dv}{dx}\right]_A^B - [v]_A^B = \int_A^B \frac{M}{EI} x\, dx$$

Hence, inserting the limits we have

$$x_B\left(\frac{dv}{dx}\right)_B - x_A\left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI} x\, dx \tag{13.10}$$

in which x_B and x_A represent the x coordinate of each of the sections B and A, respectively, while $(dv/dx)_B$ and $(dv/dx)_A$ are the respective slopes; v_B and v_A are the corresponding deflections. The right-hand side of Eq. (13.10) represents the moment of the area of the *M/EI* diagram between the sections A and B *about* A.

Equations (13.7) and (13.10) may be used to determine values of slope and deflection at any section of a beam. We note that in both equations we are concerned with the geometry of the *M/EI* diagram. This will be identical in shape to the bending moment diagram unless there is a change of section. Furthermore, the form of the right-hand side of both Eqs (13.7) and (13.10) allows two alternative methods of solution. In cases where the geometry of the *M/EI* diagram is relatively simple, we can employ a *semi-graphical* approach based on the actual geometry of the *M/EI* diagram. Alternatively, in complex problems, the bending moment may be expressed as a function of x and a completely analytical solution obtained. Both methods are illustrated in the following examples.

EXAMPLE 13.12

Determine the slope and deflection of the free end of the cantilever beam shown in Fig. 13.15.

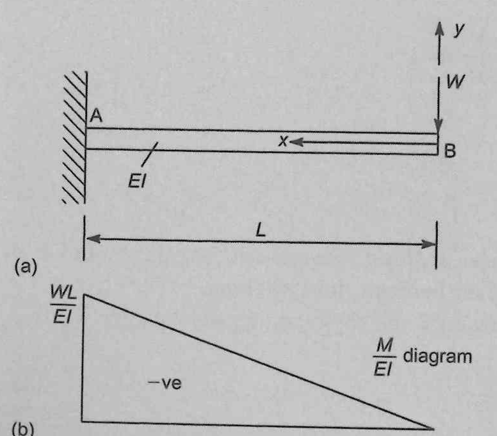


FIGURE 13.15

Moment-area method for the deflection of a cantilever (Ex. 13.12).

We choose the origin of the axes at the free end B of the cantilever. Equation (13.7) then becomes

$$\left(\frac{dv}{dx}\right)_A - \left(\frac{dv}{dx}\right)_B = \int_A^B \frac{M}{EI} dx$$

or, since $(dv/dx)_A = 0$

$$-\left(\frac{dv}{dx}\right)_B = \int_0^L \frac{M}{EI} dx \tag{i}$$

Generally at this stage we decide which approach is most suitable; however, both semi-graphical and analytical methods are illustrated here. Using the geometry of Fig. 13.15(b) we have

$$-\left(\frac{dv}{dx}\right)_B = \frac{1}{2}L\left(\frac{-WL}{EI}\right)$$

which gives

$$\left(\frac{dv}{dx}\right)_B = \frac{WL^2}{2EI}$$

(compare with the value given by Eq. (iii) of Ex. 13.1. Note the change in sign due to the different origin for x).

Alternatively, since the bending moment at any section a distance x from B is $-Wx$ we have, from Eq. (i)

$$-\left(\frac{dv}{dx}\right)_B = \int_0^L -\frac{Wx}{EI} dx$$

which again gives

$$\left(\frac{dv}{dx}\right)_B = \frac{WL^2}{2EI}$$

With the origin for x at B, Eq. (13.10) becomes

$$x_A\left(\frac{dv}{dx}\right)_A - x_B\left(\frac{dv}{dx}\right)_B - (v_A - v_B) = \int_B^A \frac{M}{EI} x\, dx \tag{ii}$$

Since $(dv/dx)_A = 0$ and $x_B = 0$ and $v_A = 0$, Eq. (ii) reduces to

$$v_B = \int_0^L \frac{M}{EI} x\, dx \tag{iii}$$

Again we can now decide whether to proceed semi-graphically or analytically. Using the former approach and taking the moment of the area of the *M/EI* diagram about B, we have

$$v_B = \frac{1}{2}L\left(\frac{-WL}{EI}\right)\frac{2}{3}L$$

which gives

$$v_B = -\frac{WL^3}{3EI} \quad \text{(compare with Eq. (v) of Ex. 13.1)}$$

Alternatively we have

$$v_B = \int_0^L \frac{(-Wx)}{EI} x\,dx = -\int_0^L \frac{Wx^2}{EI}\,dx$$

which gives

$$v_B = -\frac{WL^3}{3EI}$$

as before.

Note that if the built-in end had been selected as the origin for x, we could not have determined v_B directly since the term $x_B(dv/dx)_B$ in Eq. (ii) would not have vanished. The solution for v_B would then have consisted of two parts, first the determination of $(dv/dx)_B$ and then the calculation of v_B.

EXAMPLE 13.13

Determine the maximum deflection in the simply supported beam shown in Fig. 13.16(a).

From symmetry we deduce that the beam reactions are each $wL/2$; the M/EI diagram has the geometry shown in Fig. 13.16(b).

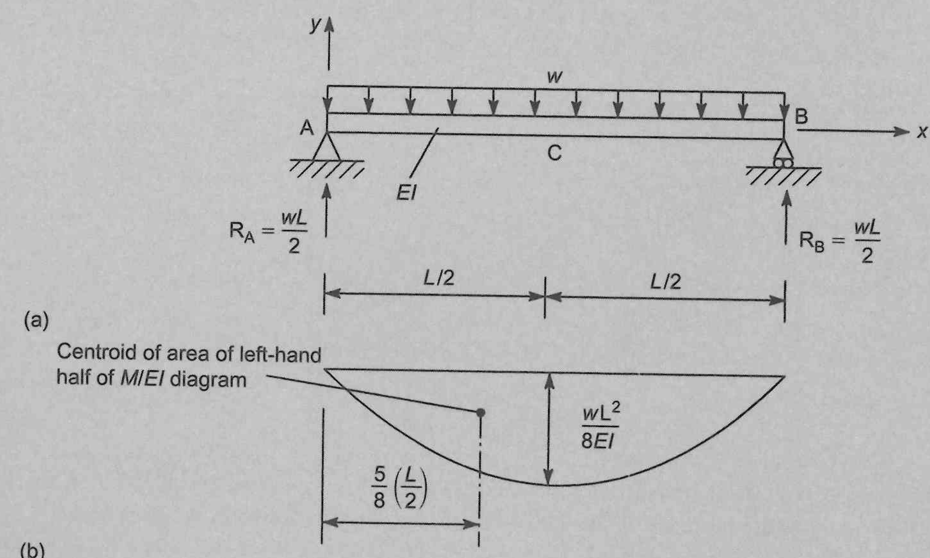


FIGURE 13.16

Moment-area method for a simply supported beam carrying a uniformly distributed load.

If we take the origin of axes to be at A and consider the half-span AC, Eq. (13.10) becomes

$$x_C\left(\frac{dv}{dx}\right)_C - x_A\left(\frac{dv}{dx}\right)_A - (v_C - v_A) = \int_A^C \frac{M}{EI} x\,dx \quad \text{(i)}$$

In this problem $(dv/dx)_C = 0$, $x_A = 0$ and $v_A = 0$; hence Eq. (i) reduces to

$$v_C = -\int_0^{L/2} \frac{M}{EI} x\,dx \quad \text{(ii)}$$

Using the geometry of the M/EI diagram, i.e. the semi-graphical approach, and taking the moment of the area of the M/EI diagram between A and C about A we have from Eq. (ii)

$$v_C = -\frac{2}{3}\frac{wL^2}{8EI}\frac{L}{2}\frac{5}{8}\left(\frac{L}{2}\right)$$

which gives

$$v_C = -\frac{5wL^4}{384EI} \quad \text{(see Eq. (}v\text{) of Ex. 13.4).}$$

For the completely analytical approach we express the bending moment M as a function of x; thus

$$M = \frac{wL}{2}x - \frac{wx^2}{2}$$

or

$$M = \frac{w}{2}(Lx - x^2)$$

Substituting for M in Eq. (ii) we have

$$v_C = -\int_0^{L/2} \frac{w}{2EI}(Lx^2 - x^3)\,dx$$

which gives

$$v_C = -\frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/2}$$

Then

$$v_C = -\frac{5wL^4}{384EI}$$

EXAMPLE 13.14

Figure 13.17(a) shows a cantilever beam of length L carrying a concentrated load W at its free end. The section of the beam changes midway along its length so that the second moment of area of its cross section is reduced by half. Determine the deflection of the free end.

In this problem the bending moment and M/EI diagrams have different geometrical shapes. Choosing the origin of axes at C, Eq. (13.10) becomes

$$x_A\left(\frac{dv}{dx}\right)_A - x_C\left(\frac{dv}{dx}\right)_C - (v_A - v_C) = \int_C^A \frac{M}{EI} x\,dx \qquad \text{(i)}$$

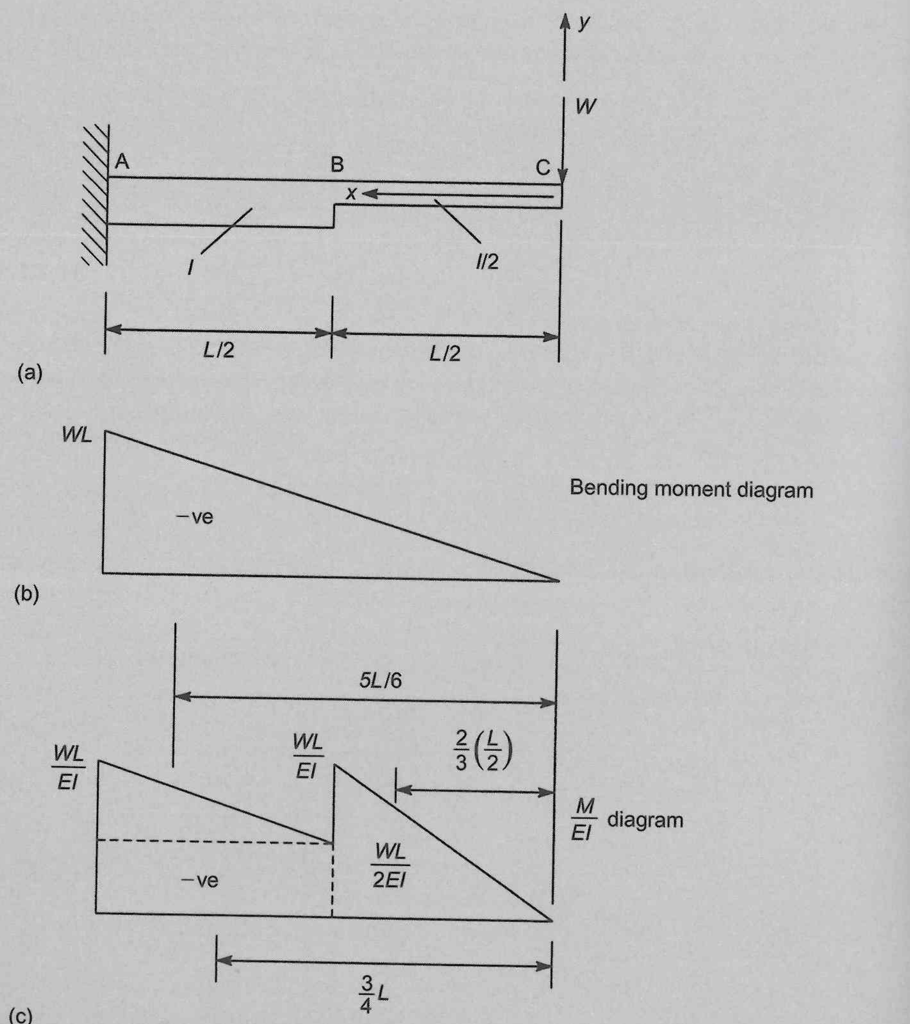


FIGURE 13.17

Deflection of a cantilever of varying section.

in which $(dv/dx)_A = 0$, $x_C = 0$, $v_A = 0$. Hence

$$v_C = \int_0^L \frac{M}{EI} x\,dx \qquad \text{(ii)}$$

From the geometry of the M/EI diagram (Fig. 13.17(c)) and taking moments of areas about C we have

$$v_C = \left[\left(\frac{-W}{2EI}\right)\frac{L}{2}\frac{3L}{4} + \frac{1}{2}\left(\frac{-WL}{2EI}\right)\frac{L}{2}\frac{5L}{6} + \frac{1}{2}\left(\frac{-WL}{EI}\right)\frac{L}{2}\frac{2}{3}\frac{L}{2}\right]$$

which gives

$$v_C = -\frac{3WL^3}{8EI}$$

Analytically we have

$$v_C = \left[\int_0^{L/2} \frac{-Wx^2}{EI/2}\,dx + \int_{L/2}^{L} \frac{-Wx^2}{EI}\,dx\right]$$

or

$$v_C = -\frac{W}{EI}\left\{\left[\frac{2x^3}{3}\right]_0^{L/2} + \left[\frac{x^3}{3}\right]_{L/2}^{L}\right\}$$

Hence

$$v_C = -\frac{3WL^3}{8EI}$$

as before.

EXAMPLE 13.15

The cantilever beam shown in Fig. 13.18 tapers along its length so that the second moment of area of its cross section varies linearly from its value I_0 at the free end to $2I_0$ at the built-in end. Determine the deflection at the free end when the cantilever carries a concentrated load W.

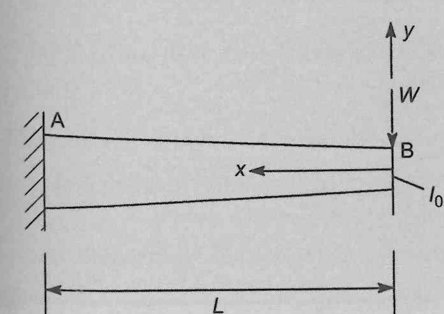


FIGURE 13.18

Deflection of a cantilever of tapering section.

Choosing the origin of axes at the free end B we have, from Eq. (13.10)

$$x_A\left(\frac{dv}{dx}\right)_A - x_B\left(\frac{dv}{dx}\right)_B - (v_A - v_B) = \int_B^A \frac{M}{EI_X} x\,dx \qquad \text{(i)}$$

in which I_x, the second moment of area at any section X, is given by

$$I_X = I_0\left(1 + \frac{x}{L}\right)$$

Also $(dv/dx)_A = 0$, $x_B = 0$, $v_A = 0$ so that Eq. (i) reduces to

$$v_B = \int_0^L \frac{Mx}{EI_0\left(1+\frac{x}{L}\right)} dx \qquad \text{(ii)}$$

The geometry of the M/EI diagram in this case will be complicated so that the analytical approach is most suitable. Therefore since $M = -Wx$, Eq. (ii) becomes

$$v_B = -\int_0^L \frac{Wx^2}{EI_0\left(1+\frac{x}{L}\right)} dx$$

or

$$v_B = -\frac{WL}{EI_0}\int_0^L \frac{x^2}{L+x} dx \qquad \text{(iii)}$$

Rearranging Eq. (iii) we have

$$v_B = -\frac{WL}{EI_0}\left[\int_0^L (x-L)\,dx + \int_0^L \frac{L^2}{L+x} dx\right]$$

which may be written in the form

$$v_B = -\frac{WL}{EI_0}\left[\int_0^L (x-L)\,dx + L^2\int_0^L \frac{dx}{L(1+x/L)}\right]$$

Hence

$$v_B = -\frac{WL}{EI_0}\left[\left(\frac{x^2}{2} - Lx\right) + L^2 \log_e(1+x/L)\right]_0^L$$

so that

$$v_B = -\frac{WL^3}{EI_0}\left(-\frac{1}{2} + \log_e 2\right)$$

i.e.

$$v_B = -\frac{0.19WL^3}{EI_0}$$

13.4 Deflections due to unsymmetrical bending

We noted in Chapter 9 that a beam bends about its neutral axis whose inclination to arbitrary centroidal axes is determined from Eq. (9.33). Beam deflections, therefore, are always perpendicular in direction to the neutral axis.

Suppose that at some section of a beam, the deflection normal to the neutral axis (and therefore an absolute deflection) is ζ. Then, as shown in Fig. 13.19, the centroid G is displaced to G'. The components of ζ, u and v, are given by

$$u = \zeta \sin\alpha \quad v = \zeta\cos\alpha \qquad (13.11)$$

The centre of curvature of the beam lies in a longitudinal plane perpendicular to the neutral axis of the beam and passing through the centroid of any section. Hence for a radius of curvature R, we see, by direct comparison with Eq. (13.2) that

$$\frac{1}{R} = \frac{d^2\zeta}{dx^2} \qquad (13.12)$$

or, substituting from Eq. (13.11)

$$\frac{\sin\alpha}{R} = \frac{d^2u}{dx^2} \quad \frac{\cos\alpha}{R} = \frac{d^2v}{dx^2} \qquad (13.13)$$

We observe from the derivation of Eq. (9.31) that

$$\frac{E\sin\alpha}{R} = \frac{M_yI_z - M_zI_{zy}}{I_zI_y - I_{zy}^2}$$

$$\frac{E\cos\alpha}{R} = \frac{M_zI_y - M_yI_{zy}}{I_zI_y - I_{zy}^2}$$

Therefore, from Eq. (13.13)

$$\frac{d^2u}{dx^2} = \frac{M_yI_z - M_zI_{zy}}{E(I_zI_y - I_{zy}^2)} \qquad (13.14)$$

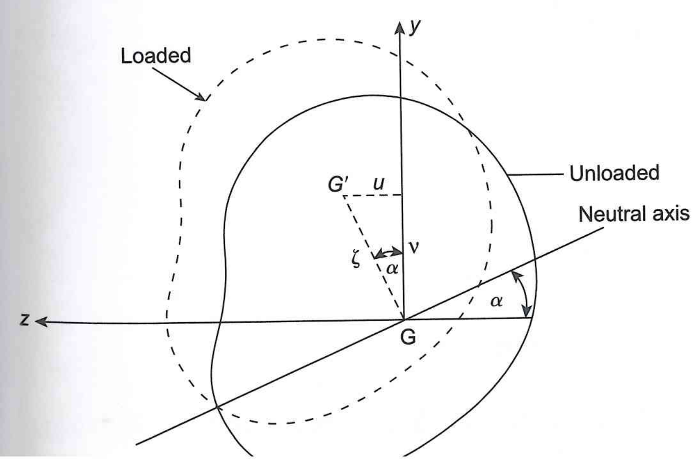


FIGURE 13.19
Deflection of a beam of

$$\frac{d^2v}{dx^2} = \frac{M_z I_y - M_y I_{zy}}{E(I_z I_y - I_{zy}^2)} \tag{13.15}$$

EXAMPLE 13.16

Determine the horizontal and vertical components of the deflection of the free end of the cantilever shown in Fig. 13.20. The second moments of area of its unsymmetrical section are I_z, I_y and I_{zy}.

The bending moments at any section of the beam due to the applied load W are

$$M_z = -W(L-x), \quad M_y = 0$$

Then Eq. (13.14) reduces to

$$\frac{d^2u}{dx^2} = \frac{W(L-x)I_{zy}}{E(I_z I_y - I_{zy}^2)} \tag{i}$$

Integrating with respect to x

$$\frac{du}{dx} = \frac{WI_{zy}}{E(I_z I_y - I_{zy}^2)}\left(Lx - \frac{x^2}{2} + C_1\right)$$

When $x = 0$, $(du/dx) = 0$ so that $C_1 = 0$ and

$$\frac{du}{dx} = \frac{WI_{zy}}{E(I_z I_y - I_{zy}^2)}\left(Lx - \frac{x^2}{2}\right) \tag{ii}$$

Integrating Eq. (ii) with respect to x

$$u = \frac{WI_{zy}}{E(I_z I_y - I_{zy}^2)}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_2\right)$$

When $x = 0$, $u = 0$ so that $C_2 = 0$. Therefore

$$u = \frac{WI_{zy}}{6E(I_z I_y - I_{zy}^2)}(3Lx^2 - x^3) \tag{iii}$$

At the free end of the cantilever where $x = L$

$$u_{fe} = \frac{WI_{zy}L^3}{3E(I_z I_y - I_{zy}^2)} \tag{iv}$$

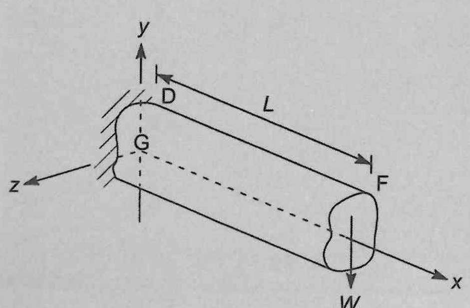


FIGURE 13.20
Deflection of a cantilever of unsymmetrical cross section carrying a concentrated load at its free end (Ex. 13.16).

The deflected shape of the beam in the xy plane is found in an identical manner from Eq. (13.15) and is

$$v = -\frac{WI_y}{6E(I_z I_y - I_{zy}^2)}(3Lx^2 - x^3) \tag{v}$$

from which the deflection at the free end is

$$v_{fe} = -\frac{WI_y L^3}{3E(I_z I_y - I_{zy}^2)} \tag{vi}$$

The absolute deflection, δ_{fe}, at the free end is given by

$$\delta_{fe} = (u_{fe}^2 + v_{fe}^2)^{\frac{1}{2}} \tag{vii}$$

and its direction is at $\tan^{-1}(u_{fe}/v_{fe})$ to the vertical.

Note that if either Gz or Gy is an axis of symmetry $I_{zy} = 0$ and Eqs. (iv) and (vi) reduce to

$$u_{fe} = 0 \quad v_{fe} = -\frac{WL^3}{3EI_z} \quad \text{(compare with Eq. (v) of Ex. 13.1)}$$

EXAMPLE 13.17

Determine the deflection of the free end of the cantilever beam shown in Fig. 13.21. The second moments of area of its cross section about a horizontal and vertical system of centroidal axes are I_z, I_y and I_{zy}.

The method of solution is identical to that in Ex. 13.16 except that the bending moments M_z and M_y are given by

$$M_z = -w(L-x)^2/2 \quad M_y = 0$$

The values of the components of the deflection at the free end of the cantilever are

$$u_{fe} = \frac{wI_{zy}L^4}{8E(I_z I_y - I_{zy}^2)} \quad v_{fe} = -\frac{wI_y L^4}{8E(I_z I_y - I_{zy}^2)}$$

Again, if either Gz or Gy is an axis of symmetry, $I_{zy} = 0$ and these expressions reduce to $u_{fe} = 0$, $v_{fe} = -\frac{wL^4}{8EI_z}$ (compare with Eq. (v) of Ex. 13.2)

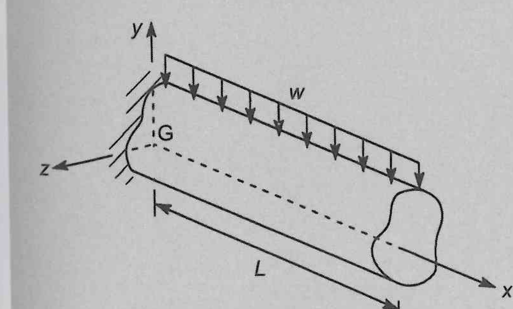


FIGURE 13.21
Deflection of a cantilever of unsymmetrical cross section carrying a uniformly distributed load (Ex. 13.17).

EXAMPLE 13.18

Determine the vertical and horizontal components of the displacement midway between the supports B and C of the thin-walled beam shown in Fig. 13.22(a). Young's modulus for the material of the beam is E and its cross section is shown in Fig. 13.22(b).

The centroid of the beam section coincides with the centre of the web 34. The second moments of area are calculated using the methods described in Section 9.6 and are:

$$I_z = 3.25a^3t, \quad I_y = 1.67a^3t, \quad I_{zy} = 1.75a^3t$$

Since only a vertical load is applied there will only be vertical support reactions at B and C. Therefore, taking moments about B

$$R_C \times 2L + WL = 0$$

so that

$$R_C = -W/2 \text{ (ie downwards)}$$

Taking the origin for x at C the bending moments at any section between B and C are given by

$$M_z = R_C x = -Wx/2, \; M_y = 0$$

Substituting these values in Eq. (13.14)

$$\frac{d^2u}{dx^2} = -\frac{(-Wx/2)I_{zy}}{E(I_zI_y - I_{zy}^2)}$$

Integrating with respect to x

$$\frac{du}{dx} = \frac{WI_{zy}}{2E(I_zI_y - I_{zy}^2)}\left(\frac{x^2}{2} + C_1\right)$$

and

$$u = \frac{WI_{zy}}{2E(I_zI_y - I_{zy}^2)}\left(\frac{x^3}{6} + C_1x + C_2\right) \quad \text{(i)}$$

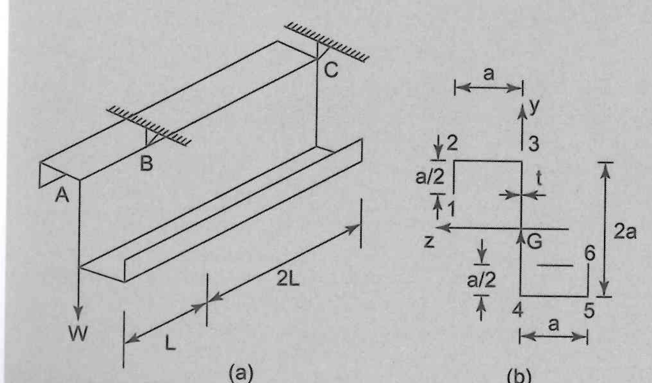


FIGURE 13.22

Beam of Ex. 13.18

When $x = 0$, $u = 0$ so that $C_2 = 0$. Also, when $x = 2L$, $u = 0$ so that, from Eq. (i)

$$\frac{8L^3}{6} + 2LC_1 = 0$$

which gives

$$C_1 = -2L^2/3$$

Eq. (i) may then be written

$$u = \frac{WI_{zy}}{12E(I_zI_y - I_{zy}^2)}(x^3 - 4L^2x) \quad \text{(ii)}$$

At the mid-point of BC, $x = L$ so that

$$u \text{ (mid-point of BC)} = -\frac{WI_{zy}L^3}{4E(I_zI_y - I_{zy}^2)}$$

Substituting the values of I_z etc gives

$$u \text{ (mid-point of BC)} = -\frac{0.186\,WL^3}{Ea^3t} \quad \text{(ie to the right)}$$

Similarly

$$v \text{ (mid-point of BC)} = +\frac{0.177\,WL^3}{Ea^3t} \quad \text{(ie upwards)}$$

Note that in this particular example the vertical displacement of the mid-point of BC may be obtained directly by replacing I_{zy} in Eq. (ii) by I_y and making allowance for the change in sign of the term involving M_z in Eq. (13.15).

13.5 Deflection due to shear

So far in this chapter we have been concerned with deflections produced by the bending action of shear loads. These shear loads however, as we saw in Chapter 10, induce shear stress distributions throughout beam sections which in turn produce shear strains and therefore shear deflections. Generally, shear deflections are small compared with bending deflections, but in some cases of deep beams they can be comparable. In the following we shall use strain energy to derive an expression for the deflection due to shear in a beam having a cross section which is at least singly symmetrical.

In Chapter 10 we showed that the strain energy U of a piece of material subjected to a uniform shear stress τ is given by

$$U = \frac{\tau^2}{2G} \times \text{volume} \quad \text{(Eq. (10.20))}$$

However, we also showed in Chapter 10 that shear stress distributions are not uniform throughout beam sections. We therefore write Eq. (10.20) as

$$U = \frac{\beta}{2G} \times \left(\frac{S}{A}\right)^2 \times \text{volume} \quad (13.16)$$

in which S is the applied shear force, A is the cross-sectional area of the beam section and β is a con-
stant which depends upon the distribution of shear stress through the beam section; β is known as the
form factor.

To determine β we consider an element $b_0\,\delta y$ in an elemental length δx of a beam subjected to a vertical shear load S_y (Fig. 13.22); we shall suppose that the beam section has a vertical axis of symmetry. The shear stress τ is constant across the width, b_0, of the element (see Section 10.2). The strain energy, δU, of the element $b_0\,\delta y\,\delta x$, from Eq. (10.20) is

$$\delta U = \frac{\tau^2}{2G} \times b_0\,\delta y\,\delta x \tag{13.17}$$

Therefore the total strain energy U in the elemental length of beam is given by

$$U = \frac{\delta x}{2G}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y \tag{13.18}$$

Alternatively U for the elemental length of beam is obtained using Eq. (13.16); thus

$$U = \frac{\beta}{2G} \times \left(\frac{S_y}{A}\right)^2 \times A\delta x \tag{13.19}$$

Equating Eqs (13.19) and (13.18) we have

$$\frac{\beta}{2G} \times \left(\frac{S_y}{A}\right)^2 \times A\delta x = \frac{\delta x}{2G}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y$$

hence

$$\beta = \frac{A}{S_y^2}\int_{y_1}^{y_2} \tau^2 b_0\,\mathrm{d}y \tag{13.20}$$

The shear stress distribution in a beam having a singly or doubly symmetrical cross section and subjected to a vertical shear force, S_y, is given by Eq. (10.4), i.e.

$$\tau = -\frac{S_y A'\bar{y}}{b_0 I_z}$$

Substituting this expression for τ in Eq. (13.20) we obtain

$$\beta = \frac{A}{S_y^2}\int_{y_1}^{y_2} \left(\frac{S_y A'\bar{y}}{b_0 I_z}\right)^2 b_0\,\mathrm{d}y$$

which gives

$$\beta = \frac{A}{I_z^2}\int_{y_1}^{y_2} \frac{(A'\bar{y})^2}{b_0}\,\mathrm{d}y \tag{13.21}$$

Suppose now that δv_s is the deflection due to shear in the elemental length of beam of Fig. 13.23. The work done by the shear force S_y (assuming it to be constant over the length δx and gradually applied) is then

$$\frac{1}{2} S_y\,\delta v_s$$

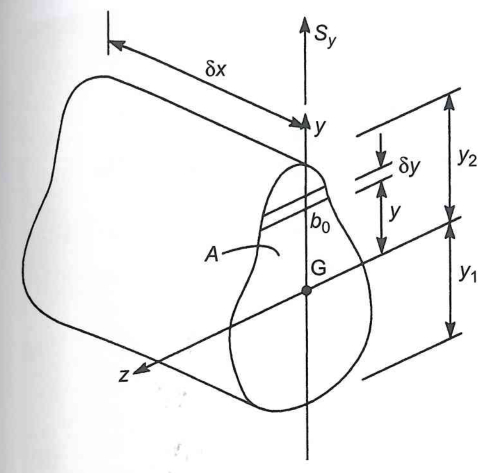


FIGURE 13.23

Determination of form factor β.

which is equal to the strain energy stored. Hence

$$\frac{1}{2} S_y\,\delta v_s = \frac{\beta}{2G} \times \left(\frac{S}{A}\right)^2 \times A\,\delta x$$

which gives

$$\delta v_s = \frac{\beta}{G}\left(\frac{S}{A}\right)\delta x$$

The total deflection due to shear in a beam of length L subjected to a vertical shear force S_y is then

$$v_s = \frac{\beta}{G}\int_L \left(\frac{S_y}{A}\right)\mathrm{d}x \tag{13.22}$$

EXAMPLE 13.19

A cantilever beam of length L has a rectangular cross section of breadth B and depth D and carries a vertical concentrated load, W, at its free end. Determine the deflection of the free end, including the effects of both bending and shear. The flexural rigidity of the cantilever is EI and its shear modulus G.

Using Eq. (13.21) we obtain the form factor β for the cross section of the beam directly. Thus

$$\beta = \frac{BD}{(BD^3/12)^2}\int_{-D/2}^{D/2} \frac{1}{B}\left[B\left(\frac{D}{2}-y\right)\frac{1}{2}\left(\frac{D}{2}+y\right)\right]^2 \mathrm{d}y \quad \text{(see Ex. 10.1)}$$

which simplifies to

$$\beta = \frac{36}{D^5}\int_{-D/2}^{D/2}\left(\frac{D^4}{16} - \frac{D^2y^2}{2} + y^4\right)\mathrm{d}y$$

Integrating we obtain

$$\beta = \frac{36}{D^5}\left[\frac{D^4 y}{16} - \frac{D^2y^3}{6} + \frac{y^5}{5}\right]_{-D/2}^{D/2}$$

which gives

$$\beta = \frac{6}{5}$$

Note that the dimensions of the cross section do not feature in the expression for β. The form factor for any rectangular cross section is therefore 6/5 or 1.2.

Let us suppose that v_s is the vertical deflection of the free end of the cantilever due to shear. Hence, from Eq. (13.22) we have

$$v_s = \frac{6}{5G}\int_0^L \left(\frac{-W}{BD}\right) dx$$

so that

$$v_s = -\frac{6WL}{5GBD} \tag{i}$$

The vertical deflection due to bending of the free end of a cantilever carrying a concentrated load has previously been determined in Ex. 13.1 and is $-WL^3/3EI$. The total deflection, v_T, produced by bending and shear is then

$$v_T = -\frac{WL^3}{3EI} - \frac{6WL}{5GBD} \tag{ii}$$

Rewriting Eq. (ii) we obtain

$$v_T = -\frac{WL^3}{3EI}\left[1 + \frac{3}{10}\frac{E}{G}\left(\frac{D}{L}\right)^2\right] \tag{iii}$$

For many materials (3E/10 G) is approximately unity so that the contribution of shear to the total deflection is $(D/L)^2$ of the bending deflection. Clearly this term only becomes significant for short, deep beams.

13.6 Statically indeterminate beams

The beams we have considered so far have been supported in such a way that the support reactions could be determined using the equations of statical equilibrium; such beams are therefore *statically determinate.* However, many practical cases arise in which additional supports are provided so that there are a greater number of unknowns than the possible number of independent equations of equilibrium; the support systems of such beams are therefore *statically indeterminate.* Simple examples are shown in Fig. 13.24 where, in Fig. 13.24(a), the cantilever does not, theoretically, require the additional support at its free end and in Fig. 13.24(b) any one of the three supports is again, theoretically, *redundant.* A beam such as that shown in Fig. 13.24(b) is known as a *continuous beam* since it has more than one span and is continuous over one or more supports.

We shall now use the results of the previous work in this chapter to investigate methods of solving statically indeterminate beam systems. Having determined the reactions, diagrams of shear force and bending moment follow in the normal manner.

The examples given below are relatively simple cases of statically indeterminate beams. We shall

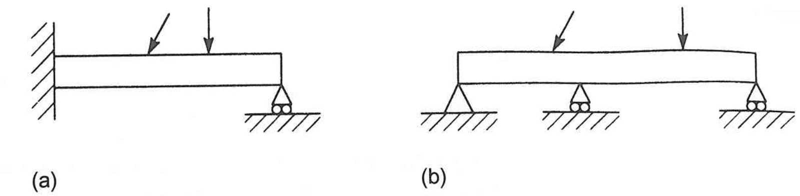


FIGURE 13.24

Examples of statically indeterminate beams.

Method of superposition

In Section 3.7 we discussed the principle of superposition and saw that the combined effect of a number of forces on a structural system may be found by the addition of their separate effects. The principle may be applied to the determination of support reactions in relatively simple statically indeterminate beams. We shall illustrate the method by examples.

EXAMPLE 13.20

The cantilever AB shown in Fig. 13.25(a) carries a uniformly distributed load and is provided with an additional support at its free end. Determine the reaction at the additional support.

Suppose that the reaction at the support B is R_B. Using the principle of superposition we can represent the combined effect of the distributed load and the reaction R_B as the sum of the two loads acting separately as shown in Fig. 13.25(b) and (c). Also, since the vertical deflection of B in Fig. 13.25(a) is zero, it follows that the vertical downward deflection of B in Fig. 13.25(b) must be numerically equal to the vertically upward deflection of B in Fig. 13.25(c). Therefore using the results of Exs (13.1) and (13.2) we have

$$\left|\frac{R_B L^3}{3EI}\right| = \left|\frac{wL^4}{8EI}\right|$$

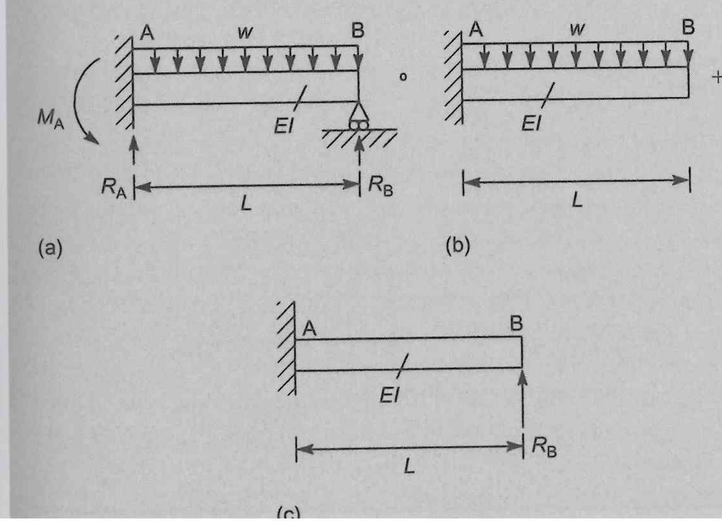


FIGURE 13.25

Propped cantilever of Ex. 13.20.

whence

$$R_B = \frac{3}{8}wL$$

It is now possible to determine the reactions R_A and M_A at the built-in end using the equations of simple statics. Taking moments about A for the beam in Fig. 13.25(a) we have

$$M_A = \frac{wL^2}{2} - R_B L = \frac{wL^2}{2} - \frac{3}{8}wL^2 = \frac{1}{8}wL^2$$

Resolving vertically

$$R_A = wL - R_B = wL - \frac{3}{8}wL = \frac{5}{8}wL$$

In the solution of Ex. 13.20 we selected R_B as the *redundancy;* in fact, any one of the three support reactions, M_A, R_A or R_B, could have been chosen. Let us suppose that M_A is taken to be the redundant reaction. We now represent the combined loading of Fig. 13.25(a) as the sum of the separate loading systems shown in Fig. 13.26(a) and (b) and work in terms of the rotations of the beam at A due to the distributed load and the applied moment, M_A. Clearly, since there is no rotation at the built-in end of a cantilever, the rotations produced separately in Fig. 13.26(a) and (b) must be numerically equal but opposite in direction. Using the method of Section 13.1 it may be shown that

$$\theta_A(\text{due to } w) = \frac{wL^3}{24EI} \quad \text{(clockwise)}$$

and

$$\theta_A(\text{due to } M_A) = \frac{M_A L}{3EI} \quad \text{(anticlockwise)}$$

Since

$$|\theta_A(M_A)| = |\theta_A(w)|$$

we have

$$M_A = \frac{wL^2}{8}$$

as before.

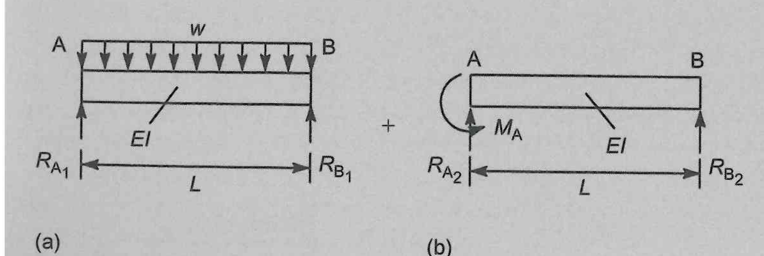


FIGURE 13.26

Alternative solution of Ex. 13.21.

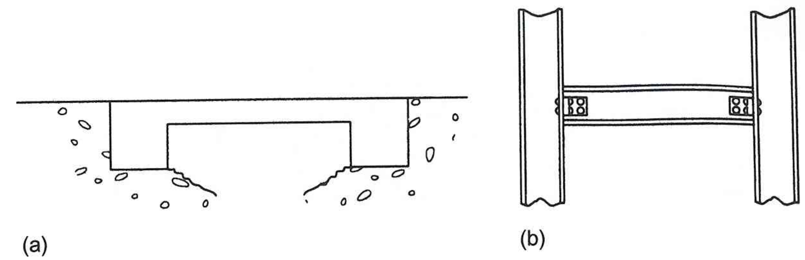


FIGURE 13.27

Practical examples of fixed beams.

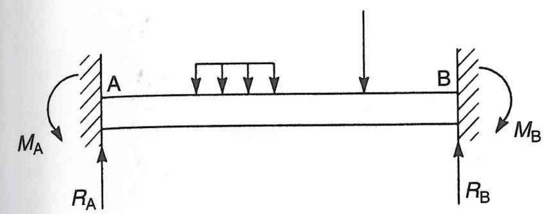


FIGURE 13.28

Support reactions in a fixed beam.

Built-in or fixed-end beams

In practice single-span beams may not be free to rotate about their supports but are connected to them in a manner that prevents rotation. Thus a reinforced concrete beam may be cast integrally with its supports as shown in Fig. 13.27(a) or a steel beam may be bolted at its ends to steel columns (Fig. 13.27 (b)). Clearly neither of the beams of Fig. 13.27(a) or (b) can be regarded as simply supported.

Consider the fixed beam of Fig. 13.28. Any system of vertical loads induces reactions of force and moment, the latter arising from the constraint against rotation provided by the supports. There are then four unknown reactions and only two possible equations of statical equilibrium; the beam is therefore statically indeterminate and has two redundancies. A solution is obtained by considering known values of slope and deflection at particular beam sections.

EXAMPLE 13.21

Figure 13.29(a) shows a fixed beam carrying a central concentrated load, *W*. Determine the value of the fixed-end moments, M_A and M_B.

Since the ends A and B of the beam are prevented from rotating, moments M_A and M_B are induced in the supports; these are termed fixed-end moments. From symmetry we see that $M_A = M_B$ and $R_A = R_B = W/2$.

The beam AB in Fig. 13.29(a) may be regarded as a simply supported beam carrying a central concentrated load with moments M_A and M_B applied at the supports. The bending moment diagrams corresponding to these two loading cases are shown in Fig. 13.29(b) and (c) and are known as the *free bending moment diagram* and the *fixed-end moment diagram*, respectively. Clearly the concentrated load produces sagging (positive) bending moments, while the fixed-end moments induce hogging (negative) bending moments. The resultant or final bending moment diagram is constructed by superimposing the free and fixed-end moment diagrams as shown in Fig. 13.29(d).

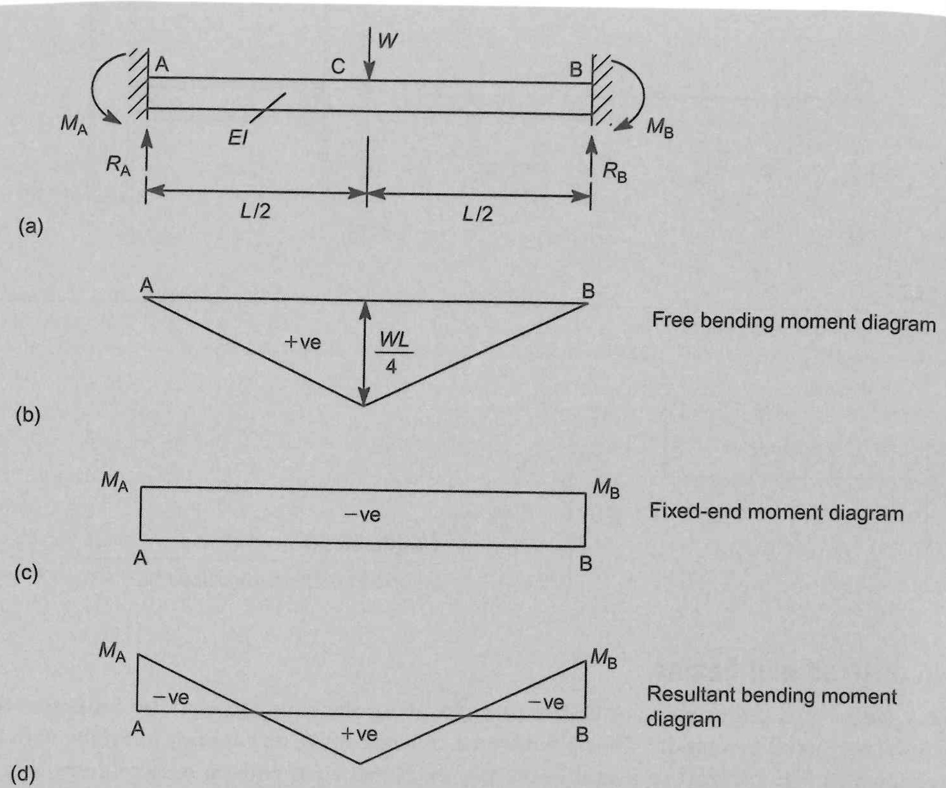


FIGURE 13.29

Bending moment diagram for a fixed beam (Ex. 13.21).

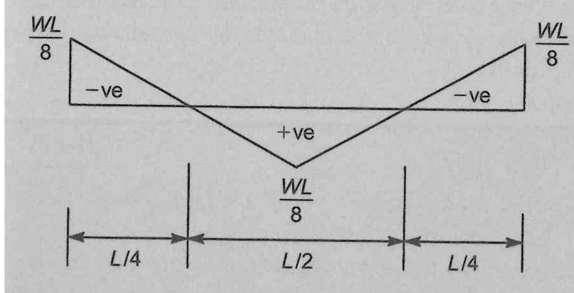


FIGURE 13.30

Complete bending moment diagram for fixed beam of Ex. 13.21.

The moment-area method is now used to determine the fixed-end moments, M_A and M_B. From Eq. (13.7) the change in slope between any two sections of a beam is equal to the area of the M/EI diagram between those sections. Therefore, the net area of the bending moment diagram of Fig. 13.29(d) must be zero since the change of slope between the ends of the beam is zero. It follows that the area of the free bending moment diagram is numerically equal to the area of the fixed-end moment diagram; thus

$$M_A L = \frac{1}{2}\frac{WL}{4}L$$

which gives

$$M_A = M_B = \frac{WL}{8}$$

and the resultant bending moment diagram has principal values as shown in Fig. 13.30. Note that the maximum positive bending moment is equal in magnitude to the maximum negative bending moment and that points of contraflexure (i.e. where the bending moment changes sign) occur at the quarter-span points.

Having determined the support reactions, the deflected shape of the beam may be found by any of the methods described in the previous part of this chapter.

EXAMPLE 13.22

Determine the fixed-end moments and the fixed-end reactions for the beam shown in Fig. 13.31(a).

The resultant bending moment diagram is shown in Fig. 13.31(b) where the line AB represents the datum from which values of bending moment are measured. Again the net area of the resultant bending moment diagram is zero since the change in slope between the ends of the beam is zero. Hence

$$\frac{1}{2}(M_A + M_B)L = \frac{1}{2}L\frac{Wab}{L}$$

which gives

$$M_A + M_B = \frac{Wab}{L} \qquad \text{(i)}$$

We require a further equation to solve for M_A and M_B. This we obtain using Eq. 13.10 and taking the origin for x at A; hence we have

$$x_B\left(\frac{dv}{dx}\right)_B - x_A\left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI}x\,dx \qquad \text{(ii)}$$

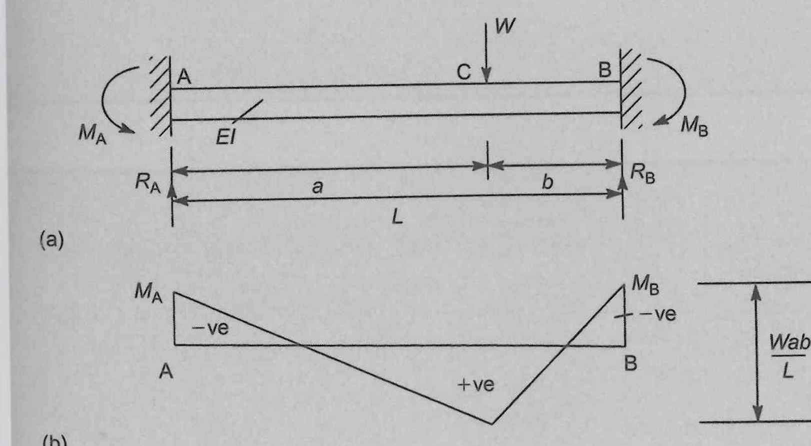


FIGURE 13.31

Fixed beam of Ex. 13.22.

In Eq. (ii) $(dv/dx)_B = (dv/dx)_A = 0$ and $v_B = v_A = 0$ so that

$$0 = \int_A^B \frac{M}{EI} x \, dx \qquad \text{(iii)}$$

nd the moment of the area of the M/EI diagram between A and B about A is zero. Since EI is constant or the beam, we need only consider the bending moment diagram. Therefore from Fig. 13.31(b)

$$M_A L \frac{L}{2} + (M_B - M_A) \frac{L}{2} \frac{2}{3} L = \frac{1}{2} a \frac{Wab}{L} \frac{2a}{3} + \frac{1}{2} b \frac{Wab}{L} \left(a + \frac{1}{3} b \right)$$

Simplifying, we obtain

$$M_A + 2M_B = \frac{Wab}{L^2}(2a + b) \qquad \text{(iv)}$$

Solving Eqs (i) and (iv) simultaneously we obtain

$$M_A = \frac{Wab^2}{L^2} \quad M_B = \frac{Wa^2 b}{L^2} \qquad \text{(v)}$$

We can now use statics to obtain R_A and R_B; hence, taking moments about B

$$R_A L - M_A + M_B - Wb = 0$$

Substituting for M_A and M_B from Eq. (v) we have

$$R_A L = \frac{Wab^2}{L^2} - \frac{Wa^2 b}{L^2} + Wb$$

hence

$$R_A = \frac{Wb^2}{L^3}(3a + b)$$

Similarly

$$R_B = \frac{Wa^2}{L^3}(a + 3b)$$

(AMPLE 13.23

ne fixed beam shown in Fig. 13.32(a) carries a uniformly distributed load of intensity w. etermine the support reactions.

From symmetry, $M_A = M_B$ and $R_A = R_B$. Again the net area of the bending moment diagram ust be zero since the change of slope between the ends of the beam is zero (Eq. (13.7)). Hence

$$M_A L = \frac{2}{3} \frac{wL^2}{8} L$$

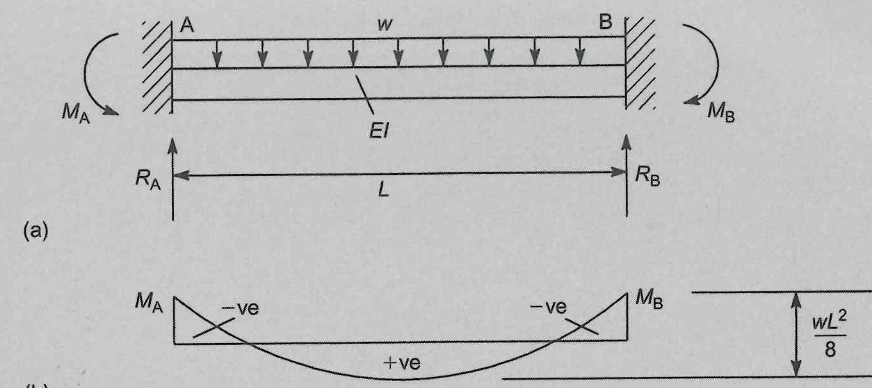


FIGURE 13.32

Fixed beam carrying a uniformly distributed load (Ex. 13.23).

so that

$$M_A = M_B = \frac{wL^2}{12}$$

From statics

$$R_A = R_B = \frac{wL}{2}$$

EXAMPLE 13.24

The fixed beam of Fig. 13.33 carries a uniformly distributed load over part of its span. Determine the values of the fixed-end moments.

Consider a small element δx of the distributed load. We can use the results of Ex. 13.22 to write down the fixed-end moments produced by this elemental load since it may be regarded, in the limit as $\delta x \to 0$, as a concentrated load. Therefore from Eq. (v) of Ex. 13.22 we have

$$\delta M_A = w \, \delta x \frac{x(L - x)^2}{L^2}$$

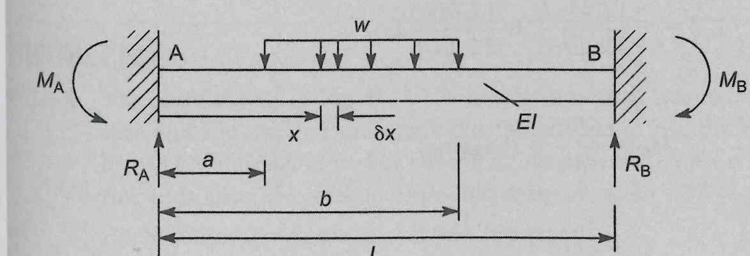


FIGURE 13.33

Fixed beam with part span uniformly distributed load (Ex. 13.24).

The total moment at A, M_A, due to all such elemental loads is then

$$M_A = \int_a^b \frac{w}{L^2} x(L-x)^2 \, dx$$

which gives

$$M_A = \frac{w}{L^2}\left[\frac{L^2}{2}(b^2 - a^2) - \frac{2}{3}L(b^3 - a^3) + \frac{1}{4}(b^4 - a^4)\right] \quad \text{(i)}$$

Similarly

$$M_B = \frac{wb^3}{L^2}\left(\frac{L}{3} - \frac{b}{4}\right) \quad \text{(ii)}$$

If the load covers the complete span, $a = 0$, $b = L$ and Eqs (i) and (ii) reduce to

$$M_A = M_B = \frac{wL^2}{12} \quad \text{(as in Ex. 13.21.)}$$

Fixed beam with a sinking support

In most practical situations the ends of a fixed beam would not remain perfectly aligned indefinitely. Since the ends of such a beam are prevented from rotating, a deflection of one end of the beam relative to the other induces fixed-end moments as shown in Fig. 13.34(a). These are in the same sense and for the relative displacement shown produce a total anticlockwise moment equal to $M_A + M_B$ on the beam. This moment is equilibrated by a clockwise couple formed by the force reactions at the supports. The resultant bending moment diagram is shown in Fig. 13.34(b) and, as in previous examples, its net area is zero since there is no change of slope between the ends of the beam and EI is constant (see Eq. (13.7)). This condition is satisfied by $M_A = M_B$.

Let us now assume an origin for x at A; Eq. (13.10) becomes

$$x_B\left(\frac{dv}{dx}\right)_B - x_A\left(\frac{dv}{dx}\right)_A - (v_B - v_A) = \int_A^B \frac{M}{EI} x \, dx \quad \text{(i)}$$

in which $(dv/dx)_A = (dv/dx)_B = 0$, $v_A = 0$ and $v_B = -\delta$. Hence Eq. (i) reduces to

$$\delta = \int_0^L \frac{M}{EI} x \, dx$$

Using the semi-graphical approach and taking moments of areas about A we have

$$\delta = -\frac{1}{2}\frac{L}{2}\frac{M_A}{EI}\frac{L}{6} + \frac{1}{2}\frac{L}{2}\frac{M_A}{EI}\frac{5}{6}L$$

which gives

$$M_A = \frac{6EI\delta}{L^2} \quad \text{(hogging)}$$

It follows that

$$M_B = \frac{6EI\delta}{L^2} \quad \text{(sagging)}$$

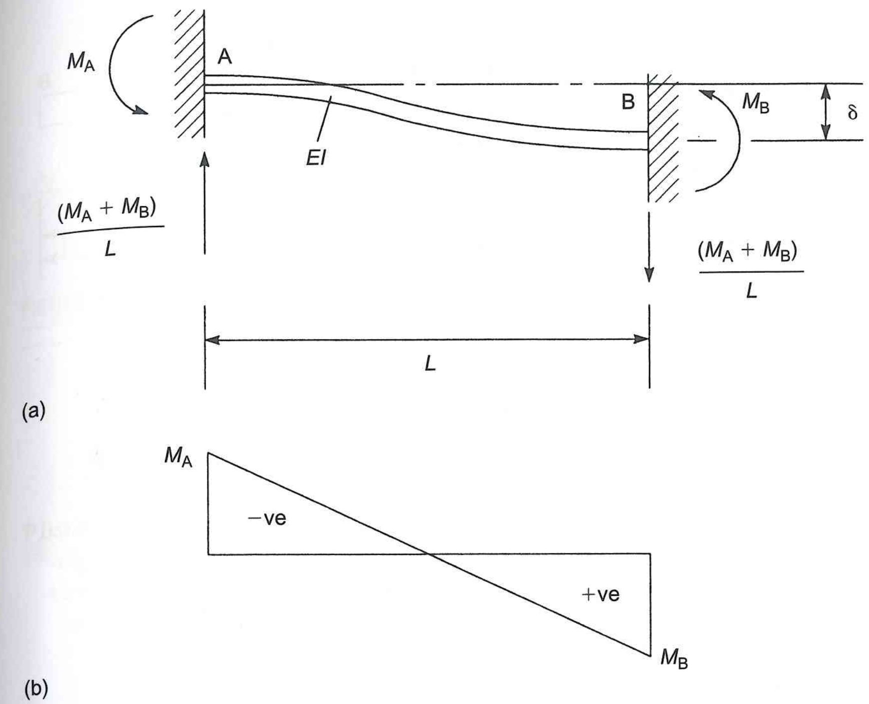


FIGURE 13.34 Fixed beam with a sinking support.

The effect of building in the ends of a beam is to increase both its strength and its stiffness. For example, the maximum bending moment in a simply supported beam carrying a central concentrated load W is $WL/4$ but it is $WL/8$ if the ends are built-in. A comparison of the maximum deflections shows a respective reduction from $WL^3/48EI$ to $WL^3/192EI$. It would therefore appear desirable for all beams to have their ends built-in if possible. However, in practice this is rarely done since, as we have seen, settlement of one of the supports induces additional bending moments in a beam. It is also clear that such moments can be induced during erection unless the supports are perfectly aligned. Furthermore, temperature changes can induce large stresses while live loads, which produce vibrations and fluctuating bending moments, can have adverse effects on the fixity of the supports.

One method of eliminating these difficulties is to employ a double cantilever construction. We have seen that points of contraflexure (i.e. zero bending moment) occur at sections along a fixed beam. Thus if hinges were positioned at these points the bending moment diagram and deflection curve would be unchanged but settlement of a support or temperature changes would have little or no effect on the beam.

PROBLEMS

P.13.1 The beam shown in Fig. P.13.1 is simply supported symmetrically at two points 2 m from each end and carries a uniformly distributed load of 5 kN/m together with two concentrated loads of 2 kN each at its free ends. Calculate the deflection at the mid-span point and at its free ends using the method of double integration. $EI = 43 \times 10^{12}$ Nmm^2.

Ans. 3.6 mm (downwards), 2.0 mm (upwards).

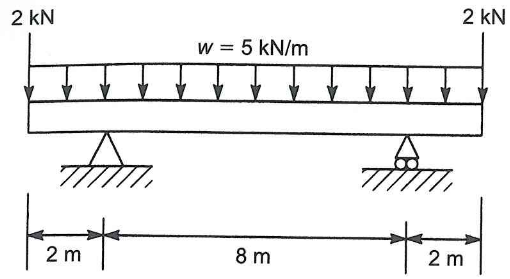


FIGURE P.13.1

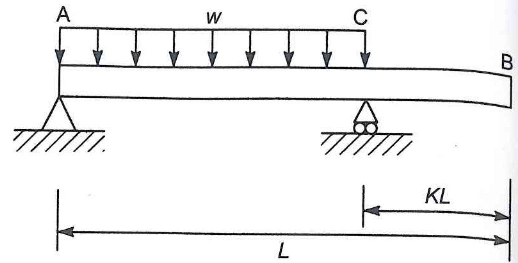


FIGURE P.13.2

on AC, find the value of K which will cause the upward deflection of B to equal the downward deflection midway between A and C.

Ans. 0.24.

P.13.3 A uniform beam is simply supported over a span of 6 m. It carries a trapezoidally distributed load with intensity varying from 30 kN/m at the left-hand support to 90 kN/m at the right-hand support. Find the equation of the deflection curve and hence the deflection at the mid-span point. The second moment of area of the cross section of the beam is 120×10^6 mm^4 and Young's modulus $E = 206\ 000$ N/mm^2.

Ans. 41 mm (downwards).

P.13.4 A cantilever of length L and having a flexural rigidity EI carries a distributed load that varies in intensity from w per unit length at the built-in end to zero at the free end. Find the deflection of the free end.

Ans. $wL^4/30EI$ (downwards).

P.13.5 Determine the position and magnitude of the maximum deflection of the simply supported beam shown in Fig. P.13.5 in terms of its flexural rigidity EI.

Ans. 38.8/EI m downwards at 2.9 m from left-hand support.

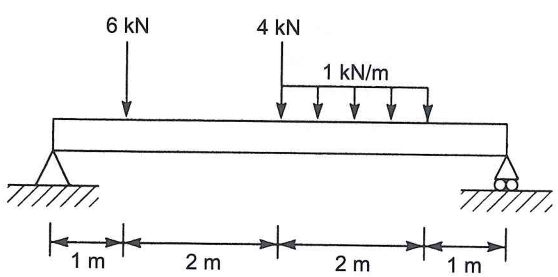


FIGURE P.13.5

P.13.6 Calculate the position and magnitude (in terms of EI) of the maximum deflection in the beam shown in Fig. P.13.6.

Ans. 1309.2/EI m downwards at 13.3 m from left-hand support.

P.13.7 Determine the equation of the deflection curve of the beam shown in Fig. P.13.7. The flexural rigidity of the beam is EI.

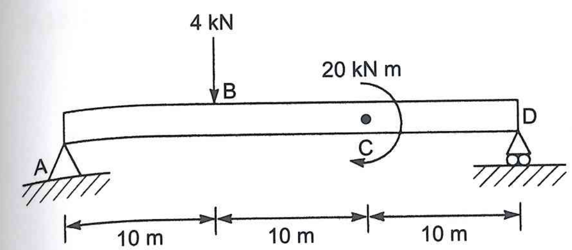


FIGURE P.13.6

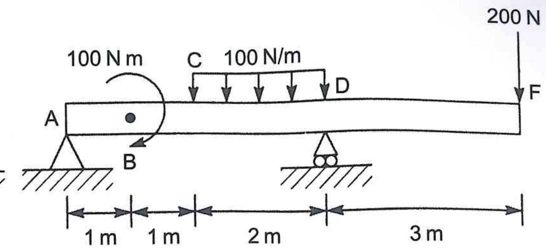


FIGURE P.13.7

Ans.

$$v = -\frac{1}{EI}\left\{\frac{125}{6}x^3 - 50[x-1]^2 + \frac{50}{12}[x-2]^4 - \frac{50}{12}[x-4]^4 - \frac{525}{6}[x-4]^3 + 237.5x\right\}.$$

P.13.8 The beam shown in Fig. P.13.8 has its central portion reinforced so that its flexural rigidity is twice that of the outer portions. Use the moment-area method to determine the central deflection.

Ans. $3WL^3/256EI$ (downwards).

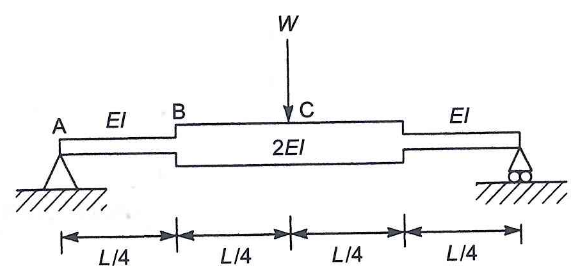


FIGURE P.13.8

P.13.9 A simply supported beam of flexural rigidity EI carries a triangularly distributed load as shown in Fig. P.13.9. Determine the deflection of the mid-point of the beam.

Ans. $w_0L^4/120EI$ (downwards).

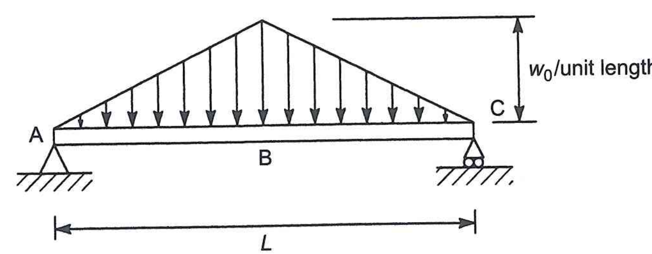


FIGURE P.13.9

P.13.10 The simply supported beam shown in Fig. P.13.10 has its outer regions reinforced so that their flexural rigidity may be regarded as infinite compared with the central region. Determine the central deflection.