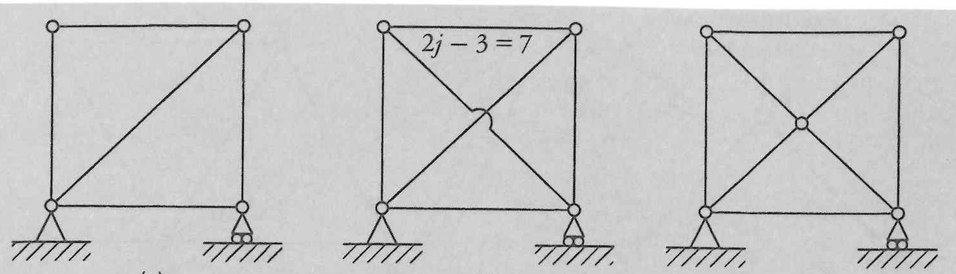


FIGURE 4.7

Statical determinacy of trusses.

so that

$$m > 2j - 3$$

and the truss is statically indeterminate. In fact any single member may be removed and the truss would retain its stability under any loading system in its own plane.

Unfortunately, in some cases, Eq. (4.1) is satisfied but the truss may be statically indeterminate or a mechanism. For example, the truss in Fig. 4.8 has nine members and six joints so that Eq. (4.1) is satisfied. However, clearly the left-hand half is a mechanism and the right-hand half is statically indeterminate. Theoretically, assuming that the truss members are weightless, the truss could support vertical loads applied to the left- and/or right-hand vertical members; this would, of course, be an unstable condition. Any other form of loading would cause a collapse of the left hand half of the truss and consequently of the truss itself.

The presence of a rectangular region in a truss such as that in the truss in Fig. 4.8 does not necessarily result in collapse. The truss in Fig. 4.9 has nine members and six joints so that Eq. (4.1) is satisfied. This does not, as we have seen, guarantee either a stable or statically determinate truss. If, therefore, there is some doubt we can return to the procedure of building up a truss from a single triangular unit as demonstrated in Fig. 4.6. Then, remembering that each additional triangle is created by adding two members and one joint and that the resulting truss is stable and statically determinate, we can examine the truss in Fig. 4.9 as follows.

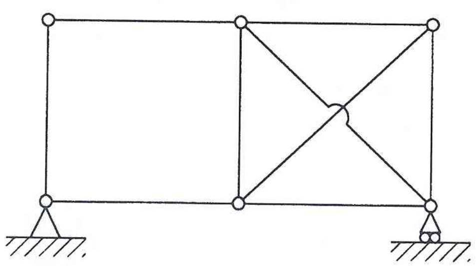


FIGURE 4.8

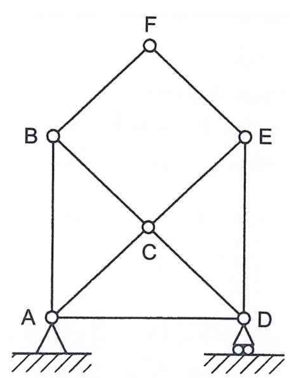


FIGURE 4.9

Suppose that ACD is the initial triangle. The additional triangle ACB is formed by adding the two members AB and BC and the single joint B. The triangle DCE follows by adding the two members CE and DE and the joint E. Finally, the two members BF and EF and the joint F are added to form the rectangular portion CBFE. We therefore conclude that the truss in Fig. 4.9 is stable and statically determinate. Compare the construction of this truss with that of the statically indeterminate truss in Fig. 4.7(c).

A condition, similar to Eq. (4.1), applies to space trusses; the result for a space truss having m members and j pinned joints is

$$m = 3j - 6 \qquad (4.2)$$

EXAMPLE 4.2

Investigate the determinacy and stability of the trusses shown in Fig. 4.10 under loads applied in their plane.

(a) In this case there are 7 members and 5 joints so that $m = 7$ and $j = 5$. Then

$$2j - 3 = 7$$

and Eq. (4.1) is satisfied and the truss is statically determinate. Also, by inspection, the truss is stable.

(b) For this truss $m = 9$ and $j = 6$ so that $2j - 3 = 9$ and Eq. (4.1) is satisfied but, by inspection, the outer half of the truss is statically indeterminate while the inner half is a mechanism.

(c) In this case $m = 13$ and $j = 9$ so that $2j - 3 = 15$ and $m < 2j - 3$, the truss is therefore a mechanism.

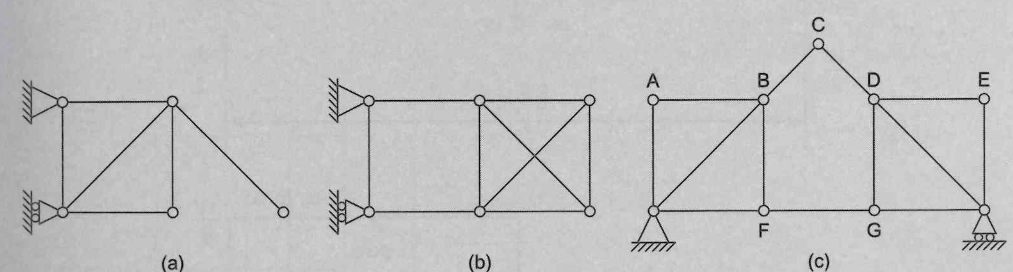


FIGURE 4.10

EXAMPLE 4.3

Suggest two ways in which the truss in Fig. 4.10(c) could be made stable and remain statically determinate.

i. Add members BD and FD (or BG).

ii. Add members CF and CG.

4.5 Resistance of a truss to shear force and bending moment

Although the members of a truss carry only axial loads, the truss itself acts as a beam and is subjected to shear forces and bending moments. Therefore, before we consider methods of analysis of trusses, it will be instructive to examine the manner in which a truss resists shear forces and bending moments.

The Pratt truss shown in Fig. 4.11(a) carries a concentrated load W applied at a joint on the bottom chord at mid-span. Using the methods described in Section 3.4, the shear force and bending moment diagrams for the truss are constructed as shown in Fig. 4.11(b) and (c), respectively.

First we shall consider the shear force. In the bay ABCD the shear force is $W/2$ and is negative. Thus at any section mm between A and B (Fig. 4.12) we see that the internal shear force is $-W/2$. Since the horizontal members AB and DC are unable to resist shear forces, the internal shear force can only be equilibrated by the vertical component of the force F_{AC} in the member AC. Figure 4.12 shows the direction of the internal shear force applied at the section mm so that F_{AC} is tensile. Then

$$F_{AC} \cos 45^\circ = \frac{W}{2}$$

The same result applies to all the internal diagonals whether to the right or left of the mid-span point since the shear force is constant, although reversed in sign, either side of the load. The two outer diagonals

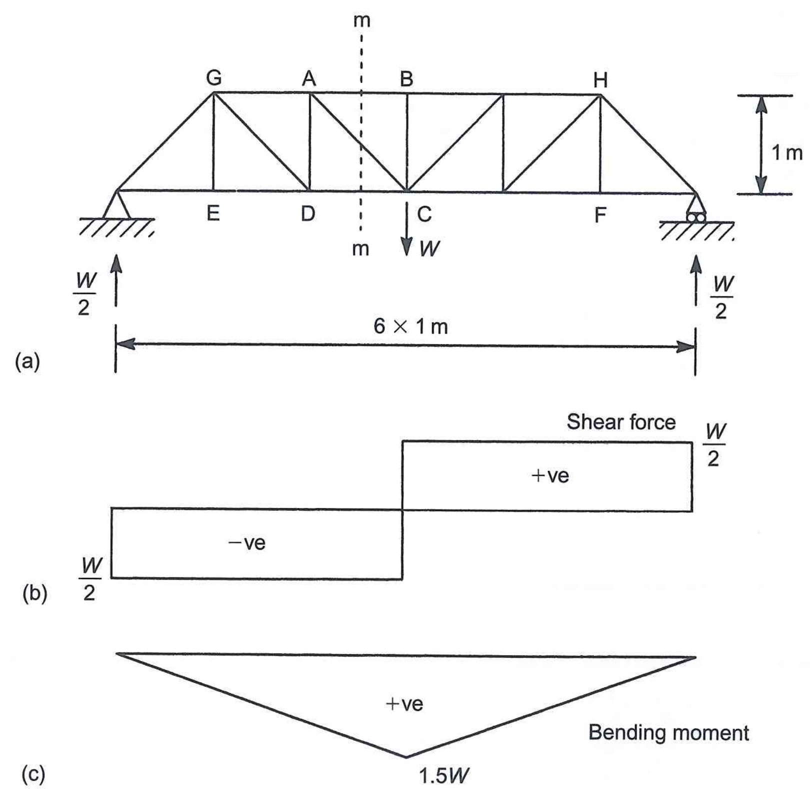


FIGURE 4.11

Shear forces and bending moments in a truss.

are in compression since their vertical components must be in equilibrium with the vertically upward support reactions. Alternatively, we arrive at the same result by considering the internal shear force at a section just to the right of the left-hand support and just to the left of the right-hand support.

If the diagonal AC was repositioned to span between D and B it would be subjected to an axial compressive load. This situation would be undesirable since the longer a compression member, the smaller the load required to cause buckling (see Chapter 21). Therefore, the aim of truss design is to ensure that the forces in the longest members, the diagonals in this case, are predominantly tensile. So we can see now why the Howe truss (Fig. 4.1(b)), whose diagonals for downward loads would be in compression, is no longer in use.

In some situations the loading on a truss could be reversed so that a diagonal that is usually in tension would be in compression. To counter this an extra diagonal inclined in the opposite direction is included (spanning, say, from D to B in Fig. 4.13). This, as we have seen, would result in the truss becoming statically indeterminate. However, if it is assumed that the original diagonal (AC in Fig. 4.13) has buckled under the compressive load and therefore carries no load, the truss is once again statically determinate.

We shall now consider the manner in which a truss resists bending moments. The bending moment at a section immediately to the left of the mid-span vertical BC in the truss in Fig. 4.11(a) is, from Fig. 4.11(c), 1.5 W and is positive, as shown in Fig. 4.13. This bending moment is equivalent to the moment resultant, about any point in their plane, of the member forces at this section. In Fig. 4.13, analysis by the method of sections (Section 4.7) gives $F_{BA} = 1.5\ W$ (compression), $F_{AC} = 0.707\ W$ (tension) and $F_{DC} = 1.0\ W$ (tension). Therefore at C, F_{DC} plus the horizontal component of F_{AC} is equal to 1.5 W which, together with F_{BA}, produces a couple of magnitude $1.5\ W \times 1$ which is equal to the applied bending moment. Alternatively, we could take moments of the internal forces about B (or C). Hence

$$M_B = F_{DC} \times 1 + F_{AC} \times 1 \sin 45^\circ = 1.0\ W \times 1 + 0.707\ W \times 1 \sin 45^\circ = 1.5\ W$$

as before. Note that in Fig. 4.13 the moment resultant of the internal force system is *equivalent* to the applied moment, i.e. it is in the same sense as the applied moment.

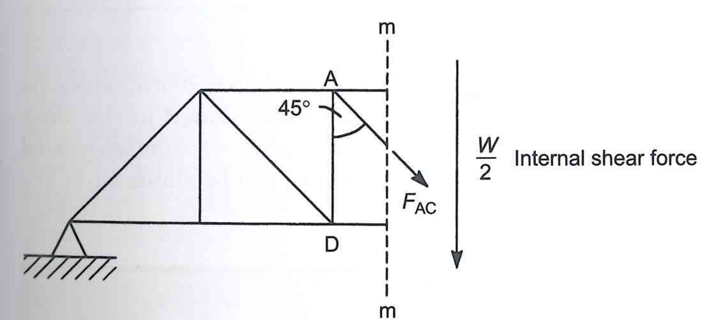


FIGURE 4.12

Internal shear force in a truss.

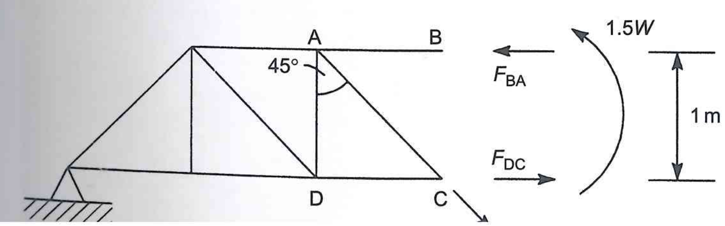


FIGURE 4.13

Internal bending moment in a

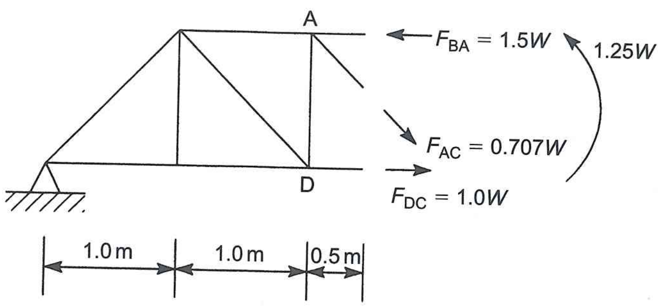


FIGURE 4.14

Resistance of a bending moment at a mid-bay point.

Now let us consider the bending moment at, say, the mid-point of the bay AB, where its magnitude is, from Fig. 4.11(c), 1.25 W. The internal force system is shown in Fig. 4.14 in which F_{BA}, F_{AC} and F_{DC} have the same values as before. Then, taking moments about, say, the mid-point of the top chord member AB, we have

$$M = F_{DC} \times 1 + F_{AC} \times 0.5 \sin 45^\circ = 1.0\ W \times 1 + 0.707\ W \times 0.5 \sin 45^\circ = 1.25\ W$$

the value of the applied moment.

From the discussion above it is clear that, in trusses, shear loads are resisted by inclined members, while all members combine to resist bending moments. Furthermore, positive (sagging) bending moments induce compression in upper chord members and tension in lower chord members.

Finally, note that in the truss in Fig. 4.11 the forces in the members GE, BC and HF are all zero, as can be seen by considering the vertical equilibrium of joints E, B and F. Forces would only be induced in these members if external loads were applied directly at the joints E, B and F. Generally, if three coplanar members meet at a joint and two of them are collinear, the force in the third member is zero if no external force is applied at the joint.

4.6 Method of joints

We have seen in Section 4.4 that the axial forces in the members of a simple pin-jointed triangular structure may be found by examining the equilibrium of their connecting pins or hinges in two directions at right angles (Eq. (2.10)). This approach may be extended to plane trusses to determine the axial forces in all their members; the method is known as the *method of joints* and will be illustrated by the following example.

EXAMPLE 4.4

Determine the forces in the members of the Warren truss shown in Fig. 4.15; all members are 1 m long.

Generally, although not always, the support reactions must be calculated first. So, taking moments about D for the truss in Fig. 4.15 we obtain

$$R_A \times 2 - 2 \times 1.5 - 1 \times 1 - 3 \times 0.5 = 0$$

which gives

$$R_A = 2.75 \text{ kN}$$

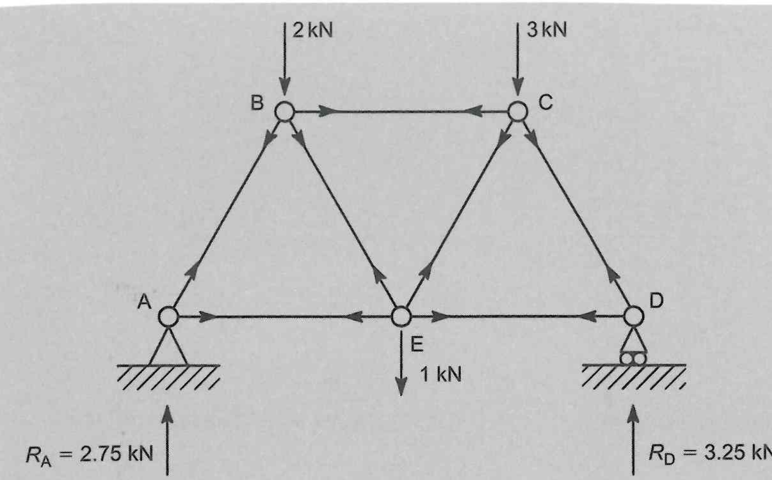


FIGURE 4.15

Analysis of a Warren truss.

Then, resolving vertically

$$R_D + R_A - 2 - 1 - 3 = 0$$

so that

$$R_D = 3.25 \text{ kN}$$

Note that there will be no horizontal reaction at A (D is a roller support) since no horizontal loads are applied.

The next step is to assign directions to the forces *acting on each joint*. In one approach the truss is examined to determine whether the force in a member is tensile or compressive. For some members this is straightforward. For example, in Fig. 4.15, the vertical reaction at A, R_A, can only be equilibrated by the vertical component of the force in AB which must therefore act downwards, indicating that the member is in compression (a compressive force in a member will push towards a joint whereas a tensile force will pull away from a joint). In some cases, where several members meet at a joint, the nature of the force in a particular member is difficult, if not impossible, to determine by inspection. Then a direction must be assumed which, if incorrect, will result in a negative value for the member force. It follows that, in the same truss, both positive and negative values may be obtained for tensile forces and also for compressive forces, a situation leading to possible confusion. Therefore, if every member in a truss is initially assumed to be in tension, negative values will always indicate compression and the solution will then agree with the sign convention adopted in Section 3.2.

We now assign tensile forces to the members of the truss in Fig. 4.15 using arrows to indicate the *action of the force in the member on the joint*; then all arrows are shown to pull away from the adjacent joint.

The analysis, as we have seen, is based on a consideration of the equilibrium of each pin or hinge under the action of *all* the forces at the joint. Thus for each pin or hinge we can write down two equations of equilibrium. It follows that a solution can only be obtained if there are no more than two unknown forces acting at the joint. In Fig. 4.15, therefore, we can only begin the analysis at the joints A or D since at each of the joints B and C there are three unknown forces while at E there are four

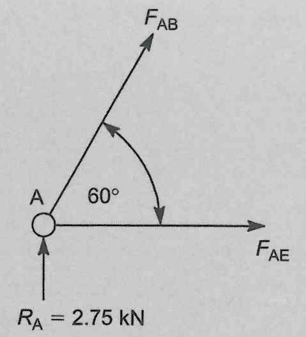


FIGURE 4.16

Equilibrium of forces at joint A.

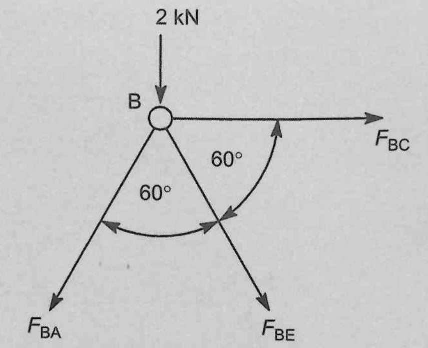


FIGURE 4.17

Equilibrium of forces at joint B.

Consider joint A. The forces acting on the pin at A are shown in the free body diagram in Fig. 4.16. F_{AB} may be determined directly by resolving forces vertically.

Hence

$$F_{AB} \sin 60^\circ + 2.75 = 0 \qquad \text{(i)}$$

so that

$$F_{AB} = -3.18 \text{ kN}$$

the negative sign indicating that AB is in compression as expected.

Referring again to Fig. 4.16 and resolving forces horizontally

$$F_{AE} + F_{AB} \cos 60^\circ = 0 \qquad \text{(ii)}$$

Substituting the *negative* value of F_{AB} in Eq. (ii) we obtain

$$F_{AE} - 3.18 \cos 60^\circ = 0$$

which gives

$$F_{AE} = +1.59 \text{ kN}$$

the positive sign indicating that F_{AB} is a tensile force.

We now inspect the truss to determine the next joint at which there are no more than two unknown forces. At joint E there remain three unknowns since only F_{EA} $(=F_{AE})$ has yet been determined. At joint B there are now two unknowns since F_{BA} $(=F_{AB})$ has been determined; we can therefore proceed to joint B. The forces acting at B are shown in Fig. 4.17. Since F_{BA} is now known we can resolve forces vertically and therefore obtain F_{BE} directly. Thus

$$F_{BE} \cos 30^\circ + F_{BA} \cos 30^\circ + 2 = 0 \qquad \text{(iii)}$$

Substituting the negative value of F_{BA} in Eq. (iii) gives

$$F_{BE} = +0.87 \text{ kN}$$

which is positive and therefore tensile.

Resolving forces horizontally at the joint B we have

$$F_{BC} + F_{BE} \cos 60^\circ - F_{BA} \cos 60^\circ = 0 \qquad \text{(iv)}$$

Substituting the positive value of F_{BE} and the negative value of F_{BA} in Eq. (iv) gives

$$F_{BC} = -2.03 \text{ kN}$$

the negative sign indicating that the member BC is in compression.

We have now calculated four of the seven unknown member forces. There are in fact just two unknown forces at each of the remaining joints C, D and E so that, theoretically, it is immaterial which joint we consider next. From a solution viewpoint there are three forces at D, four at C and five at E so that the arithmetic will be slightly simpler if we next consider D to obtain F_{DC} and F_{DE} and then C to obtain F_{CE}. At C, F_{CE} could be determined by resolving forces in the direction CE rather than horizontally or vertically. Carrying out this procedure gives

$$F_{DC} = -3.75 \text{ kN (compression)}$$

$$F_{DE} = +1.88 \text{ kN (tension)}$$

$$F_{CE} = +0.29 \text{ kN (tension)}$$

The reader should verify these values using the method suggested above.

It may be noted that in this example we could write down 10 equations of equilibrium, two for each of the five joints, and yet there are only seven unknown member forces. The apparently extra three equations result from the use of overall equilibrium to calculate the support reactions. An alternative approach would therefore be to write down the 10 equilibrium equations which would include the three unknown support reactions (there would be a horizontal reaction at A if horizontal as well as vertical loads were applied) and solve the resulting 10 equations simultaneously. Overall equilibrium could then be examined to check the accuracy of the solution. Generally, however, the method adopted above produces a quicker solution.

4.7 Method of sections

It will be appreciated from Section 4.5 that in many trusses the maximum member forces, particularly in horizontal members, will occur in the central region where the applied bending moment would possibly have its maximum value. It will also be appreciated from Ex. 4.4 that the calculation of member forces in the central region of a multibay truss such as the Pratt truss shown in Fig. 4.1(a) would be extremely tedious since the calculation must begin at an outside support and then proceed inwards joint by joint. This approach may be circumvented by using the *method of sections.*

The method is based on the premise that if a structure is in equilibrium, any portion or component of the structure will also be in equilibrium under the action of any external forces and the internal forces acting between the portion or component and the remainder of the structure. We shall illustrate

EXAMPLE 4.5

Calculate the forces in the members CD, CF and EF in the Pratt truss shown in Fig. 4.18.

Initially the support reactions are calculated and are readily shown to be

$$R_{A,V} = 4.5 \text{ kN} \quad R_{A,H} = 2 \text{ kN} \quad R_B = 5.5 \text{ kN}$$

We now 'cut' the members CD, CF and EF by a section mm, thereby dividing the truss into two separate parts. Consider the left-hand part shown in Fig. 4.19 (equally we could consider the right-hand part). Clearly, if we actually cut the members CD, CF and EF, both the left- and right-hand parts would collapse. However, the equilibrium of the left-hand part, say, could be maintained by applying the forces F_{CD}, F_{CF} and F_{EF} to the cut ends of the members. Therefore, in Fig. 4.19, the left-hand part of the truss is in equilibrium under the action of the externally applied loads, the support reactions and the forces F_{CD}, F_{CF} and F_{EF} which are, as in the method of joints, initially assumed to be tensile; Eq. (2.10) are then used to calculate the three unknown forces.

Resolving vertically gives

$$F_{CF} \cos 45^\circ + 4 - 4.5 = 0 \qquad \text{(i)}$$

so that

$$F_{CF} = +0.71 \text{ kN}$$

and is tensile.

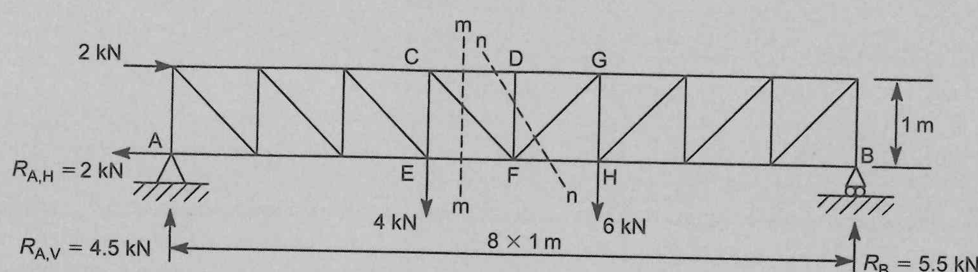


FIGURE 4.18

Calculation of member forces using the method of sections.

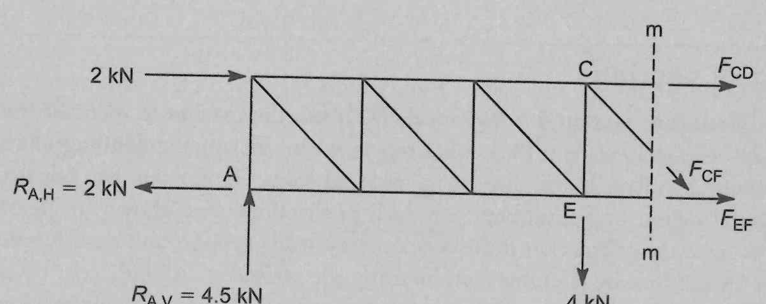


FIGURE 4.19

Equilibrium of a portion of a truss.

Now taking moments about the point of intersection of F_{CF} and F_{EF} we have

$$F_{CD} \times 1 + 2 \times 1 + 4.5 \times 4 - 4 \times 1 = 0 \qquad \text{(ii)}$$

so that

$$F_{CD} = -16 \text{ kN}$$

and is compressive.

Finally F_{EF} is obtained by taking moments about C, thereby eliminating F_{CF} and F_{CD} from the equation. Alternatively, we could resolve forces horizontally since F_{CF} and F_{CD} are now known; however, this approach would involve a slightly lengthier calculation. Hence

$$F_{EF} \times 1 - 4.5 \times 3 - 2 \times 1 = 0 \qquad \text{(iii)}$$

which gives

$$F_{EF} = +15.5 \text{ kN}$$

the positive sign indicating tension.

Note that Eqs (i)–(iii) each include just one of the unknown member forces so that it is immaterial which is calculated first. In some problems, however, a preliminary examination is worthwhile to determine the optimum order of solution.

In Ex. 4.5 we see that there are just three possible equations of equilibrium so that we cannot solve for more than three unknown forces. It follows that a section such as mm which *must divide the frame into two separate parts* must also *not cut through more than three members in which the forces are unknown.* For example, if we wished to determine the forces in CD, DF, FG and FH we would first calculate F_{CD} using the section mm as above and then determine F_{DF}, F_{FG} and F_{FH} using the section nn. Actually, in this particular example F_{DF} may be seen to be zero by inspection (see Section 4.5) but the principle holds.

4.8 Method of tension coefficients

An alternative form of the method of joints which is particularly useful in the analysis of space trusses is the *method of tension coefficients.*

Consider the member AB, shown in Fig. 4.20, which connects two pinned joints A and B whose coordinates, referred to arbitrary xy axes, are (x_A, y_A) and (x_B, y_B) respectively; the member carries a *tensile* force, T_{AB}, is of length L_{AB} and is inclined at an angle α to the x axis. The component of T_{AB} parallel to the x axis at A is given by

$$T_{AB} \cos \alpha = T_{AB} \frac{(x_B - x_A)}{L_{AB}} = \frac{T_{AB}}{L_{AB}}(x_B - x_A)$$

Similarly the component of T_{AB} at A parallel to the y axis is

$$T_{AB} \sin \alpha = \frac{T_{AB}}{L_{AB}}(y_B - y_A)$$

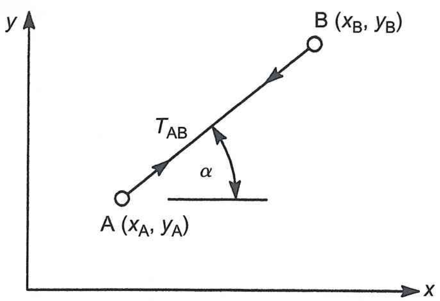


FIGURE 4.20

Method of tension coefficients.

We now define a *tension coefficient* $t_{AB} = T_{AB}/L_{AB}$ so that the above components of T_{AB} become

$$\text{parallel to the } x \text{ axis: } t_{AB}(x_B - x_A) \tag{4.3}$$

$$\text{parallel to the } y \text{ axis: } t_{AB}(y_B - y_A) \tag{4.4}$$

Equilibrium equations may be written down for each joint in turn in terms of tension coefficients and joint coordinates referred to some convenient axis system. The solution of these equations gives t_{AB}, etc, whence $T_{AB} = t_{AB}L_{AB}$ in which L_{AB}, unless given, may be calculated using Pythagoras' theorem, i.e. $L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$. Again the initial assumption of tension in a member results in negative values corresponding to compression. Note the order of suffixes in Eqs (4.3) and (4.4).

EXAMPLE 4.6

Determine the forces in the members of the pin-jointed truss shown in Fig. 4.21.

The support reactions are first calculated and are as shown in Fig. 4.21.

The next step is to choose an xy axis system and then insert the joint coordinates in the diagram. In Fig. 4.21 we shall choose the support point A as the origin of axes although, in fact, any joint would suffice; the joint coordinates are then as shown.

Again, as in the method of joints, the solution can only begin at a joint where there are no more than two unknown member forces, in this case joints A and E. Theoretically it is immaterial at which of these joints the analysis begins but since A is the origin of axes we shall start at A. Note that it is unnecessary to insert arrows to indicate the directions of the member forces since the members are assumed to be in tension and the directions of the components of the member forces are automatically specified when written in terms of tension coefficients and joint coordinates (Eqs (4.3) and (4.4)).

The equations of equilibrium at joint A are

$$x \text{ direction: } t_{AB}(x_B - x_A) + t_{AC}(x_C - x_A) - R_{A,H} = 0 \tag{i}$$

$$y \text{ direction: } t_{AB}(y_B - y_A) + t_{AC}(y_C - y_A) - R_{A,V} = 0 \tag{ii}$$

Substituting the values of $R_{A,H}$, $R_{A,V}$ and the joint coordinates in Eqs (i) and (ii) we obtain, from Eq. (i),

$$t_{AB}(0 - 0) + t_{AC}(1.5 - 0) - 3 = 0$$

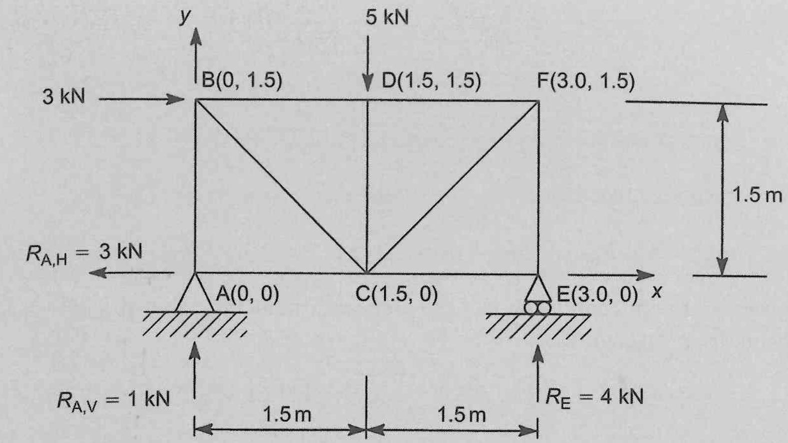


FIGURE 4.21

Analysis of a truss using tension coefficients (Ex. 4.6).

whence

$$t_{AC} = +2.0$$

and from Eq. (ii)

$$t_{AB}(1.5 - 0) + t_{AC}(0 - 0) + 1 = 0$$

so that

$$t_{AB} = -0.67$$

We see from the derivation of Eqs (4.3) and (4.4) that the units of a tension coefficient are force/unit length, in this case kN/m. Generally, however, we shall omit the units.

We can now proceed to joint B at which, since t_{BA} ($=t_{AB}$) has been calculated, there are two unknowns

$$x \text{ direction: } t_{AB}(x_A - x_B) + t_{BC}(x_C - x_B) + t_{BD}(x_D - x_B) + 3 = 0 \tag{iii}$$

$$y \text{ direction: } t_{BA}(y_A - y_B) + t_{BC}(y_C - y_B) + t_{BD}(y_D - y_B) = 0 \tag{iv}$$

Substituting the values of the joint coordinates and t_{BA} in Eqs (iii) and (iv) we have, from Eq. (iii)

$$-0.67\,(0 - 0) + t_{BC}(1.5 - 0) + t_{BD}(1.5 - 0) + 3 = 0$$

which simplifies to

$$1.5t_{BC} + 1.5t_{BD} + 3 = 0 \tag{v}$$

and from Eq. (iv)

$$-0.67\,(0 - 1.5) + t_{BC}(0 - 1.5) + t_{BD}(1.5 - 1.5) = 0$$

whence

$$t_{BC} = +0.67$$

Hence, from Eq. (v)

$$t_{BD} = -2.67$$

There are now just two unknown member forces at joint D. Hence, at D

$$x \text{ direction: } t_{DB}(x_B - x_D) + t_{DF}(x_F - x_D) + t_{DC}(x_C - x_D) = 0 \quad \text{(vi)}$$

$$y \text{ direction: } t_{DB}(y_B - y_D) + t_{DF}(y_F - y_D) + t_{DC}(y_C - y_D) - 5 = 0 \quad \text{(vii)}$$

Substituting values of joint coordinates and the previously calculated value of t_{DB} ($= t_{BD}$) in Eqs (vi) and (vii) we obtain, from Eq. (vi)

$$-2.67\,(0 - 1.5) + t_{DF}(3.0 - 1.5) + t_{DC}(1.5 - 1.5) - 5 = 0$$

so that

$$t_{DF} = -2.67$$

and from Eq. (vii)

$$-2.67(1.5 - 1.5) + t_{DF}(1.5 - 1.5) + t_{DC}(0 - 1.5) = 0$$

from which

$$t_{DC} = -3.33$$

The solution then proceeds to joint C to obtain t_{CF} and t_{CE} or to joint F to determine t_{FC} and t_{FE}; joint F would be preferable since fewer members meet at F than at C. Finally, the remaining unknown tension coefficient (t_{EC} or t_{EF}) is found by considering the equilibrium of joint E. Then

$$t_{FC} = +2.67, \quad t_{FE} = -2.67, \quad t_{EC} = 0$$

which the reader should verify.

The forces in the truss members are now calculated by multiplying the tension coefficients by the member lengths, i.e.

$$T_{AB} = t_{AB}L_{AB} = -0.67 \times 1.5 = -1.0 \text{ kN (compression)}$$
$$T_{AC} = t_{AC}L_{AC} = +2.0 \times 1.5 = +3.0 \text{ kN (tension)}$$
$$T_{BC} = t_{BC}L_{BC}$$

in which

$$L_{BC} = \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} = \sqrt{(0-1.5)^2 + (1.5-0)^2} = 2.12 \text{ m}$$

Then

$$T_{BC} = +0.67 \times 2.12 = +1.42 \text{ kN (tension)}$$

Note that in the calculation of member lengths it is immaterial in which order the joint coordinates occur in the brackets since the brackets are squared. Also

$$T_{BD} = t_{BD}L_{BD} = -2.67 \times 1.5 = -4.0 \text{ kN (compression)}$$

Similarly

$$T_{DF} = -4.0 \text{ kN (compression)}$$
$$T_{DC} = -5.0 \text{ kN (compression)}$$
$$T_{FC} = +5.67 \text{ kN (tension)}$$
$$T_{FE} = -4.0 \text{ kN (compression)}$$
$$T_{EC} = 0$$

4.9 Graphical method of solution

In some instances, particularly when a rapid solution is required, the member forces in a truss may be found using a graphical method.

The method is based upon the condition that each joint in a truss is in equilibrium so that the forces acting at a joint may be represented in magnitude and direction by the sides of a closed polygon (see Section 2.1). The directions of the forces must be drawn in the same directions as the corresponding members and there must be no more than two unknown forces at a particular joint otherwise a polygon of forces cannot be constructed. The method will be illustrated by applying it to the truss in Ex. 4.4.

EXAMPLE 4.7

Determine the forces in the members of the Warren truss shown in Fig. 4.22; all members are 1 m long.

It is convenient in this approach to designate forces in members in terms of the areas between them rather than referring to the joints at their ends. Thus, in Fig. 4.22, we number the areas between all forces, both internal and external; the reason for this will become clear when the force diagram for the complete structure is constructed.

The support reactions were calculated in Ex. 4.4 and are shown in Fig. 4.22. We must start at a joint where there are no more than two unknown forces, in this example either A or D; here we select A. The force polygon for joint A is constructed by going round A in, say, a clockwise sense. We must then go round every joint in the same sense.

First we draw a vector 12 to represent the support reaction at A of 2.75 kN to a convenient scale (see Fig. 4.23). Note that we are moving clockwise from the region 1 to the region 2 so that the vector 12 is vertically upwards, the direction of the reaction at A (if we had decided to move round A in an anticlockwise sense the vector would be drawn as 21 vertically upwards). The force in the member AB at A will be represented by a vector 26 in the direction AB or BA, depending on whether it is tensile or compressive, while the force in the member AE at A is represented by the vector 61 in the direction AE or EA depending, again, on whether it is tensile or compressive. The point 6 in the force polygon is therefore located by drawing a line through the point 2 parallel to the member AB to intersect, at 6, a line drawn through the point 1 parallel to the member AE. We see from the force polygon that the direction of the vector 26 is towards A so that the member AB is in compression while the direction of the vector 61 is away from A indicating that the member AE is in tension. We now insert arrows on the members AB and AE in Fig. 4.22 to indicate compression and tension, respectively.

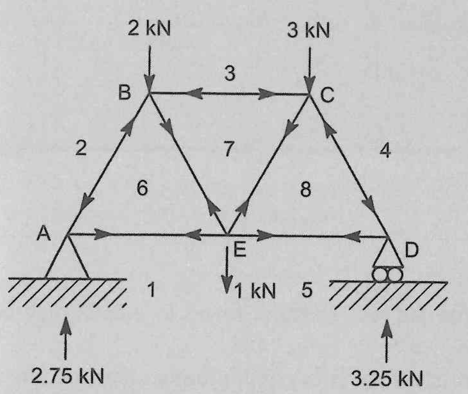


FIGURE 4.22

Analysis of a truss by a graphical method.

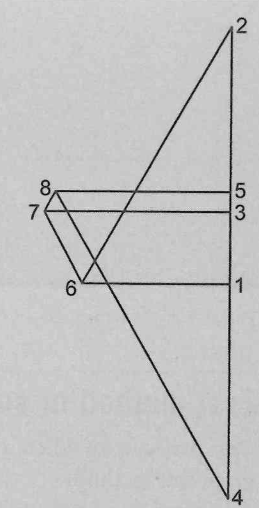


FIGURE 4.23

Force polygon for the truss of Ex. 4.7.

We next consider joint B where there are now just two unknown member forces since we have previously determined the force in the member AB; note that, moving clockwise round B, this force is represented by the vector 62, which means that it is acting towards B as it must since we have already established that AB is in compression. Rather than construct a separate force polygon for the joint B we shall superimpose the force polygon on that constructed for joint A since the vector 26 (or 62) is common to both; we thereby avoid repetition. Thus, through the point 2, we draw a vector 23 vertically downwards to represent the 2 kN load to the same scale as before. The force in the member BC is represented by the vector 37 parallel to BC (or CB) while the force in the member BE is represented by the vector 76 drawn in the direction of BE (or EB); this locates the point 7 in the force polygon. Hence we see that the force in BC (vector 37) acts towards B indicating compression, while the force in BE (vector 76) acts away from B indicating tension; again, arrows are inserted in Fig. 4.22 to show the action of the forces.

Now we consider joint C where the unknown member forces are in CD and CE. The force in the member CB at C is represented in magnitude and direction by the vector 73 in the force polygon. From the point 3 we draw a vector 34 vertically downwards to represent the 3 kN load. The vectors 48 and 87 are then drawn parallel to the members CD and CE and represent the forces in the members CD and CE, respectively. Thus we see that the force in CD (vector 48) acts towards C, i.e. CD is in compression, while the force in CE (vector 87) acts away from C indicating tension; again we insert corresponding arrows on the members in Fig. 4.22.

Finally the vector 45 is drawn vertically upwards to represent the vertical reaction (=3.25 kN) at D and the vector 58, which must be parallel to the member DE, inserted (since the points 5 and 8 are already located in the force polygon this is a useful check on the accuracy of construction). From the direction of the vector 58 we deduce that the member DE is in tension.

Note that in the force polygon the vectors may be read in both directions. Thus the vector 26 represents the force in the member AB acting at A, while the vector 62 represents the force in AB acting at B. It should also be clear why there must be consistency in the sense in which we move round each joint; e.g. the vector 26 represents the direction of the force at A in the member AB when we move in a clockwise sense round A. However, if we then move in an anticlockwise sense round the joint B the vector 26 would represent the magnitude and direction of the force in AB at B and would indicate that AB is in tension, but clearly it is not.

4.10 Compound trusses

In some situations simple trusses are connected together to form a compound truss, in which case it is generally not possible to calculate the forces in all the members by the method of joints even though the truss is statically determinate.

Figure 4.24 shows a compound truss comprising two simple trusses AGC and BJC connected at the apex C and by the linking bar GJ; all the joints are pinned and we shall suppose that the truss carries loads at all its joints. We note that the truss has 27 members and 15 joints so that Eq. (4.1) is satisfied and the truss is statically determinate. This truss is, in fact, a Fink truss (see Fig. 4.1(c)).

Initially we would calculate the support reactions at A and B and commence a method of joints solution at the joint A (or at the joint B) where there are no more than two unknown member forces. Thus the magnitudes of F_{AD} and F_{AE} would be obtained. Then, by considering the equilibrium of joint D, we would calculate F_{DE} and F_{DF} and then F_{EF} and F_{EG} by considering the equilibrium of joint E. At this stage, however, the analysis can proceed no further, since at each of the next joints to be considered, F and G, there are three unknown member forces: F_{FG}, F_{FI} and F_{FH} at F, and F_{GF}, F_{GI} and F_{GJ} at G. An identical situation would have arisen if the analysis had commenced in the right-hand half of the truss at B. This difficulty is overcome by taking a section mm to cut the three members HC, IC and GJ and using the method of sections to calculate the corresponding member forces. Having obtained F_{GJ} we can consider the equilibrium of joint G to calculate F_{GI} and F_{GF}. Hence F_{FI} and F_{FH} follow by considering the equilibrium of joint F; the remaining unknown member forces follow. Note that obtaining F_{GI} by taking the section mm allows all the member forces in the right-hand half of the truss to be found by the method of joints.

The method of sections could be used to solve for all the member forces. First we could obtain F_{HC}, F_{IC} and F_{GJ} by taking the section mm and then F_{FH}, F_{FI} and F_{GI} by taking the section nn where F_{GJ} is known, and so on.

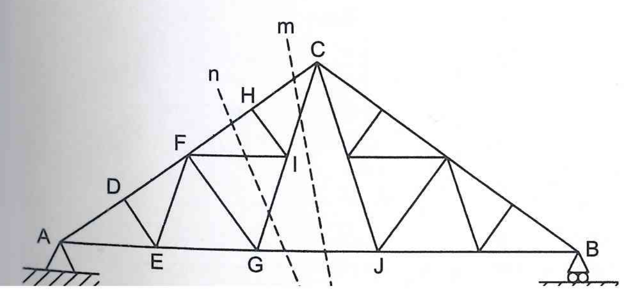


FIGURE 4.24

4.11 Space trusses

The most convenient method of analysing statically determinate stable space trusses (see Eq. (4.2)) is that of tension coefficients. In the case of space trusses, however, there are three possible equations of equilibrium for each joint (Eq. (2.11)); the moment equations (Eq. (2.12)) are automatically satisfied since, as in the case of plane trusses, the lines of action of all the forces in the members meeting at a joint pass through the joint and the pin cannot transmit moments. Therefore the analysis must begin at a joint where there are no more than three unknown forces.

The calculation of the reactions at supports in space frames can be complex. If a space frame has a statically determinate support system, a maximum of six reaction components can exist since there are a maximum of six equations of overall equilibrium (Eqs (2.11) and (2.12)). However, for the truss to be stable the reactions must be orientated in such a way that they can resist the components of the forces and moments about each of the three coordinate axes. Fortunately, in many problems, it is unnecessary to calculate support reactions since there is usually one joint at which there are no more than three unknown member forces.

EXAMPLE 4.8

Calculate the forces in the members of the space truss whose elevations and plan are shown in Fig. 4.25.

In this particular problem the exact nature of the support points is not specified so that the support reactions cannot be calculated. However, we note that at joint F there are just three unknown member forces so that the analysis may begin at F.

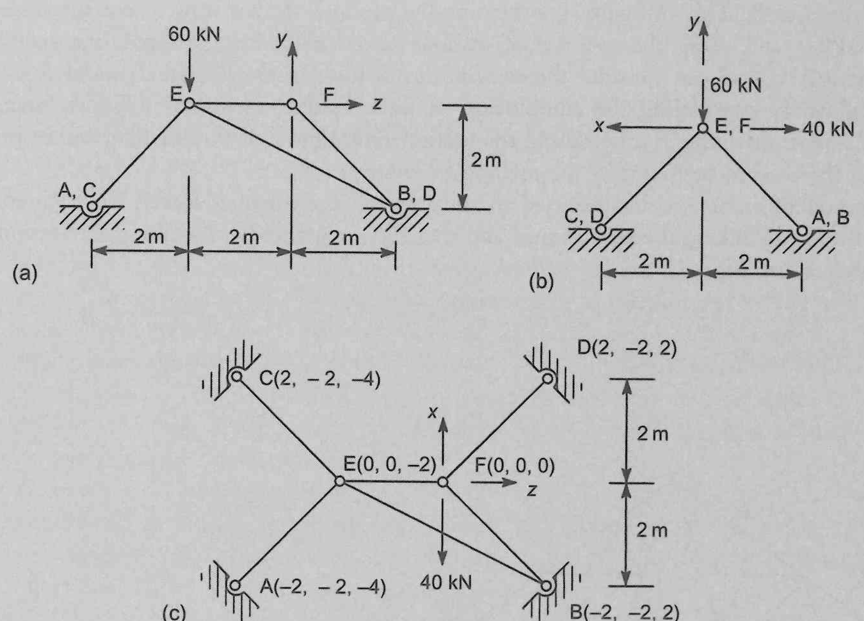


FIGURE 4.25

Elevations and plan of space frame of Ex. 4.8.

The first step is to choose an axis system and an origin of axes. Any system may be chosen so long as care is taken to ensure that there is agreement between the axis directions in each of the three views. Also, any point may be chosen as the origin of axes and need not necessarily coincide with a joint. In this problem it would appear logical to choose F, since the analysis will begin at F. Furthermore, it will be helpful to sketch the axis directions on each of the three views as shown and to insert the joint coordinates on the plan view (Fig. 4.25(c)).

At joint F

$$x \text{ direction: } t_{FD}(x_D - x_F) + t_{FB}(x_B - x_F) + t_{FE}(x_E - x_F) - 40 = 0 \quad \text{(i)}$$

$$y \text{ direction: } t_{FD}(y_D - y_F) + t_{FB}(y_B - y_F) + t_{FE}(y_E - y_F) = 0 \quad \text{(ii)}$$

$$z \text{ direction: } t_{FD}(z_D - z_F) + t_{FB}(z_B - z_F) + t_{FE}(z_E - z_F) = 0 \quad \text{(iii)}$$

Substituting the values of the joint coordinates in Eqs (i)–(iii) in turn we obtain, from Eq. (i)

$$t_{FD}(2-0) + t_{FB}(-2-0) + t_{FE}(0-0) - 40 = 0$$

whence

$$t_{FD} - t_{FB} - 20 = 0 \quad \text{(iv)}$$

from Eq. (ii)

$$t_{FD}(-2-0) + t_{FB}(-2-0) + t_{FE}(0-0) = 0$$

which gives

$$t_{FD} + t_{FB} = 0 \quad \text{(v)}$$

and from Eq. (iii)

$$t_{FD}(2-0) + t_{FB}(2-0) + t_{FE}(-2-0) = 0$$

so that

$$t_{FD} + t_{FB} - t_{FE} = 0 \quad \text{(vi)}$$

From Eqs (v) and (vi) we see by inspection that

$$t_{FE} = 0$$

Now adding Eqs (iv) and (v)

$$2t_{FD} - 20 = 0$$

whence

$$t_{FD} = 10$$

Therefore, from Eq. (v)

$$t_{FB} = -10$$

We now proceed to joint E where, since $t_{EF} = t_{FE}$, there are just three unknown member forces

$$x \text{ direction: } t_{ED}(x_D - x_E) + t_{EC}(x_C - x_E) + t_{EA}(x_A - x_E) + t_{EF}(x_F - x_E) = 0 \quad \text{(vii)}$$

$$y \text{ direction: } t_{EB}(y_B - y_E) + t_{EC}(y_C - y_E) + t_{EA}(y_A - y_E) + t_{EF}(y_F - y_E) - 60 = 0 \quad \text{(viii)}$$

$$z \text{ direction: } t_{EB}(z_B - z_E) + t_{EC}(z_C - z_E) + t_{EA}(z_A - z_E) + t_{EF}(z_F - z_E) = 0 \quad \text{(ix)}$$

Substituting the values of the coordinates and t_{EF} (=0) in Eqs (vii)–(ix) in turn gives, from Eq. (vii)

$$t_{EB}(-2-0) + t_{EC}(2-0) + t_{EA}(-2-0) = 0$$

so that

$$t_{EB} - t_{EC} + t_{EA} = 0 \quad \text{(x)}$$

from Eq. (viii)

$$t_{EB}(-2-0) + t_{EC}(-2-0) + t_{EA}(-2-0) - 60 = 0$$

whence

$$t_{EB} + t_{EC} + t_{EA} + 30 = 0 \quad \text{(xi)}$$

and from Eq. (ix)

$$t_{EB}(2+2) + t_{EC}(-4+2) + t_{EA}(-4+2) = 0$$

which gives

$$t_{EB} - 0.5t_{EC} - 0.5t_{EA} = 0 \quad \text{(xii)}$$

Subtracting Eq. (xi) from Eq. (x) we have

$$-2t_{EC} - 30 = 0$$

so that

$$t_{EC} = -15$$

Now subtracting Eq. (xii) from Eq. (xi) (or Eq. (x)) yields

$$1.5t_{EC} + 1.5t_{EA} + 30 = 0$$

which gives

$$t_{EA} = -5$$

Finally, from any of Eqs (x)–(xii),

$$t_{EB} = -10$$

The length of each of the members is now calculated, except that of EF which is given (=2 m). Using Pythagoras' theorem

$$L_{FB} = \sqrt{(x_B - x_F)^2 + (y_B - y_F)^2 + (z_B - z_F)^2}$$

whence

$$L_{FB} = \sqrt{(-2-0)^2 + (-2-0)^2 + (2-0)^2} = 3.46 \text{ m}$$

Similarly

$$L_{FD} = L_{EC} = L_{EA} = 3.46 \text{ m} \quad L_{FB} = 4.90 \text{ m}$$

The forces in the members then follow

$$T_{FB} = t_{FB} L_{FB} = -10 \times 3.46 \text{ kN} = -34.6 \text{ kN(compression)}$$

Similarly

$$T_{FD} = +34.6 \text{ kN (tension)}$$
$$T_{FE} = 0$$
$$T_{EC} = -51.9 \text{ kN (compression)}$$
$$T_{EA} = -17.3 \text{ kN (compression)}$$
$$T_{EB} = -49.0 \text{ kN (compression)}$$

The solution of Eqs (iv)–(vi) and (x)–(xii) in Ex. 4.8 was relatively straightforward in that many of the coefficients of the tension coefficients could be reduced to unity. This is not always the case, so that it is possible that the solution of three simultaneous equations must be carried out. In this situation an elimination method, described in standard mathematical texts, may be used.

4.12 A computer-based approach

The calculation of the member forces in trusses generally involves, as we have seen, in the solution of a number of simultaneous equations; clearly, the greater the number of members the greater the number of equations. For a truss with N members and R reactions $N+R$ equations are required for a solution provided that the truss and the support systems are both statically determinate. However, in some cases such as in Ex. 4.8, it is possible to solve for member forces without first calculating the support reactions. This still could mean that there would be a large number of equations to solve so that a more mechanical approach, such as the use of a computer, would be time and labour saving. For this we need to express the equations in matrix form.

At the joint F in Ex. 4.8 suppose that, instead of the 40 kN load, there are external loads X_F, Y_F and Z_F applied in the positive directions of the respective axes. Eqs (i)–(iii) are then written as

$$t_{FD}(x_D - x_F) + t_{FB}(x_B - x_F) + t_{FE}(x_E - x_F) + X_F = 0$$
$$t_{FD}(y_D - y_F) + t_{FB}(y_B - y_F) + t_{FE}(y_E - y_F) + Y_F = 0$$
$$t_{FD}(z_D - z_F) + t_{FB}(z_B - z_F) + t_{FE}(z_E - z_F) + Z_F = 0$$

In matrix form these become

$$\begin{bmatrix} x_D - x_F & x_B - x_F & x_E - x_F \\ y_D - y_F & y_B - y_F & y_E - y_F \\ z_D - z_F & z_B - z_F & z_E - z_F \end{bmatrix} \begin{bmatrix} t_{FD} \\ t_{FB} \\ t_{FE} \end{bmatrix} = \begin{bmatrix} -X_F \\ -Y_F \\ -Z_F \end{bmatrix}$$

or

$$[C][t] = [F]$$

where $[C]$ is the coordinate matrix, $[t]$ the tension coefficient matrix and $[F]$ the force matrix. Then

$$[t] = [C]^{-1}[F]$$

Computer programs exist which will carry out the inversion of $[C]$ so that the tension coefficients are easily obtained.

In the above the matrix equation only represents the equilibrium of joint F. There are, in fact, six members in the truss so that a total of six equations are required. The additional equations are Eqs (vii)–(ix) in Ex. 4.8. Therefore, to obtain a complete solution, these equations would be incorporated giving a 6×6 square matrix for $[C]$.

In practice plane and space frame programs exist which, after the relevant data have been input, give the member forces directly. It is, however, important that the fundamentals on which these programs are based are understood. We shall return to matrix methods later.

PROBLEMS

P.4.1 Investigate the determinacy and stability of the trusses shown in Fig.P.4.1(a)–(d).

Ans.

a. Statically determinate and stable.
b. Statically determinate and stable.
c. Statically in determinate and stable.
d. Statically determinate but unstable unless ABC is in tension.

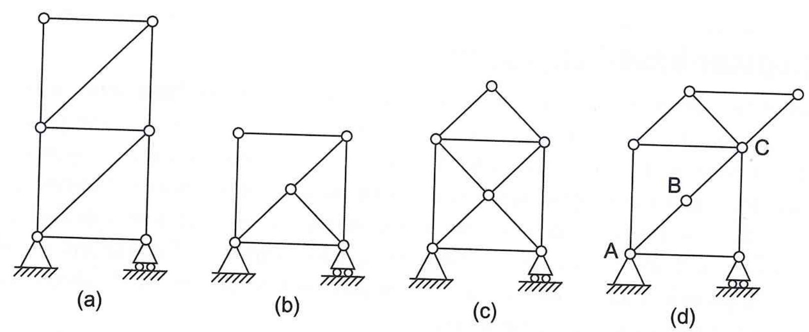
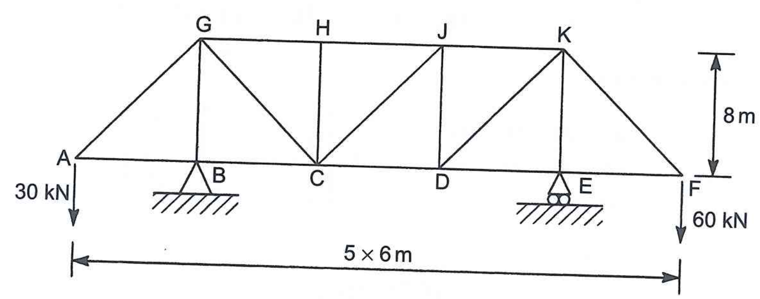


FIGURE P.4.1

P.4.2 Determine the forces in the members of the truss shown in Fig. P.4.2 using the method of joints and check the forces in the members JK, JD and DE by the method of sections.



Ans. AG = +37.5, AB = −22.5, BG = −20.0, BC = −22.5, GC = −12.5, GH = +30.0, HC = 0, HJ = +30.0, CJ = +12.5, CD = −37.5, JD = −10.0, JK = +37.5, DK = +12.5, DE = −45.0, EK = −70.0, FE = −45.0, FK = +75.0. All in kN.

P.4.3 Calculate the forces in the members of the truss shown in Fig. P.4.3.

Ans. AC = −30.0, AP = +26.0, CP = −8.7, CE = −25.0, PE = +8.7, PF = +17.3, FE = −17.3, GE = −20.0, HE = +8.7, FH = +17.3, GH = −8.7, JG = −15.0, HJ = +26.0, FB = 0, BJ = −15.0. All in kN.

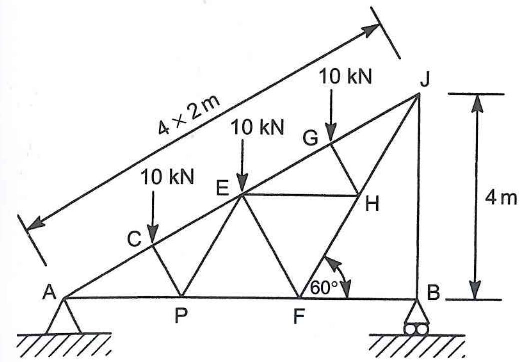


FIGURE P.4.3

P.4.4 Calculate the forces in the members EF, EG, EH and FH of the truss shown in Fig. P.4.4. Note that the horizontal load of 4 kN is applied at the joint C.

Ans. EF = −20.0, EG = −80.0, EH = −33.3, FH = +106.6. All in kN.

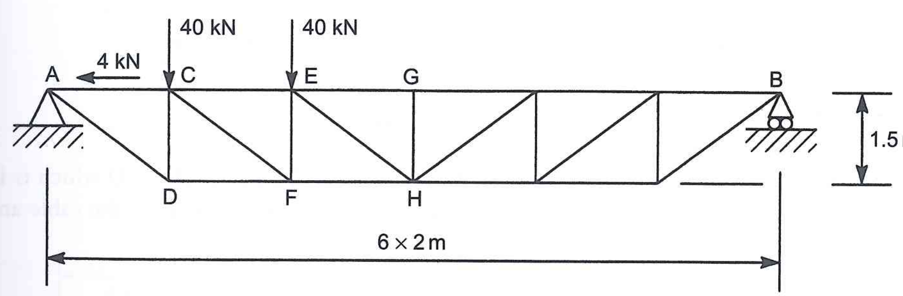


FIGURE P.4.4

P.4.5 The roof truss shown in Fig. P.4.5 is comprised entirely of equilateral triangles; the wind loads of 6 kN at J and B act perpendicularly to the member JB. Calculate the forces in the members DF, EF, EG and EK.

Ans. DF = +106.4, EF = +1.7, EG = −107.3, KE = −20.8. All in kN.

P.4.6 The upper chord joints of the bowstring truss shown in Fig. P.4.6 lie on a parabola whose equation has the form $y = kx^2$ referred to axes whose origin coincides with the uppermost joint.

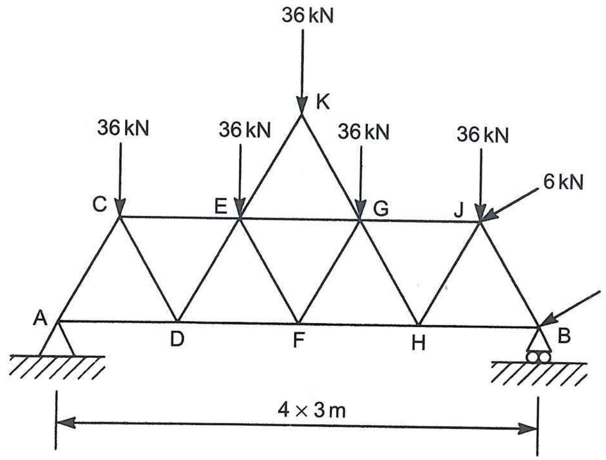


FIGURE P.4.5

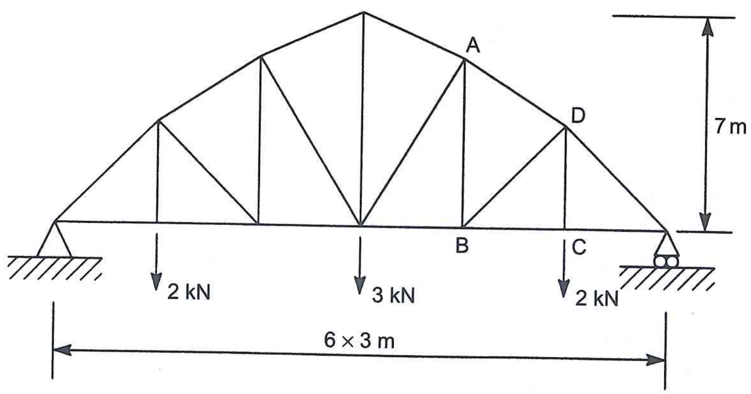
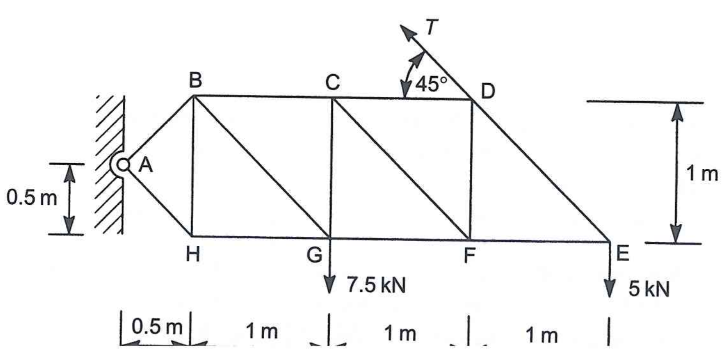


FIGURE P.4.6

P.4.7 The truss shown in Fig. P.4.7 is supported by a hinge at A and a cable at D which is inclined at an angle of 45° to the horizontal members. Calculate the tension, T, in the cable and hence the forces in all the members by the method of tension coefficients.



Ans. $T = 13.6$, BA = −8.9, CB = −9.2, DC = −4.6, ED = +7.1, EF = −5.0, FG = −0.4, GH = −3.3, HA = −4.7, BH = +3.4, GB = +4.1, FC = −6.5, CG = +4.6, DF = +4.6. All in kN.

P.4.8 Check your answers to problems P.4.2 and P.4.3 using a graphical method.

P.4.9 Determine the force in the member BC of the crane shown in Fig. P.4.9.

Ans. 707 kN. (tension)

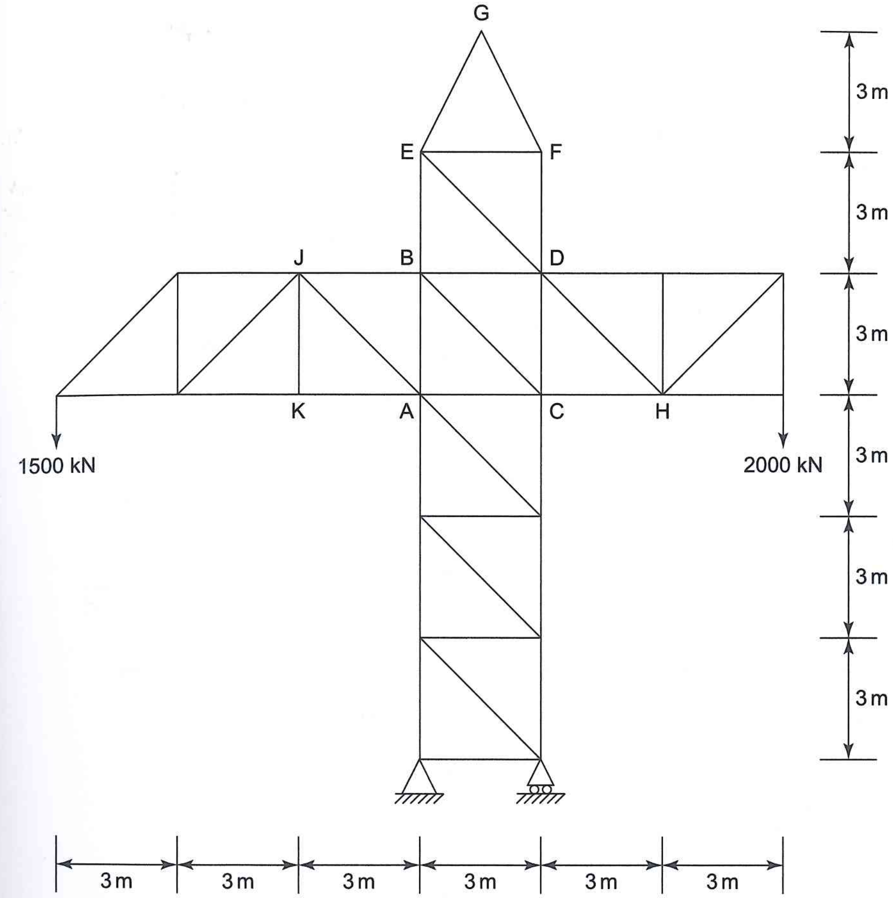


FIGURE P.4.9

P.4.10 Find the forces in the members of the space truss shown in Fig. P.4.10; suggested axes are also shown.

Ans. OA = +24.2, OB = +11.9, OC = −40.2. All in kN.

P.4.11 Use the method of tension coefficients to calculate the forces in the members of the space truss shown in Fig. P.4.11. Note that the loads P_2 and P_3 act in a horizontal plane and at angles of 45° to the vertical plane BAD.

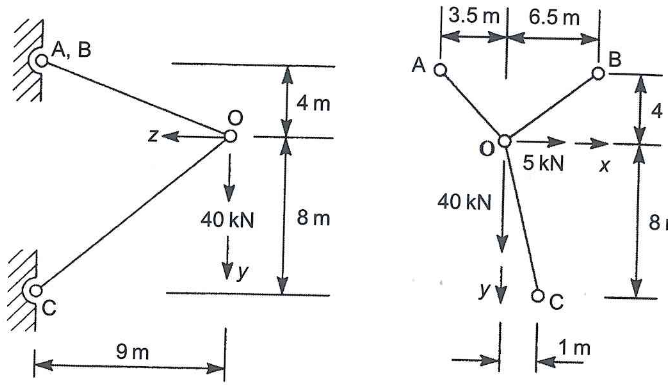


FIGURE P.4.10

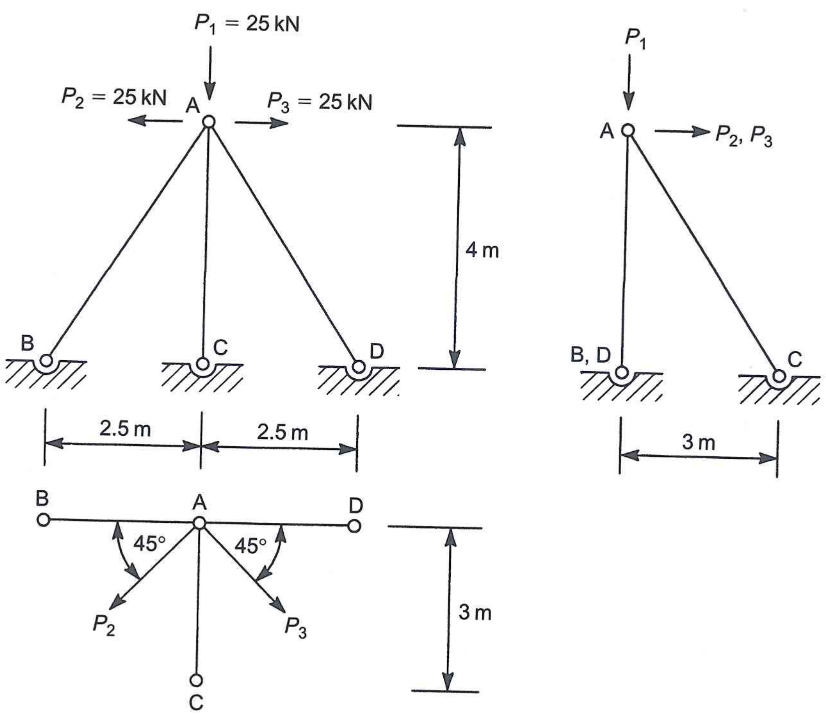


FIGURE P.4.11

P.4.12 The pin-jointed truss shown in Fig. P.4.12 is attached to a vertical wall at the points A, B, C and D; the members BE, BF, EF and AF are in the same horizontal plane. The truss supports vertically downward loads of 9 and 6 kN at E and F, respectively, and a horizontal load of 3 kN at E in the direction EF.

Calculate the forces in the members of the truss using the method of tension coefficients.

Ans. EF = −3.0, EC = −15.0, EB = +12.0, ...

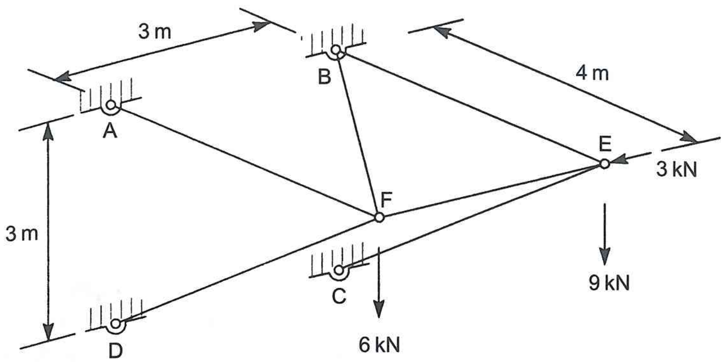


FIGURE P.4.12

P.4.13 Fig. P.4.13 shows the plan of a space truss which consists of six pin-jointed members. The member DE is horizontal and 4 m above the horizontal plane containing A, B and C while the loads applied at D and E act in a horizontal plane. Calculate the forces in the members.

Ans. AD = 0, DC = 0, DE = +40.0, AE = 0, CE = −60.0, BE = +60.0. All in kN.

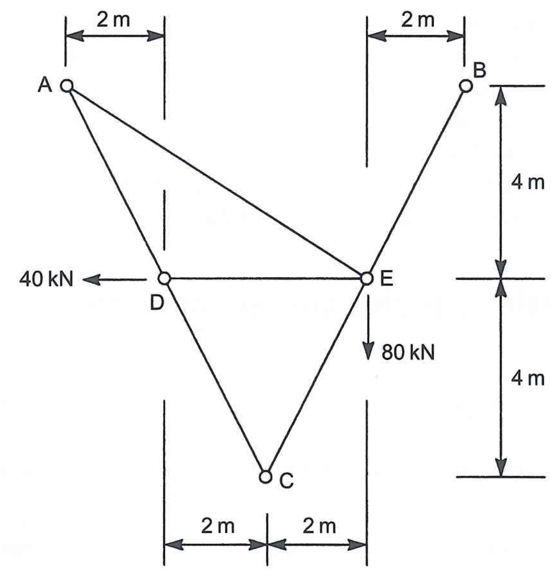


FIGURE P.4.13

CHAPTER

Cables

5

Flexible cables have been used to form structural systems for many centuries. Some of the earliest man-made structures of any size were hanging bridges constructed from jungle vines and creepers, and spanning ravines and rivers. In European literature the earliest description of an iron suspension bridge was published by Verantius in 1607, while ropes have been used in military bridging from at least 1600. In modern times, cables formed by binding a large number of steel wires together are employed in bridge construction where the bridge deck is suspended on hangers from the cables themselves. The cables in turn pass over the tops of towers and are fixed to anchor blocks embedded in the ground; in this manner large, clear spans are achieved. Cables are also used in cable-stayed bridges, as part of roof support systems, for prestressing in concrete beams and for guyed structures such as pylons and television masts.

Structurally, cables are extremely efficient because they make the most effective use of structural material in that their loads are carried solely through tension. Therefore, there is no tendency for buckling to occur either from bending or from compressive axial loads (see Chapter 21). However, many of the structures mentioned above are statically indeterminate to a high degree. In other situations, particularly in guyed towers and cable-stayed bridges, the extension of the cables affects the internal force system and the analysis becomes non-linear. Such considerations are outside the scope of this book so that we shall concentrate on cables in which loads are suspended directly from the cable.

Two categories of cable arise; the first is relatively lightweight and carries a limited number of concentrated loads, while the second is heavier with a more uniform distribution of load. We shall also examine, in the case of suspension bridges, the effects of different forms of cable support at the towers.

5.1 Lightweight cables carrying concentrated loads

In the analysis of this type of cable we shall assume that the self-weight of the cable is negligible, that it can only carry tensile forces and that the extension of the cable does not affect the geometry of the system. We shall illustrate the method by examples.

EXAMPLE 5.1

The cable shown in Fig. 5.1 is pinned to supports at A and B and carries a concentrated load of 10 kN at a point C. Calculate the tension in each part of the cable and the reactions at the supports.

Since the cable is weightless the lengths AC and CB are straight. The tensions T_{CA} and T_{CB} in the parts AC and CB, respectively, may be found by considering the equilibrium of the forces acting at C where, from Fig. 5.1, we see that

$$\alpha = \tan^{-1} 1/3 = 18.4^\circ \quad \beta = \tan^{-1} 1/2 = 26.6^\circ$$

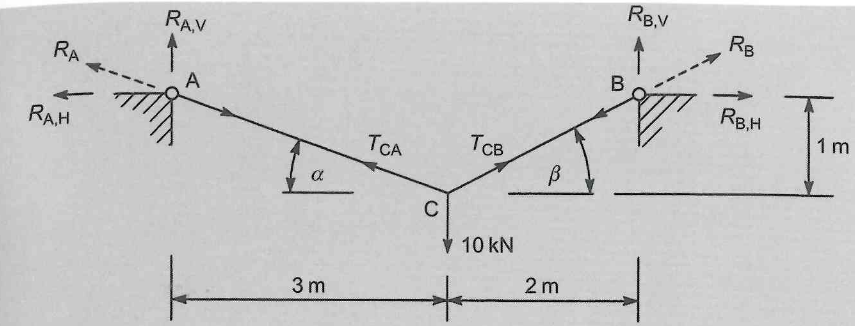


FIGURE 5.1

Lightweight cable carrying a concentrated load.

Resolving forces in a direction *perpendicular* to CB (thereby eliminating T_{CB}) we have, since $\alpha + \beta = 45^\circ$

$$T_{CA} \cos 45^\circ - 10 \cos 26.6^\circ = 0$$

from which

$$T_{CA} = 12.6 \text{ kN}$$

Now resolving forces horizontally (or alternatively vertically or perpendicular to CA) gives

$$T_{CB} \cos 26.6^\circ - T_{CA} \cos 18.4^\circ = 0$$

so that

$$T_{CB} = 13.4 \text{ kN}$$

Since the bending moment in the cable is everywhere zero we can take moments about B (or A) to find the vertical component of the reaction at A, $R_{A,V}$ (or $R_{B,V}$) directly. Then

$$R_{A,V} \times 5 - 10 \times 2 = 0 \quad \text{(i)}$$

so that

$$R_{A,V} = 4 \text{ kN}$$

Now resolving forces vertically for the complete cable

$$R_{B,V} + R_{A,V} - 10 = 0 \quad \text{(ii)}$$

which gives

$$R_{B,V} = 6 \text{ kN}$$

From the horizontal equilibrium of the cable the horizontal components of the reactions at A and B are equal, i.e. $R_{A,H} = R_{B,H}$. Thus, taking moments about C for the forces to the left of C

$$R_{A,H} \times 1 - R_{A,V} \times 3 = 0 \quad \text{(iii)}$$

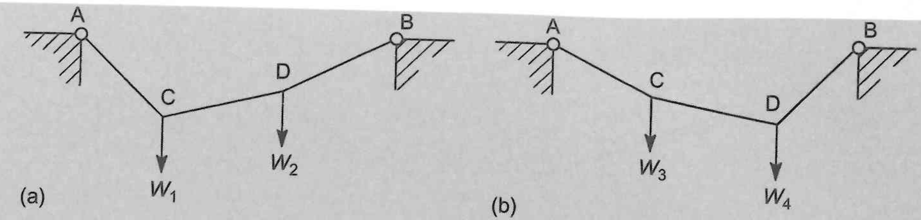


FIGURE 5.2

Effect on cable geometry of load variation.

from which

$$R_{A,H} = 12 \text{ kN}(=R_{B,H})$$

Note that the horizontal component of the reaction at A, $R_{A,H}$, would be included in the moment equation (Eq. (i)) if the support points A and B were on different levels. In this case Eqs (i) and (iii) could be solved simultaneously for $R_{A,V}$ and $R_{A,H}$. Note also that the tensions T_{CA} and T_{CB} could be found from the components of the support reactions since the resultant reaction at each support, R_A at A and R_B at B, must be equal and opposite in direction to the tension in the cable otherwise the cable would be subjected to shear forces, which we have assumed is not possible. Hence

$$T_{CA} = R_A = \sqrt{R_{A,V}^2 + R_{A,H}^2} = \sqrt{4^2 + 12^2} = 12.6 \text{ kN}$$

$$T_{CB} = R_B = \sqrt{R_{B,V}^2 + R_{B,H}^2} = \sqrt{6^2 + 12^2} = 13.4 \text{ kN}$$

as before.

In Ex. 5.1 the geometry of the loaded cable was specified. We shall now consider the case of a cable carrying more than one load. In the cable in Fig. 5.2(a), the loads W_1 and W_2 at the points C and D produce a different deflected shape to the loads W_3 and W_4 at C and D in Fig. 5.2(b). The analysis is then affected by the change in geometry as well as the change in loading, a different situation to that in beam and truss analysis. The cable becomes, in effect, a mechanism and changes shape to maintain its equilibrium; the analysis then becomes non-linear and therefore statically indeterminate. However, if the geometry of the deflected cable is partially specified, say the maximum deflection or sag is given, the system becomes statically determinate.

EXAMPLE 5.2

Calculate the tension in each of the parts AC, CD and DB of the cable shown in Fig. 5.3.

There are different possible approaches to the solution of this problem. For example, we could investigate the equilibrium of the forces acting at the point C and resolve horizontally and vertically. We would then obtain two equations in which the unknowns would be T_{CA}, T_{CD}, α and β. From the geometry of the cable $\alpha = \tan^{-1}(0.5/1.5) = 18.4°$ so that there would be three unknowns remaining. A third equation could be obtained by examining the moment equilibrium of the length AC of the cable about A, where the moment is zero since the cable is flexible. The solution of these three simultaneous equations would be rather tedious so that a simpler approach is preferable.

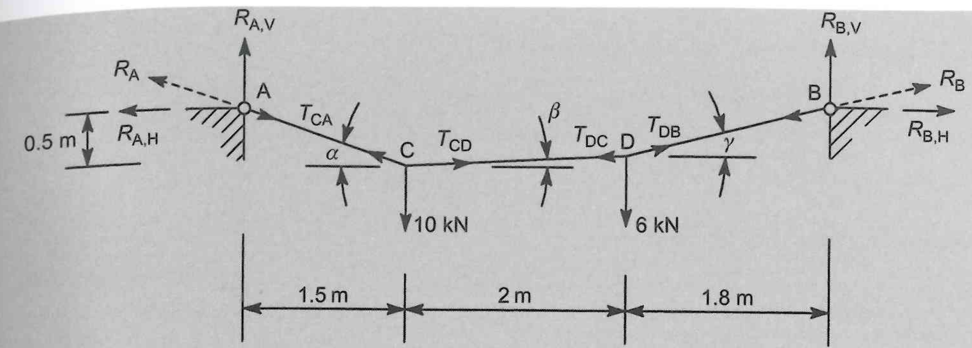


FIGURE 5.3

Cable of Ex. 5.2.

In Ex. 5.1 we saw that the resultant reaction at the supports is equal and opposite to the tension in the cable at the supports. Therefore, by determining $R_{A,V}$ and $R_{A,H}$ we can obtain T_{CA} directly. Hence, taking moments about B we have

$$R_{A,V} \times 5.3 - 10 \times 3.8 - 6 \times 1.8 = 0$$

from which

$$R_{A,V} = 9.2 \text{ kN}$$

Since the cable is perfectly flexible the internal moment at any point is zero. Therefore, taking moments of forces to the left of C about C gives

$$R_{A,H} \times 0.5 - R_{A,V} \times 1.5 = 0$$

so that

$$R_{A,H} = 27.6 \text{ kN}$$

Alternatively we could have obtained $R_{A,H}$ by using the fact that the resultant reaction, R_A, at A is in line with the cable at A, i.e. $R_{A,V}/R_{A,H} = \tan\alpha = \tan 18.4°$, which gives $R_{A,H} = 27.6$ kN as before. Having obtained $R_{A,V}$ and $R_{A,H}$, T_{CA} follows. Thus

$$T_{CA} = R_A = \sqrt{R_{A,H}^2 + R_{A,V}^2} = \sqrt{27.6^2 + 9.2^2}$$

i.e.

$$T_{CA} = 29.1 \text{ kN}$$

From a consideration of the vertical equilibrium of the forces acting at C we have

$$T_{CD}\sin\beta + T_{CA}\sin\alpha - 10 = T_{CD}\sin\beta + 29.1\sin 18.4° - 10 = 0$$

which gives

$$T_{CD}\sin\beta = 0.815 \quad \text{(i)}$$

From the horizontal equilibrium of the forces at C

$$T_{CD}\cos\beta - T_{CA}\cos\alpha = T_{CD}\cos\beta - 29.1\cos 18.4^\circ = 0$$

so that

$$T_{CD}\cos\beta = 27.612 \qquad \text{(ii)}$$

Dividing Eq. (i) by Eq. (ii) yields

$$\tan\beta = 0.0295$$

from which

$$\beta = 1.69^\circ$$

Therefore from either of Eq. (i) or (ii)

$$T_{CD} = 27.6 \text{ kN}$$

We can obtain the tension in DB in a similar manner. Thus, from the vertical equilibrium of the forces at D, we have

$$T_{DB}\sin\gamma - T_{DC}\sin\beta - 6 = T_{DB}\sin\gamma - 27.6\sin 1.69^\circ - 6 = 0$$

from which

$$T_{DB}\sin\gamma = 6.815 \qquad \text{(iii)}$$

From the horizontal equilibrium of the forces at D we see that

$$T_{DB}\cos\gamma - T_{CB}\cos\beta = T_{DB}\cos\gamma - 27.6\cos 1.69^\circ = 0$$

from which

$$T_{DB}\cos\gamma = 27.618 \qquad \text{(iv)}$$

Dividing Eq. (iii) by Eq. (iv) we obtain

$$\tan\gamma = 0.2468$$

so that

$$\gamma = 13.86^\circ$$

T_{DB} follows from either of Eq. (iii) or (iv) and is

$$T_{DB} = 28.4 \text{ kN}$$

Alternatively we could have calculated T_{DB} by determining $R_{B,H}$ $(=R_{A,H})$ and $R_{B,V}$. Then

$$T_{DB} = R_B = \sqrt{R_{B,H}^2 + R_{B,V}^2}$$

and

$$\gamma = \tan^{-1}\left(\frac{R_{B,V}}{R_{B,H}}\right)$$

This approach would, in fact, be a little shorter than the one given above. However, in the case where the cable carries more than two loads, the above method must be used at loading points adjacent to the support points.

5.2 Heavy cables

We shall now consider the more practical case of cables having a significant self-weight.

Governing equation for deflected shape

The cable AB shown in Fig. 5.4(a) carries a distributed load $w(x)$ per unit of its horizontally projected length. An element of the cable, whose horizontal projection is δx, is shown, together with the forces acting on it, in Fig. 5.4(b). Since δx is infinitesimally small, the load intensity may be regarded as constant over the length of the element.

Suppose that T is the tension in the cable at the point x and that $T + \delta T$ is the tension at the point $x + \delta x$; the vertical and horizontal components of T are V and H, respectively. In the absence of any externally applied horizontal loads we see that

$$H = \text{constant}$$

and from the vertical equilibrium of the element we have

$$V + \delta V - w(x)\delta x - V = 0$$

so that, in the limit as $\delta x \rightarrow 0$

$$\frac{dV}{dx} = w(x) \qquad (5.1)$$

From Fig. 5.4(b)

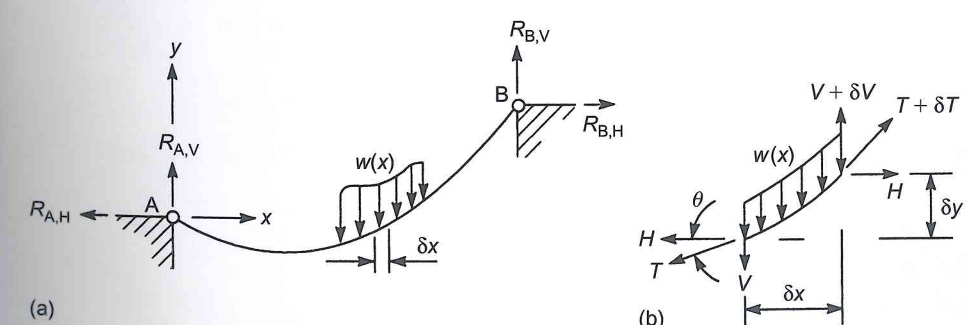


FIGURE 5.4

Cable subjected ...

$$\frac{V}{H} = \tan\theta = +\frac{dy}{dx}$$

where y is the vertical deflection of the cable at any point referred to the x axis.

Hence

$$V = +H\frac{dy}{dx}$$

so that

$$\frac{dV}{dx} = +H\frac{d^2y}{dx^2} \tag{5.2}$$

Substituting for dV/dx from Eq. (5.1) into Eq. (5.2) we obtain the *governing equation* for the deflected shape of the cable. Thus

$$H\frac{d^2y}{dx^2} = +w(x) \tag{5.3}$$

We are now in a position to investigate cables subjected to different load applications.

Cable under its own weight

In this case let us suppose that the weight per actual unit length of the cable is w_s. Then, by referring to Fig. 5.5, we see that the weight per unit of the horizontally projected length of the cable, $w(x)$, is given by

$$w(x)\delta x = w_s \delta s \tag{5.4}$$

Now, in the limit as $\delta s \to 0$, $ds = (dx^2 + dy^2)^{1/2}$

Whence, from Eq. (5.4)

$$w(x) = w_s\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{5.5}$$

Substituting for $w(x)$ from Eq. (5.5) in Eq. (5.3) gives

$$H\frac{d^2y}{dx^2} = +w_s\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{5.6}$$

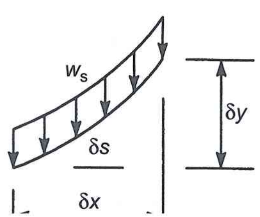


FIGURE 5.5

Let $dy/dx = p$. Then Eq. (5.6) may be written

$$H\frac{dp}{dx} = +w_s(1 + p^2)^{1/2}$$

or, rearranging and integrating

$$\int \frac{dp}{(1+p^2)^{1/2}} = +\int \frac{w_s}{H}dx \tag{5.7}$$

The term on the left-hand side of Eq. (5.7) is a standard integral. Thus

$$\sinh^{-1}p = +\frac{w_s}{H}x + C_1$$

in which C_1 is a constant of integration. Then

$$p = \sinh\left(+\frac{w_s}{H}x + C_1\right)$$

Now substituting for p ($=dy/dx$) we obtain

$$\frac{dy}{dx} = \sinh\left(+\frac{w_s}{H}x + C_1\right)$$

which, when integrated, becomes

$$y = +\frac{H}{w_s}\cosh\left(+\frac{w_s}{H}x + C_1\right) + C_2 \tag{5.8}$$

in which C_2 is a second constant of integration.

The deflected shape defined by Eq. (5.8) is known as a *catenary*; the constants C_1 and C_2 may be found using the boundary conditions of a particular problem.

EXAMPLE 5.3

Determine the equation of the deflected shape of the symmetrically supported cable shown in Fig. 5.6, if its self-weight is w_s per unit of its actual length.

The equation of its deflected shape is given by Eq. (5.8), i.e.

$$y = +\frac{H}{w_s}\cosh\left(+\frac{w_s}{H}x + C_1\right) + C_2 \tag{i}$$

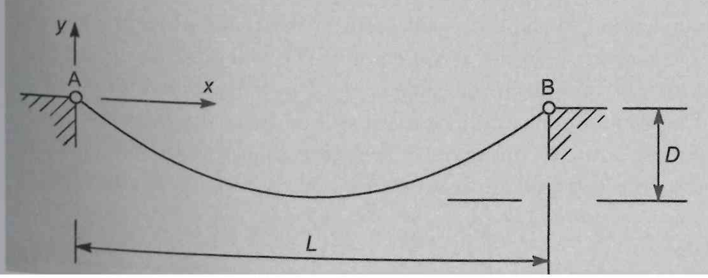


FIGURE 5.6

Deflected shape of a symmetrically supported cable.

Differentiating Eq. (i) with respect to x we have

$$\frac{dy}{dx} = \sinh\left(+\frac{w_s}{H}x + C_1\right) \qquad \text{(ii)}$$

From symmetry, the slope of the cable at mid-span is zero, i.e. $dy/dx = 0$ when $x = L/2$. Thus, from Eq. (ii)

$$0 = \sinh\left(+\frac{w_s}{H}\frac{L}{2} + C_1\right)$$

from which

$$C_1 = -\frac{w_s}{H}\frac{L}{2}$$

Eq. (i) then becomes

$$y = +\frac{H}{w_s}\cosh\left[+\frac{w_s}{H}\left(x - \frac{L}{2}\right)\right] + C_2 \qquad \text{(iii)}$$

The deflection of the cable at its supports is zero, i.e. $y = 0$ when $x = 0$ and $x = L$. From the first of these conditions

$$0 = +\frac{H}{w_s}\cosh\left(-\frac{w_s}{H}\frac{L}{2}\right) + C_2$$

so that

$$C_2 = -\frac{H}{w_s}\cosh\left(-\frac{w_s}{H}\frac{L}{2}\right) = -\frac{H}{w_s}\cosh\left(\frac{w_s}{H}\frac{L}{2}\right) \quad \text{(note: } \cosh(-x) \equiv \cosh(x)\text{)}$$

Eq. (iii) is then written as

$$y = +\frac{H}{w_s}\left\{\cosh\left[+\frac{w_s}{H}\left(x - \frac{L}{2}\right)\right] - \cosh\left(\frac{w_s}{H}\frac{L}{2}\right)\right\} \qquad \text{(iv)}$$

Equation (iv) gives the deflected shape of the cable in terms of its self-weight, its length and the horizontal component, H, of the tension in the cable. In a particular case where, say, w_s, L and H are specified, the sag, D, of the cable is obtained directly from Eq. (iv). Alternatively if, instead of H, the sag D is fixed, H is obtained from Eq. (iv) which then becomes a transcendental equation which may be solved graphically.

Since H is constant the maximum tension in the cable will occur at the point where the vertical component of the tension in the cable is greatest. In the above example this will occur at the support points where the vertical component of the tension in the cable is equal to half its total weight. For a cable having supports at different heights, the maximum tension will occur at the highest support since the length of cable from its lowest point to this support is greater than that on the opposite side of the lowest point. Furthermore, the slope of the cable at the highest support is a maximum (see Fig. 5.4(a)).

Cable subjected to a uniform horizontally distributed load

This loading condition is, as we shall see when we consider suspension bridges, more representative of that in actual suspension structures than the previous case.

For the cable shown in Fig. 5.7, Eq. (5.3) becomes

$$H\frac{d^2y}{dx^2} = +w \qquad (5.9)$$

Integrating Eq. (5.9) with respect to x we have

$$H\frac{dy}{dx} = +wx + C_1 \qquad (5.10)$$

again integrating

$$Hy = +w\frac{x^2}{2} + C_1x + C_2 \qquad (5.11)$$

The boundary conditions are $y = 0$ at $x = 0$ and $y = h$ at $x = L$. The first of these gives $C_2 = 0$ while from the second we have

$$H(+h) = +w\frac{L^2}{2} + C_1L$$

so that

$$C_1 = -\frac{wL}{2} + H\frac{h}{L}$$

Equations (5.10) and (5.11) then become, respectively

$$\frac{dy}{dx} = +\frac{w}{H}x - \frac{wL}{2H} + \frac{h}{L} \qquad (5.12)$$

and

$$y = +\frac{w}{2H}x^2 - \left(\frac{wL}{2H} - \frac{h}{L}\right)x \qquad (5.13)$$

Thus the cable in this case takes up a parabolic shape.

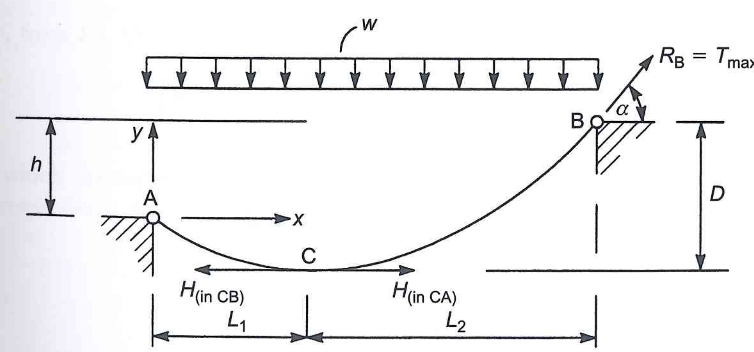


FIGURE 5.7

Equations (5.12) and (5.13) are expressed in terms of the horizontal component, H, of the tension in the cable, the applied load and the cable geometry. If, however, the maximum sag, D, of the cable is known, H may be eliminated as follows.

The position of maximum sag coincides with the point of zero slope. Thus from Eq. (5.12)

$$0 = +\frac{w}{H}x - \frac{wL}{2H} + \frac{h}{L}$$

so that

$$x = \frac{L}{2} - \frac{Hh}{wL} = L_1 \quad \text{(see Fig. 5.7)}$$

Then the horizontal distance, L_2, from the lowest point of the cable to the support at B is given by

$$L_2 = L - L_1 = \frac{L}{2} + \frac{Hh}{wL}$$

Now considering the moment equilibrium of the length CB of the cable about B we have, from Fig. 5.7

$$HD - w\frac{L_2^2}{2} = 0$$

so that

$$HD - \frac{w}{2}\left(\frac{L}{2} + \frac{Hh}{wL}\right)^2 = 0 \tag{5.14}$$

Equation (5.14) is a quadratic equation in H and may be solved for a specific case using the formula.

Alternatively, H may be determined by considering the moment equilibrium of the lengths AC and CB about A and C, respectively. Thus, for AC

$$H(D - h) - w\frac{L_1^2}{2} = 0$$

which gives

$$H = \frac{wL_1^2}{2(D - h)} \tag{5.15}$$

For CB

$$HD - \frac{wL_2^2}{2} = 0$$

so that

$$H = \frac{wL_2^2}{2D} \tag{5.16}$$

Equating Eqs (5.15) and (5.16)

$$\frac{wL_1^2}{2(D - h)} = \frac{wL_2^2}{2D}$$

which gives

$$L_1 = \sqrt{\frac{D - h}{D}}L_2$$

But

$$L_1 + L_2 = L$$

therefore

$$L_2 = \left[\sqrt{\frac{D - h}{D}} + 1\right] = L$$

from which

$$L_2 = \frac{L}{\left(\sqrt{\frac{D-h}{D}} + 1\right)} \tag{5.17}$$

Then, from Eq. (5.16)

$$H = \frac{wL^2}{2D\left[\sqrt{\frac{D-h}{D}} + 1\right]^2} \tag{5.18}$$

As in the case of the catenary the maximum tension will occur, since $H =$ constant, at the point where the vertical component of the tension is greatest. Thus, in the cable of Fig. 5.7, the maximum tension occurs at B where, as $L_2 > L_1$, the vertical component of the tension ($= wL_2$) is greatest. Hence

$$T_{\max} = \sqrt{(wL_2)^2 + H^2} \tag{5.19}$$

in which L_2 is obtained from Eq. (5.17) and H from one of Eqs (5.14), (5.16) or (5.18). At B the slope of the cable is given by

$$\alpha = \tan^{-1}\left(\frac{wL}{H}\right) \tag{5.20}$$

or, alternatively, from Eq. (5.12)

$$\left(\frac{dy}{dx}\right)_{x=L} = +\frac{w}{H}L - \frac{wL}{2H} + \frac{h}{L} = +\frac{wL}{2H} + \frac{h}{L} \tag{5.21}$$

For a cable in which the supports are on the same horizontal level, i.e. $h = 0$, Eqs (5.12)–(5.14) and (5.19) reduce, respectively, to

$$\frac{dy}{dx} = \frac{w}{H}\left(x - \frac{L}{2}\right) \tag{5.22}$$

$$y = \frac{w}{}(x^2 - Lx) \tag{5.23}$$

$$H = \frac{wL^2}{8D} \tag{5.24}$$

$$T_{max} = \frac{wL}{2}\sqrt{1 + \left(\frac{L}{4D}\right)^2} \tag{5.25}$$

We observe from the above that the analysis of a cable under its own weight, that is a catenary, yields a more complex solution than that in which the load is assumed to be uniformly distributed horizontally. However, if the sag in the cable is small relative to its length, this assumption gives results that differ only slightly from the more accurate but more complex catenary approach. Therefore, in practice, the loading is generally assumed to be uniformly distributed horizontally.

EXAMPLE 5.4

Determine the maximum tension and the maximum slope in the cable shown in Fig. 5.8 if it carries a uniform horizontally distributed load of intensity 10 kN/m.

From Eq. (5.17)

$$L_2 = \frac{200}{\left(\sqrt{\frac{18-6}{18}} + 1\right)} = 110.1 \text{ m}$$

Then, from Eq. (5.16)

$$H = \frac{10 \times 110.1^2}{2 \times 18} = 3367.2 \text{ kN}$$

The maximum tension follows from Eq. (5.19), i.e.

$$T_{max} = \sqrt{(10 \times 110.1)^2 + 3367.2^2} = 3542.6 \text{ kN}$$

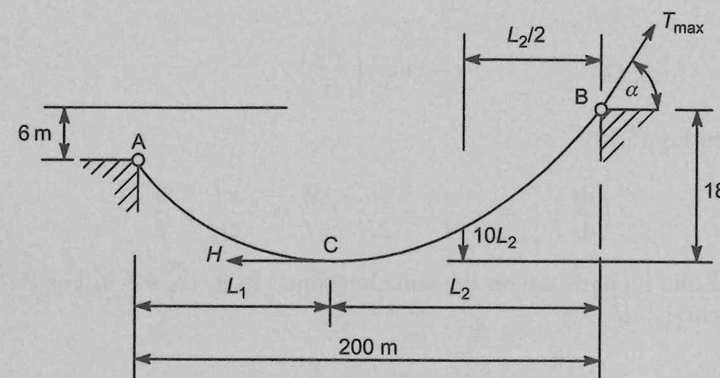


FIGURE 5.8

Suspension cable of Ex. 5.4.

Then, from Eq. (5.20)

$$\alpha_{max} = \tan^{-1}\frac{10 \times 110.1}{3367.2} = 18.1° \text{ at B}$$

Ex. 5.4 has been solved by direct substitution in Eqs (5.16), (5.17), (5.19) and (5.20). However, working from first principles obviates the necessity of remembering rather unwieldy formulae; the solution of Ex. 5.4 would then proceed as follows:

Taking moments about A for the portion AC of the cable we have

$$H \times 12 - 10L_1(L_1/2) = 0$$

from which

$$H = 5L_1^2/12 \tag{i}$$

Now taking moments about B for the portion BC

$$H \times 18 - 10L_2(L_2/2) = 0$$

so that

$$H = 5L_2^2/18 \tag{ii}$$

Equating Eqs (i) and (ii) gives

$$L_1 = (\sqrt{2/3})L_2 = 0.816\ L_2$$

But

$$L_1 + L_2 = 200 \tag{iii}$$

Substituting for L_1 in Eq. (iii) gives

$$L_2 = 110.1 \text{ m}$$

Then, from Eq. (ii)

$$H = 5 \times 110.1^2/18 = 3367.2 \text{ kN}$$

The vertical component of the support reaction at B is $10L_2 = 1101$ kN and the horizontal component is $H = 3367.2$ kN. Then

$$T_{max} = \sqrt{(1101^2 + 3367.2^2)} = 3542.6 \text{ kN}$$

Finally

$$\alpha_{max} = \tan^{-1}(1101/3367.2) = 18.1° \text{ at B}$$

Suspension bridges

A typical arrangement for a suspension bridge is shown diagrammatically in Fig. 5.9. The bridge deck is suspended by hangers from the cables which pass over the tops of the towers and are secured by massive anchor blocks embedded in the ground. The advantage of this form of bridge construction is its abili...

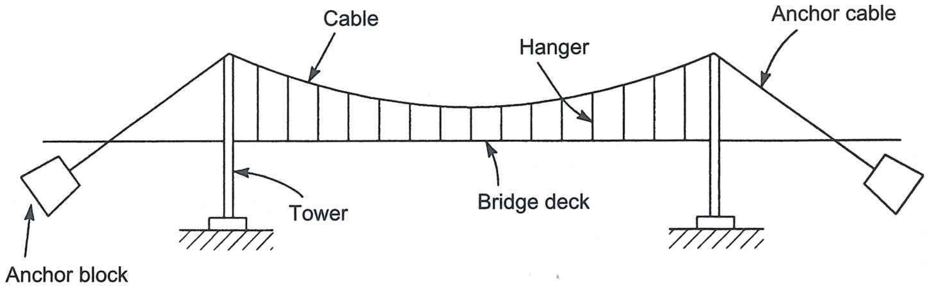


FIGURE 5.9

Diagrammatic representation of a suspension bridge.

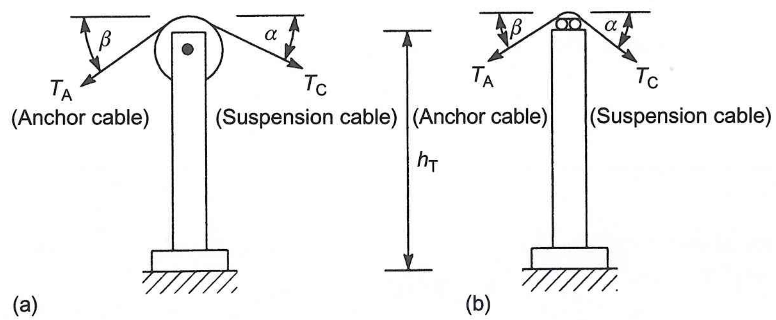


FIGURE 5.10

Idealization of cable supports.

in the UK are the suspension bridges over the rivers Humber and Severn, the Forth road bridge and the Menai Straits bridge in which the suspension cables comprise chain links rather than tightly bound wires. Suspension bridges are also used for much smaller spans such as pedestrian footbridges and for light vehicular traffic over narrow rivers.

The major portion of the load carried by the cables in a suspension bridge is due to the weight of the deck, its associated stiffening girder and the weight of the vehicles crossing the bridge. By comparison, the self-weight of the cables is negligible. We may assume therefore that the cables carry a uniform horizontally distributed load and therefore take up a parabolic shape; the analysis described in the preceding section then applies.

The cables, as can be seen from Fig. 5.9, are continuous over the tops of the towers. In practice they slide in grooves in saddles located on the tops of the towers. For convenience we shall idealize this method of support into two forms, the actual method lying somewhere between the two. In Fig. 5.10(a) the cable passes over a frictionless pulley, which means that the tension, T_A, in the anchor cable is equal to T_C, the tension at the tower in the suspension cable. Generally the inclination, β, of the anchor cable is fixed and will not be equal to the inclination, α, of the suspension cable at the tower. Therefore, there will be a resultant horizontal force, H_T, on the top of the tower given by

$$H_T = T_C \cos\alpha - T_A \cos\beta$$

or, since $T_A = T_C$

H_T, in turn, produces a bending moment, M_T, in the tower which is a maximum at the tower base. Hence

$$M_{T(max)} = H_T h_T = T_C(\cos\alpha - \cos\beta)h_T \tag{5.27}$$

Also, the vertical compressive load, V_T, on the tower is

$$V_T = T_C(\sin\alpha + \sin\beta) \tag{5.28}$$

In the arrangement shown in Fig. 5.10(b) the cable passes over a saddle which is supported on rollers on the top of the tower. The saddle therefore cannot resist a horizontal force and adjusts its position until

$$T_A \cos\beta = T_C \cos\alpha \tag{5.29}$$

For a given value of β, Eq. (5.29) determines the necessary value of T_A. Clearly, since there is no resultant horizontal force on the top of the tower, the bending moment in the tower is everywhere zero. Finally, the vertical compressive load on the tower is given by

$$V_T = T_C \sin\alpha + T_A \sin\beta \tag{5.30}$$

EXAMPLE 5.5

The cable of a suspension bridge, shown in Fig. 5.11, runs over a frictionless pulley on the top of each of the towers at A and B and is fixed to anchor blocks at D and E. If the cable carries a uniform horizontally distributed load of 120 kN/m, determine the diameter required if the permissible working stress on the gross area of the cable, including voids, is 600 N/mm^2. Also calculate the bending moment and direct load at the base of a tower and the required weight of the anchor blocks.

The tops of the towers are on the same horizontal level, so that the tension in the cable at these points is the same and will be the maximum tension in the cable. The maximum tension is found directly from Eq. (5.25) and is

$$T_{max} = \frac{120 \times 300}{2}\sqrt{1 + \left(\frac{300}{4 \times 30}\right)^2} = 48466.5 \text{ kN}$$

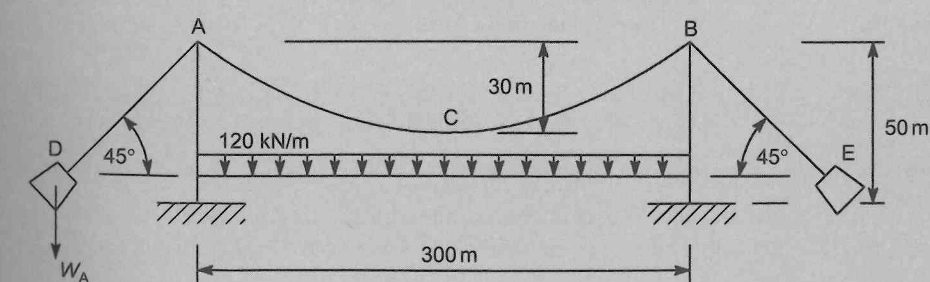


FIGURE 5.11

Suspension bridge of Ex. 5.5.

The maximum direct stress, σ_{max}, is given by

$$\sigma_{max} = \frac{T_{max}}{\pi d^2/4} \quad \text{(sec Section 7.1)}$$

in which d is the cable diameter. Hence

$$600 = \frac{48466.5 \times 10^3}{\pi d^2/4}$$

which gives

$$d = 320.7 \text{ mm}$$

The angle of inclination of the suspension cable to the horizontal at the top of the tower is obtained using Eq. (5.20) in which $L_2 = L/2$. Hence

$$\alpha = \tan^{-1}\left(\frac{wL}{2H}\right) = \tan^{-1}\left(\frac{120 \times 300}{2H}\right)$$

where H is given by Eq. (5.24). Thus

$$H = \frac{120 \times 300^2}{8 \times 30} = 45\,000 \text{ kN}$$

so that

$$\alpha = \tan^{-1}\left(\frac{120 \times 300}{2 \times 45\,000}\right) = 21.8°$$

Therefore, from Eq. (5.27), the bending moment at the base of the tower is

$$M_T = 48466.5(\cos 21.8° - \cos 45°) \times 50$$

from which

$$M_T = 536473.4 \text{ kNm}$$

The direct load at the base of the tower is found using Eq. (5.28), i.e.

$$V_T = 48466.5(\sin 21.8° + \sin 45°)$$

which gives

$$V_T = 52269.9 \text{ kN}$$

Finally the weight, W_A, of an anchor block must resist the vertical component of the tension in the anchor cable. Thus

$$W_A = T_A \cos 45° = 48466.5 \cos 45°$$

from which

$$W_A = 34271.0 \text{ kN}.$$

Again, working from first principles and taking moments about A for the portion AC of the cable

$$H \times 30 - 120 \times 150 \times 75 = 0$$

which gives

$$H = 45000 \text{ kN}$$

The horizontal component of the tension in the cable at A is equal to H and the vertical component is equal to $120 \times 150 = 18000$ kN. Then the maximum tension in the cable is

$$T_{max} = \sqrt{(45000^2 + 18000^2)} = 48466.5 \text{ kN}$$

The cable diameter then follows as before and the angle, α, the cable makes with the horizontal at the top of the tower is given by

$$\alpha = \tan^{-1}(18000/45000) = 21.8°$$

Since the cable passes over frictionless pulleys the tension in the anchor cable is equal to the tension in the suspension cable. The resultant horizontal force on the top of a tower is then

$$\text{Resultant horizontal force} = 48466.5(\cos 21.8° - \cos 45°)$$

The bending moment at the base of a tower is then given by

$$M_T = 48466.5(\cos 21.8° - \cos 45°) = 536473.4 \text{ kNm}$$

The direct load at the base of a tower is

$$V_T = 48466.5(\sin 21.8° + \sin 45°) = 52269.9 \text{ kN}$$

Finally, the weight of the anchor block is given by

$$W_A = 48466.5 \cos 45° = 34271.0 \text{ kN}$$

PROBLEMS

P.5.1 Calculate the tension in each segment of the cable shown in Fig. P.5.1 and also the vertical distance of the points B and E below the support points A and F.

Ans. $T_{AB} = T_{EF} = 26.9$ kN, $T_{CB} = T_{ED} = 25.5$ kN, $T_{CD} = 25.0$ kN, 1.0 m.

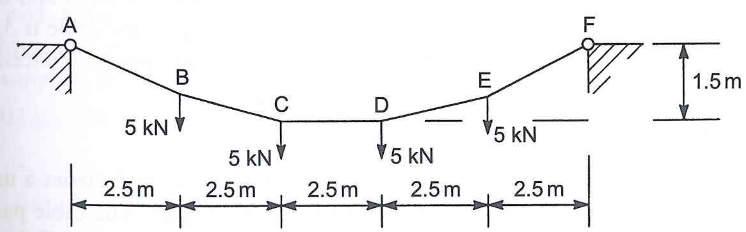


FIGURE P.5.1

P.5.2 Calculate the sag at the point B in the cable shown in Fig. P.5.2 and the tension in each of its segments.

Ans. 0.81 m relative to A. $T_{AB} = 4.9$ kN, $T_{BC} = 4.6$ kN, $T_{DC} = 4.7$ kN.

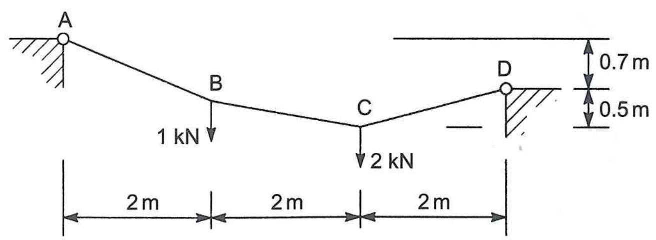


FIGURE P.5.2

P.5.3 Calculate the sag, relative to A, of the points C and D in the cable shown in Fig. P.5.3. Determine also the tension in each of its segments.

Ans. C = 4.2 m, D = 3.1 m, $T_{AB} = 10.98$ kN, $T_{BC} = 9.68$ kN, $T_{CD} = 9.43$ kN.

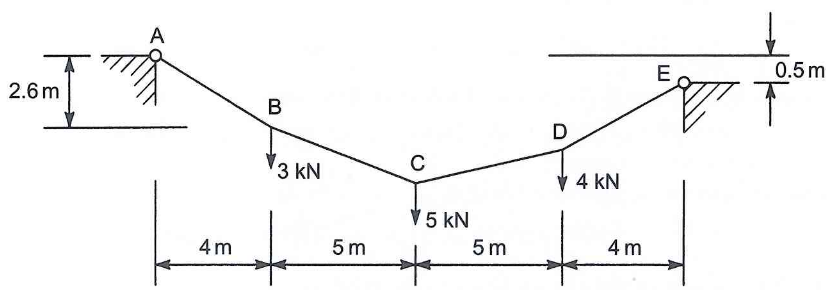


FIGURE P.5.3

P.5.4 A cable that carries a uniform horizontally distributed load of 10 kN/m is suspended between two points that are at the same level and 80 m apart. Determine the minimum sag that may be allowed at mid-span if the maximum tension in the cable is limited to 1000 kN.

Ans. 8.73 m.

P.5.5 A suspension cable is suspended from two points 102 m apart and at the same horizontal level. The self-weight of the cable can be considered to be equivalent to 36 N/m of horizontal length. If the cable carries two concentrated loads each of 10 kN at 34 m and 68 m horizontally from the left-hand support and the maximum sag in the cable is 3 m, determine the maximum tension in the cable and the vertical distance between the concentrated loads and the supports.

Ans. 129.5 kN, 2.96 m.

P.5.6 A cable of a suspension bridge has a span of 80 m, a sag of 8 m and carries a uniform horizontally distributed load of 24 kN/m over the complete span. The cable passes over frictionless pulleys at the top of each tower which are of the same height. If the anchor cables

inclination of the anchor cables to the horizontal. Calculate also the maximum tension in the cable and the vertical force on a tower.

Ans. 21.8°, 2584.9 kN, 1919.9 kN.

P.5.7 A suspension cable passes over saddles supported by roller bearings on the top of two towers 120 m apart and differing in height by 2.5 m. The maximum sag in the cable is 10 m and each anchor cable is inclined at 55° to the horizontal. If the cable carries a uniform horizontally distributed load of 25 kN/m and is to be made of steel having an allowable tensile stress of 240 N/mm^2, determine its minimum diameter. Calculate also the vertical load on the tallest tower.

Ans. 218.7 mm, 8990.9 kN.

P.5.8 A suspension cable has a sag of 40 m and is fixed to two towers of the same height and 400 m apart; the effective cross-sectional area of the cable is 0.08 m^2. However, due to corrosion, the effective cross-sectional area of the central half of the cable is reduced by 20%. If the stress in the cable is limited to 500 N/mm^2, calculate the maximum allowable distributed load the cable can support. Calculate also the inclination of the cable to the horizontal at the top of the towers.

Ans. 62.8 kN/m, 21.8°.

P.5.9 A suspension bridge with two main cables has a span of 250 m and a sag of 25 m. It carries a uniform horizontally distributed load of 25 kN/m and the allowable stress in the cables is 800 N/mm^2. If each anchor cable makes an angle of 45° with the towers, calculate:

a. the required cross-sectional area of the cables,

b. the load in an anchor cable and the overturning force on a tower, when

  i. the cables run over a pulley device,

  ii. the cables are attached to a saddle resting on rollers.

Ans. (a) 5259 mm^2, (b) (i) 4207.2 kN, 931.3 kN (ii) 5524.3 kN, 0.

P.5.10 A suspension cable passes over two towers 80 m apart and carries a load of 5 kN/m of span. If the top of the left-hand tower is 4 m below the top of the right-hand tower and the maximum sag in the cable is 16 m, calculate the maximum tension in the cables. Also, if the cable passes over saddles on rollers on the tops of the towers with the anchor cable at 45° to the horizontal, calculate the vertical thrust on the right-hand tower.

Ans. 358.3 kN, 501.5 kN.

P.5.11 A footbridge 2 m wide spans a 25 m wide river. The deck is supported by two cables and the loading on the deck is 7 kN/m^2. Find the greatest and least tension in the cables and the inclination of the cables at the towers if the dip is 3 m. If the cables pass over pulleys at the top of each tower and the anchor cables are inclined at 60° to the horizontal calculate the total thrust and maximum bending moment for a tower height of 7 m. Note that a single tower supports the cables at each end of the footbridge.

Ans. 202 kN, 182 kN, 25°64′, 525 kN, 1137 kNm.

CHAPTER

Arches

6

The Romans were the first to use arches as major structural elements, employing them, mainly in semicircular form, in bridge and aqueduct construction and for roof supports, particularly the barrel vault. Their choice of the semicircular shape was due to the ease with which such an arch could be set out. Generally these arches, as we shall see, carried mainly compressive loads and were therefore constructed from stone blocks, or *voussoirs*, where the joints were either dry or used weak mortar.

During the Middle Ages, Gothic arches, distinguished by their pointed apex, were used to a large extent in the construction of the great European cathedrals. The horizontal thrust developed at the supports, or *springings*, and caused by the tendency of an arch to 'flatten' under load was frequently resisted by *flying buttresses*. This type of arch was also used extensively in the 19th century.

In the 18th century masonry arches were used to support bridges over the large number of canals that were built in that period. Many of these bridges survive to the present day and carry loads unimagined by their designers.

Today arches are usually made of steel or of reinforced or prestressed concrete and can support both tensile as well as compressive loads. They are used to support bridge decks and roofs and vary in span from a few metres in a roof support system to several hundred metres in bridges. A fine example of a steel arch bridge is the Sydney harbour bridge in which the deck is supported by hangers suspended from the arch (see Fig. 1.6(a) and (b) for examples of bridge decks supported by arches).

Arches are constructed in a variety of forms. Their components may be straight or curved, but generally fall into two categories. The first, which we shall consider in this chapter, is the three-pinned arch which is statically determinate, whereas the second, the two-pinned arch, is statically indeterminate and will be considered in Chapter 16.

Initially we shall examine the manner in which arches carry loads.

6.1 The linear arch

There is a direct relationship between the action of a flexible cable in carrying loads and the action of an arch. In Section 5.1 we determined the tensile forces in the segments of lightweight cables carrying concentrated loads and saw that the geometry of a cable changed under different loading systems; hence, for example, the two geometries of the same cable in Fig. 5.2(a) and (b).

Let us suppose that the cable in Fig. 5.2(a) is made up of three bars or links AC, CD and DB hinged together at C and D and pinned to the supports at A and B. If the loading remains unchanged the deflected shape of the three-link structure will be identical to that of the cable in Fig. 5.2(a) and is shown in Fig. 6.1(a). Furthermore the tension in a link will be exactly the same as the tension in the corresponding segment of the cable. Now suppose that the three-link structure of Fig. 6.1(a) is inverted as shown in Fig. 6.1(b) and that the loads W_1 and W_2 are applied as before. In this situation the forces in the links will be identical in magnitude to those in Fig. 6.1(a) but will now be compressive as opposed to tensile; the structure shown in Fig. 6.1(b) is patently an arch.

The same argument can be applied to any cable and loading system so that the internal forces in an arch may be deduced by analysing a cable having exactly the same shape and carrying identical loads, a fact first realized by Robert Hooke in the 17th century. As in the example in Fig. 6.1 the internal forces in the arch will have the same magnitude as the corresponding cable forces but will be compressive, not tensile.

It is obvious from the above that the internal forces in the arch act along the axes of the different components and that the arch is therefore not subjected to internal shear forces and bending moments; an arch in which the internal forces are purely axial is called a *linear arch*. We also deduce, from Section 5.2, that the internal forces in an arch whose shape is that of a parabola and which carries a uniform horizontally distributed load are purely axial. Further, it will now have become clear why the internal members of a bowstring truss (Section 4.1) carrying loads of equal magnitude along its upper chord joints carry zero force.

However, there is a major difference between the behaviour of the two structures in Fig. 6.1(a) and (b). A change in the values of the loads W_1 and W_2 will merely result in a change in the geometry of the structure in Fig. 6.1(a), whereas the slightest changes in the values of W_1 and W_2 in Fig. 6.1(b) will result in the collapse of the arch as a mechanism. In this particular case collapse could be prevented by replacing the pinned joint at C (or D) by a rigid joint as shown in Fig. 6.2. The forces in the members remain unchanged since the geometry of the structure is unchanged, but the arch is now stable and has become a *three-pinned arch* which, as we shall see, is statically determinate.

If now the pinned joint at D was replaced by a rigid joint, the forces in the members would remain the same, but the arch has become a *two-pinned arch*. In this case, because of the tension cable equivalence, the arch is statically determinate. It is important to realize, however, that the above arguments only apply for the set of loads W_1 and W_2 which produce the particular shape of cable shown in Fig. 6.1(a). If the loads were repositioned or changed in magnitude, the two-pinned arch would become statically indeterminate and would probably cease to be a linear arch so that bending moments and shear forces

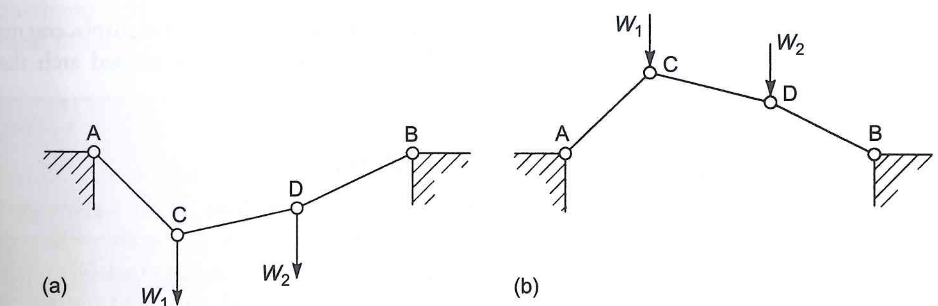


FIGURE 6.1

Equivalence of cable and arch structures.

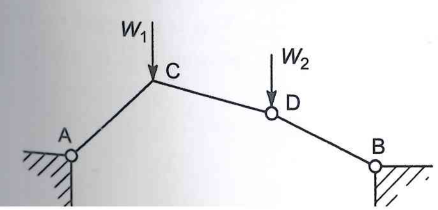


FIGURE 6.2

would be induced. The three-pinned arch of Fig. 6.2 would also become non-linear if the loads were repositioned or changed in magnitude.

In the above we have ignored the effect on the geometry of the arch caused by the shortening of the members. The effect of this on the three-pinned arch is negligible since the pins can accommodate the small changes in angle between the members which this causes. This is not the case in a two-pinned arch or in an arch with no pins at all (in effect a portal frame) so that bending moments and shear forces are induced. However, so long as the loads (W_1 and W_2 in this case) remain unchanged in magnitude and position, the corresponding stresses are 'secondary' and will have little effect on the axial forces.

The linear arch, in which the internal forces are purely axial, is important for the structural designer since the linear arch shape gives the smallest stresses. If, however, the thrust line is not axial, bending stresses are induced and these can cause tension on the inner or outer faces (the *intrados* and *extrados*) of the arch. In a masonry arch in which the joints are either dry or made using a weak mortar, this can lead to cracking and possible failure. Furthermore, if the thrust line lies outside the faces of the arch, instability leading to collapse can also occur. We shall deduce in Section 9.2 that for no tension to be developed in a rectangular cross section, the compressive force on the section must lie within the middle third of the section.

In small-span arch bridges, these factors are not of great importance since the greatest loads on the arch come from vehicular traffic. These loads vary with the size of the vehicle and its position on the bridge, so that it is generally impossible for the designer to achieve a linear arch. On the other hand, in large-span arch bridges, the self-weight of the arch forms the major portion of the load the arch has to carry. In Section 5.2 we saw that a cable under its own weight takes up the shape of a catenary. It follows that the ideal shape for an arch of constant thickness is an inverted catenary. However, in the analysis of the three-pinned arch we shall assume a general case in which shear forces and bending moments, as well as axial forces, are present.

6.2 The three-pinned arch

A three-pinned arch would be used in situations where there is a possibility of support displacement; this, in a two-pinned arch, would induce additional stresses. In the analysis of a three-pinned arch the first step, generally, is to determine the support reactions.

Support reactions — supports on same horizontal level

Consider the arch shown in Fig. 6.3. It carries an inclined concentrated load, W, at a given point D, a horizontal distance a from the support point A. The equation of the shape of the arch will generally be known so that the position of specified points on the arch, say D, can be obtained. We shall suppose that the third pin is positioned at the crown, C, of the arch, although this need not necessarily be the case; the height or *rise* of the arch is h.

The supports at A and B are pinned but neither can be a roller support or the arch would collapse. Therefore, in addition to the two vertical components of the reactions at A and B, there will be horizontal components $R_{A,H}$ and $R_{B,H}$. Thus, there are four unknown components of reaction but only three equations of overall equilibrium (Eq. (2.10)) so that an additional equation is required. This is obtained from the fact that the third pin at C is unable to transmit bending moments although, obviously, it is able to transmit shear forces.

Then, from the overall vertical equilibrium of the arch in Fig. 6.3, we have

$$R_{A,V} + R_{B,V} - W\cos\alpha = 0 \tag{6.1}$$

and from the horizontal equilibrium

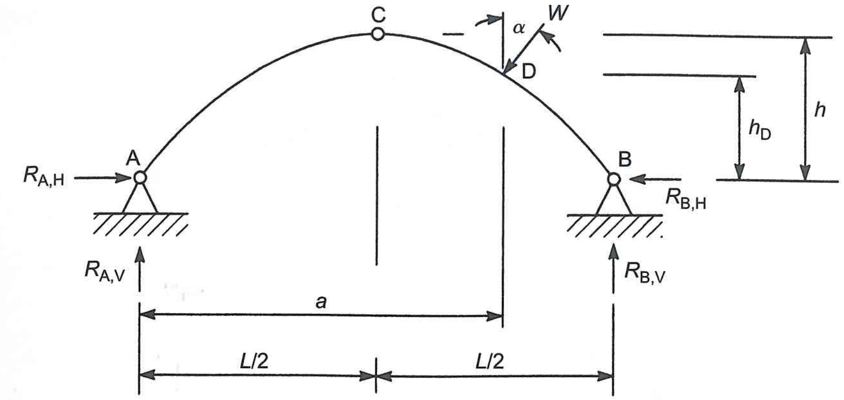


FIGURE 6.3

Three-pinned arch.

$$R_{A,H} - R_{B,H} - W\sin\alpha = 0 \tag{6.2}$$

Now taking moments about, say, B,

$$R_{A,V}L - W\cos\alpha(L-a) - W\sin\alpha h_D = 0 \tag{6.3}$$

The internal moment at C is zero so that we can take moments about C of forces to the left or right of C. A slightly simpler expression results by considering forces to the left of C, i.e.

$$R_{A,V}\frac{L}{2} - R_{A,H}h = 0 \tag{6.4}$$

Equations (6.1)–(6.4) enable the four components of reaction to be found; the normal force, shear force and bending moment at any point in the arch follow.

EXAMPLE 6.1

Calculate the normal force, shear force and bending moment at the point X in the semicircular arch shown in Fig. 6.4.

In this example we can find either vertical component of reaction directly by taking moments about one of the support points. Hence, taking moments about B, say,

$$R_{A,V} \times 12 - 60\,(6\cos 30^\circ + 6) - 100\,(6\sin 30^\circ + 6) = 0$$

which gives

$$R_{A,V} = 131.0\ \text{kN}$$

Now resolving forces vertically

$$R_{B,V} + R_{A,V} - 60 - 100 = 0$$

which, on substituting for $R_{A,V}$, gives

$$R_{B,V} = 29.0\ \text{kN}$$

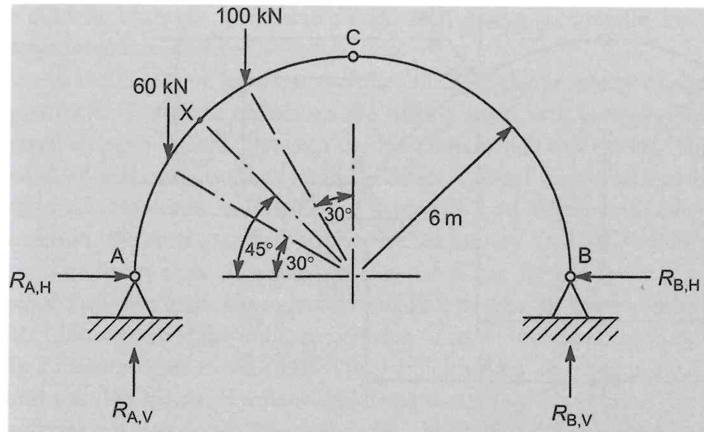


FIGURE 6.4

Three-pinned arch of Ex. 6.1.

Since no horizontal loads are present, we see by inspection that

$$R_{A,H} = R_{B,H}$$

Finally, taking moments of forces to the right of C about C (this is a little simpler than considering forces to the left of C) we have

$$R_{B,H} \times 6 - R_{B,V} \times 6 = 0$$

from which

$$R_{B,H} = 29.0 \text{ kN} = R_{A,H}$$

The normal force at the point X is obtained by resolving the forces to one side of X in a direction tangential to the arch at X. Thus, considering forces to the left of X and taking tensile forces as positive

$$N_X = -R_{A,V} \cos 45^\circ - R_{A,H} \sin 45^\circ + 60 \cos 45^\circ$$

so that

$$N_X = -70.7 \text{ kN}$$

and is compressive.

The shear force at X is found by resolving the forces to one side of X in a direction perpendicular to the tangent at X. We shall take a positive shear force as acting radially inwards when it is to the left of a section. So, considering forces to the left of X

$$S_X = -R_{A,V} \sin 45^\circ + R_{A,H} \cos 45^\circ + 60 \sin 45^\circ$$

which gives

$$S_X = -29.7 \text{ kN}$$

Now taking moments about X for forces to the left of X and regarding a positive moment as causing tension on the underside of the arch, we have

$$M_X = R_{A,V} (6 - 6 \cos 45^\circ) - R_{A,H} \times 6 \sin 45^\circ - 60 (6 \cos 30^\circ - 6 \cos 45^\circ)$$

from which

$$M_X = +50.0 \text{ kNm}$$

Note that in Ex. 6.1 the sign conventions adopted for normal force, shear force and bending moment are the same as those specified in Chapter 3.

Support reactions – supports on different levels

In the three-pinned arch shown in Fig. 6.5 the support at B is a known height, h_B, above A. Let us suppose that the equation of the shape of the arch is known so that all dimensions may be calculated. Now, resolving forces vertically gives

$$R_{A,V} + R_{B,V} - W \cos \alpha = 0 \quad (6.5)$$

and horizontally we have

$$R_{A,H} - R_{B,H} - W \sin \alpha = 0 \quad (6.6)$$

Also, taking moments about B, say,

$$R_{A,V} L - R_{A,H} h_B - W \cos \alpha (L - a) - W \sin \alpha (h_D - h_B) = 0 \quad (6.7)$$

Note that, unlike the previous case, the horizontal component of the reaction at A is included in the overall moment equation (Eq. (6.7)).

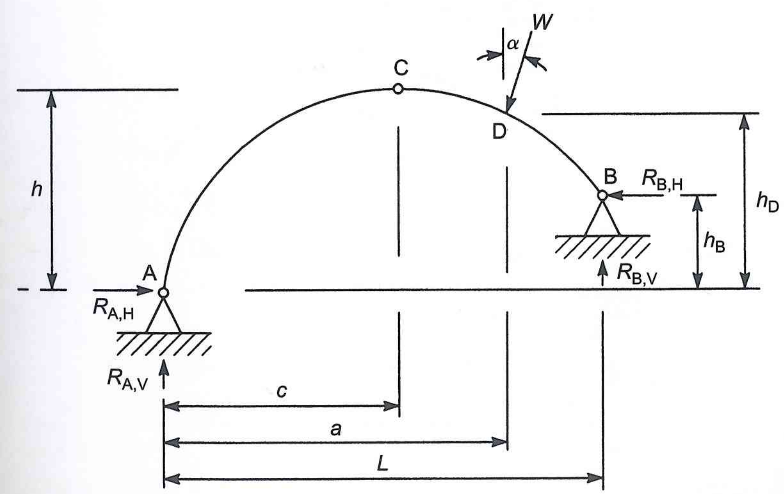


FIGURE 6.5

Three-pinned

Finally we can take moments of all the forces to the left or right of C about C since the internal moment at C is zero. In this case the overall moment equation (Eq. (6.7)) includes both components, $R_{A,V}$ and $R_{A,H}$, of the support reaction at A. If we now consider moments about C of forces to the left of C, we shall obtain a moment equation in terms of $R_{A,V}$ and $R_{A,H}$. This equation, with Eq. (6.7), provides two simultaneous equations which may be solved for $R_{A,V}$ and $R_{A,H}$. Alternatively if, when we were considering the overall moment equilibrium of the arch, we had taken moments about A, Eq. (6.7) would have been expressed in terms of $R_{B,V}$ and $R_{B,H}$. Then we would obtain the fourth equation by taking moments about C of the forces to the right of C and the two simultaneous equations would be in terms of $R_{B,V}$ and $R_{B,H}$. Theoretically this approach is not necessary but it leads to a simpler solution. Referring to Fig. 6.5

$$R_{A,V}c - R_{A,H}h = 0 \tag{6.8}$$

The solution of Eqs (6.7) and (6.8) gives $R_{A,V}$ and $R_{A,H}$, then $R_{B,V}$ and $R_{B,H}$ follow from Eqs (6.5) and (6.6), respectively.

EXAMPLE 6.2

The parabolic arch shown in Fig. 6.6 carries a uniform horizontally distributed load of intensity 10 kN/m over the portion AC of its span. Calculate the values of the normal force, shear force and bending moment at the point D.

Initially we must determine the equation of the arch so that the heights of B and D may be calculated. The simplest approach is to choose the origin of axes at C so that the equation of the parabola may be written in the form

$$y = kx^2 \tag{i}$$

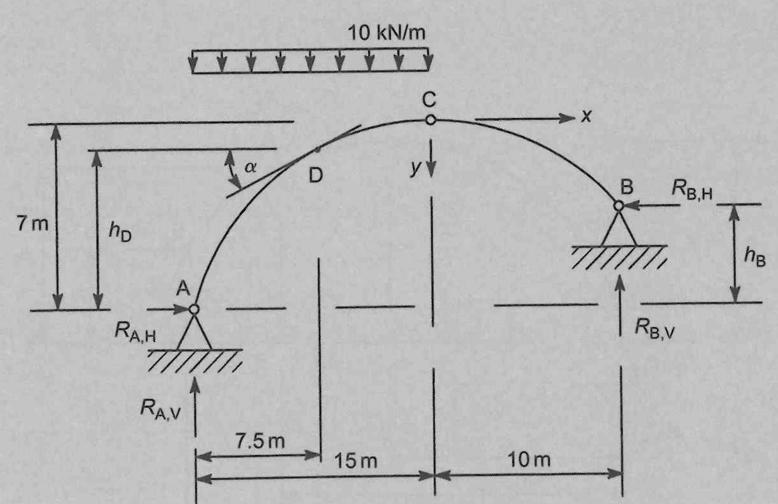


FIGURE 6.6

Parabolic arch of Ex. 6.2.

in which k is a constant. At A, $y = 7$ m when $x = -15$ m. Hence, from Eq. (i)

$$7 = k \times (-15)^2$$

whence

$$k = 0.0311$$

and Eq. (i) becomes

$$y = 0.0311x^2 \tag{ii}$$

Then

$$y_B = 0.0311 \times (10)^2 = 3.11 \text{ m}$$

Hence

$$h_B = 7 - 3.11 = 3.89 \text{ m}$$

Also

$$y_D = 0.0311 \times (-7.5)^2 = 1.75 \text{ m}$$

so that

$$h_D = 7 - 1.75 = 5.25 \text{ m}$$

Taking moments about A for the overall equilibrium of the arch we have

$$R_{B,V} \times 25 + R_{B,H} \times 3.89 - 10 \times 15 \times 7.5 = 0$$

which simplifies to

$$R_{B,V} + 0.16R_{B,H} - 45.0 = 0 \tag{iii}$$

Now taking moments about C for the forces to the right of C we obtain

$$R_{B,V} \times 10 - R_{B,H} \times 3.11 = 0$$

which gives

$$R_{B,V} - 0.311R_{B,H} = 0 \tag{iv}$$

The simultaneous solution of Eqs (iii) and (iv) gives

$$R_{B,V} = 29.7 \text{ kN} \quad R_{B,H} = 95.5 \text{ kN}$$

From the horizontal equilibrium of the arch we have

$$R_{A,H} = R_{B,H} = 95.5 \text{ kN}$$

and from the vertical equilibrium

$$R_{A,V} + R_{B,V} - 10 \times 15 = 0$$

which gives

$$R_{A,V} = 120.3 \text{ kN}$$

To calculate the normal force and shear force at the point D we require the slope of the arch at D. From Eq. (ii)

$$\left(\frac{dy}{dx}\right)_D = 2 \times 0.0311 \times (-7.5) = -0.4665 = -\tan\alpha$$

Hence

$$\alpha = 25.0°$$

Now resolving forces to the left (or right) of D in a direction parallel to the tangent at D we obtain the normal force at D. Hence

$$N_D = -R_{A,V}\sin 25.0° - R_{A,H}\cos 25.0° + 10 \times 7.5 \sin 25.0°$$

which gives

$$N_D = -105.7 \text{ kN (compression)}$$

The shear force at D is then

$$S_D = -R_{A,V}\cos 25.0° + R_{A,H}\sin 25.0° + 10 \times 7.5 \cos 25.0°$$

so that

$$S_D = -0.7 \text{ kN}$$

Finally the bending moment at D is

$$M_D = R_{A,V} \times 7.5 - R_{A,H} \times 5.25 - 10 \times 7.5 \times \frac{7.5}{2}$$

from which

$$M_D = +119.6 \text{ kNm}$$

6.3 A three-pinned parabolic arch carrying a uniform horizontally distributed load

In Section 5.2 we saw that a flexible cable carrying a uniform horizontally distributed load took up the shape of a parabola. It follows that a three-pinned parabolic arch carrying the same loading would experience zero shear force and bending moment at all sections. We shall now investigate the bending moment in the symmetrical three-pinned arch shown in Fig. 6.7.

The vertical components of the support reactions are, from symmetry,

$$R_{A,V} = R_{B,V} = \frac{wL}{2}$$

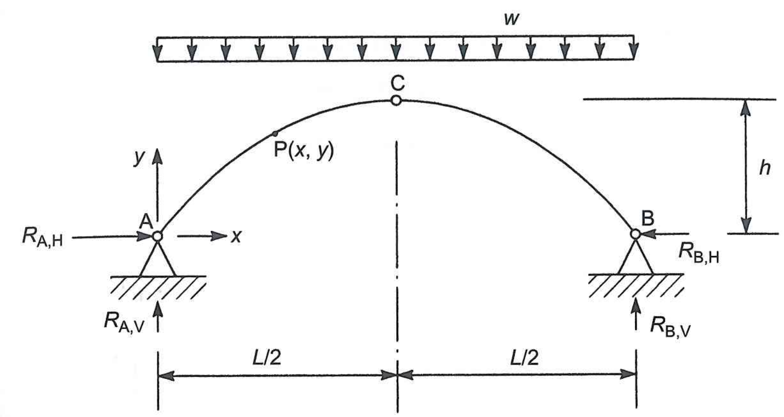


FIGURE 6.7
Parabolic arch carrying a uniform horizontally distributed load.

Also, in the absence of any horizontal loads

$$R_{A,H} = R_{B,H}$$

Now taking moments of forces to the left of C about C,

$$R_{A,H}h - R_{A,V}\frac{L}{2} + \frac{wL}{2}\frac{L}{4} = 0$$

which gives

$$R_{A,H} = \frac{wL^2}{8h}$$

With the origin of axes at A, the equation of the parabolic shape of the arch may be shown to be

$$y = \frac{4h}{L^2}(Lx - x^2)$$

The bending moment at any point P(*x*,*y*) in the arch is given by

$$M_P = R_{A,V}x - R_{A,H}y - \frac{wx^2}{2}$$

or, substituting for $R_{A,V}$ and $R_{A,H}$ and for y in terms of x,

$$M_P = \frac{wL}{2}x - \frac{wL^2}{8h}\frac{4h}{L^2}(Lx - x^2) - \frac{wx^2}{2}$$

Simplifying this expression

$$M_P = \frac{wL}{2}x - \frac{wL}{2}x + \frac{wx^2}{2} - \frac{wx^2}{2} = 0$$

as expected.

6.4 Bending moment diagram for a three-pinned arch

Consider the arch shown in Fig. 6.8; we shall suppose that the equation of the arch referred to the xy axes is known. The load W is applied at a given point $D(x_D, y_D)$ and the support reactions may be calculated by the methods previously described. The bending moment, M_{P1}, at any point $P_1(x,y)$ between A and D is given by

$$M_{P1} = R_{A,V}x - R_{A,H}y \tag{6.9}$$

and the bending moment, M_{P2}, at the point P_2, (x,y) between D and B is

$$M_{P2} = R_{A,V}x - W(x - x_D) - R_{A,H}y \tag{6.10}$$

Now let us consider a simply supported beam AB having the same span as the arch and carrying a load, W, at the same horizontal distance, x_D, from the left-hand support (Fig. 6.9(a)). The vertical reactions, R_A

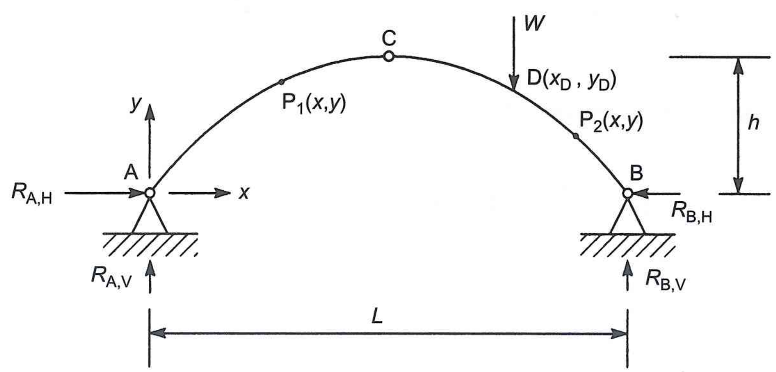


FIGURE 6.8

Determination of the bending moment diagram for a three-pinned arch.

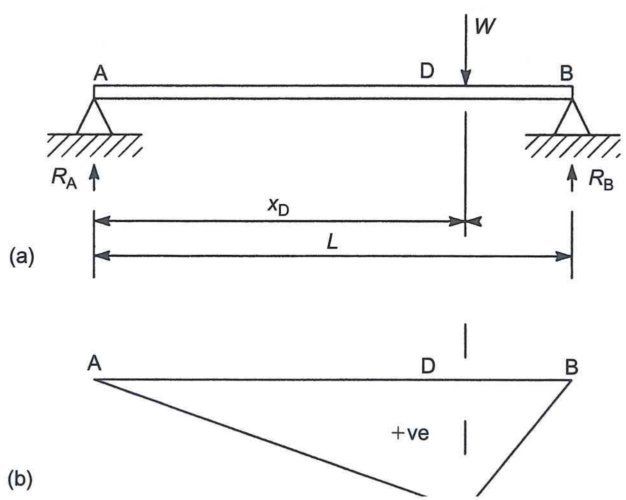


FIGURE 6.9

Bending moment diagram for a simply supported beam (tension on undersurface

and R_B will have the same magnitude as the vertical components of the support reactions in the arch. Thus the bending moment at any point between A and D and a distance x from A is

$$M_{AD} = R_A x = R_{A,V}x \tag{6.11}$$

Also the bending moment at any point between D and B a distance x from A is

$$M_{DB} = R_A x - W(x - x_D) = R_{A,V}x - W(x - x_D) \tag{6.12}$$

giving the bending moment diagram shown in Fig. 6.9(b). Comparing Eqs (6.11) and (6.12) with Eqs (6.9) and (6.10), respectively, we see that Eq. (6.9) may be written

$$M_{P1} = M_{AD} - R_{A,H}y \tag{6.13}$$

and Eq. (6.10) may be written

$$M_{P2} = M_{DB} - R_{A,H}y \tag{6.14}$$

Therefore, the complete bending moment diagram for the arch may be regarded as the sum of a 'simply supported beam' bending moment diagram and an 'arch' bending moment diagram in which the 'arch' diagram has the same shape as the arch itself, since its ordinates are equal to a constant multiplied by y. The two bending moment diagrams may be superimposed as shown in Fig. 6.10 to give the complete bending moment diagram for the arch. Note that the curve of the arch forms the baseline of the bending moment diagram and that the bending moment at the crown of the arch where the third pin is located is zero.

In the above it was assumed that the mathematical equation of the curve of the arch is known. However, in a situation where, say, only a scale drawing of the curve of the arch is available, a semigraphical

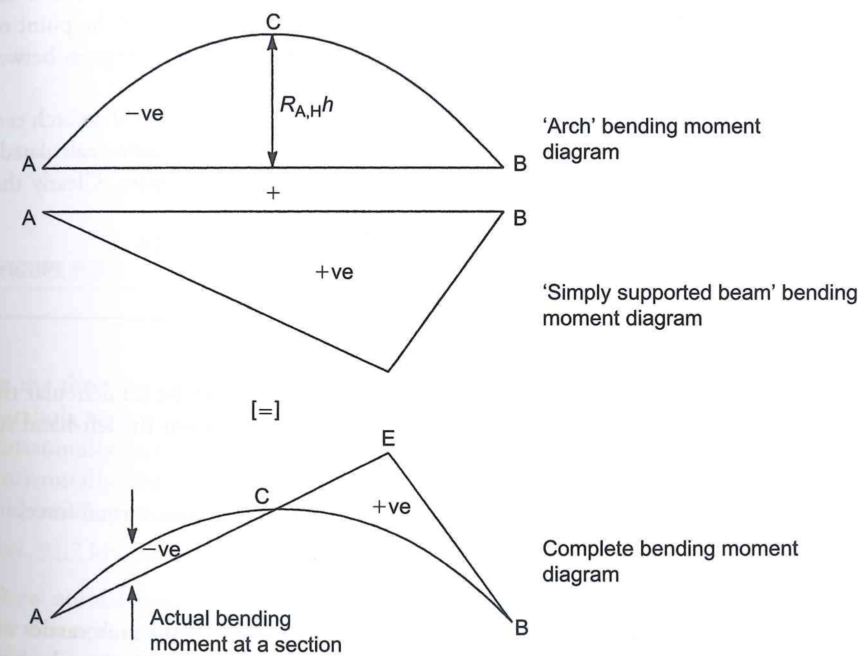


FIGURE 6.10

Complete bending

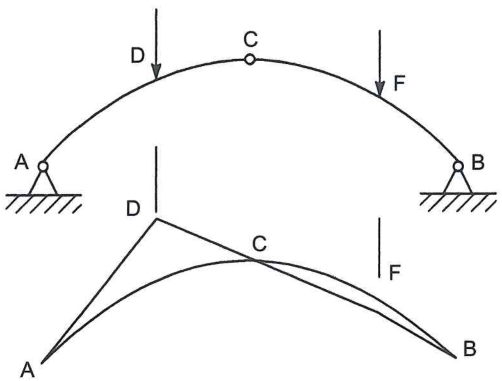


FIGURE 6.11

Bending moment diagram for a three-pinned arch carrying two loads.

procedure may be adopted if the loads are vertical. The 'arch' bending moment at the crown C of the arch is $R_{AH}h$ as shown in Fig. 6.10. The magnitude of this bending moment may be calculated so that the scale of the bending moment diagram is then fixed by the rise (at C) of the arch in the scale drawing. Also this bending moment is equal in magnitude but opposite in sign to the 'simply supported beam' bending moment at this point. Other values of 'simply supported beam' bending moment may be calculated at, say, load positions and plotted on the complete bending moment diagram to the already determined scale. The diagram is then completed, enabling values of bending moment to be scaled off as required.

In the arch of Fig. 6.8 a simple construction may be used to produce the complete bending moment diagram. In this case the arch shape is drawn as in Fig. 6.10 and this, as we have seen, fixes the scale of the bending moment diagram. Then, since the final bending moment at C is zero and is also zero at A and B, a line drawn from A through C to meet the vertical through the point of application of the load at E represents the 'simply supported beam' bending moment diagram between A and D. The bending moment diagram is then completed by drawing the line EB.

This construction is only possible when the arch carries a single load. In the case of an arch carrying two or more loads as in Fig. 6.11, the 'simply supported beam' bending moments must be calculated at D and F and their values plotted to the same scale as the 'arch' bending moment diagram. Clearly the bending moment at C remains zero.

We shall consider the statically indeterminate two-pinned arch in Chapter 16.

PROBLEMS

P.6.1 Determine the value of the bending moment in the loaded half of the semicircular three-pinned arch shown in Fig. P.6.1 at a horizontal distance of 5 m from the left-hand support.

Ans. 67.0 kN m (sagging).

P.6.2 Figure P.6.2 shows a three-pinned arch of radius 12 m. Calculate the normal force, shear force and bending moment at the point D.

Ans. 14.4 kN (compression), 5.5 kN, 21.6 kN m (hogging).

P.6.3 The three-pinned arch shown in Fig. P.6.3 is parabolic in shape. If the arch carries a uniform horizontally distributed load of intensity 40 kN/m over the part CB, calculate the bending moment at D.

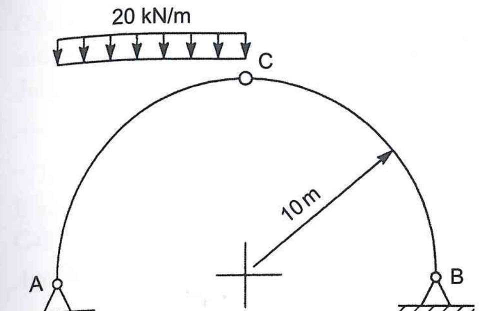


FIGURE P.6.1

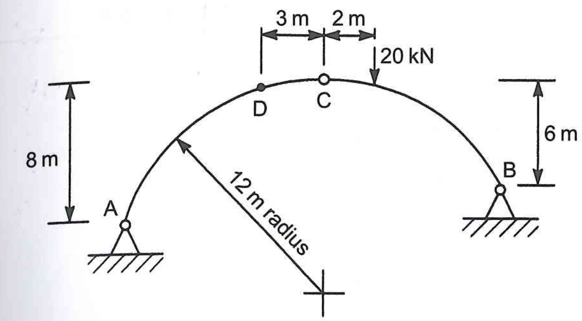


FIGURE P.6.2

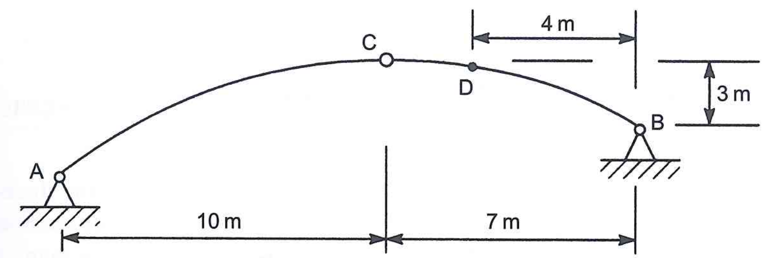


FIGURE P.6.3

P.6.4 In the three-pinned arch ACB shown in Fig. P.6.4 the portion AC has the shape of a parabola with its origin at C, while CB is straight. The portion AC carries a uniform horizontally distributed load of intensity 30 kN/m, while the portion CB carries a uniform horizontally distributed load of intensity 18 kN/m. Calculate the normal force, shear force and bending moment at the point D.

Ans. 91.2 kN (compression), 9.0 kN, 209.8 kN m (sagging).

P.6.5 Draw normal force, shear force and bending moment diagrams for the loaded half of the three-pinned arch shown in Fig. P.6.5.

Ans. $N_{BD} = 26.5$ kN, $N_{DE} = 19.4$ kN, $N_{EF} = N_{FC} = 15$ kN (all compression).
$S_{BD} = 5.3$ kN, $S_{DE} = -1.8$ kN, $S_{EF} = 2.5$ kN, $S_{FC} = -7.5$ kN.

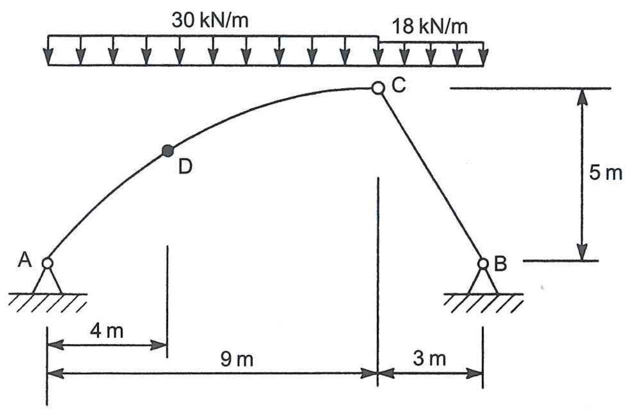


FIGURE P.6.4

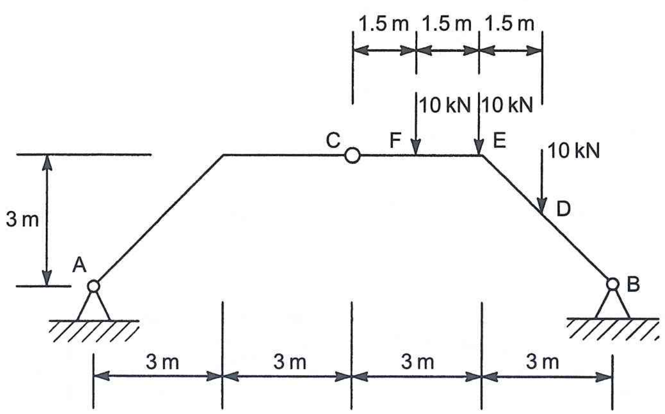
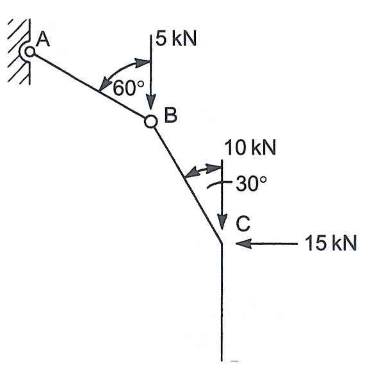


FIGURE P.6.5

P.6.6 Calculate the components of the support reactions at A and D in the three-pinned arch shown in Fig. P.6.6 and hence draw the bending moment diagram for the member DC; draw the diagram on the tension side of the member. All members are 1.5 m long.

Ans. $R_{A,V} = 6.46$ kN, $R_{A,H} = 11.13$ kN, $R_{D,V} = 21.46$ kN, $R_{D,H} = 3.87$ kN.
$M_D = 0$, $M_C = 5.81$ kN m (tension on left of CD).



P.6.7 Calculate the reactions at the supports A and D in the three-pinned arch shown in Fig. P.6.7 and obtain the bending moment diagram for the members AB and BC. In each case draw the diagram on the tension side of the member.

Ans. $M_A = M_C = 0$, $M_B = 14.1$ kNm.

P.6.8 "The three-pinned arch shown in Fig. P.6.8 carries loads of 60 kN and 120 kN at the points E and D in addition to a distributed load due to its self-weight of 10 kN/m of its true length. Calculate the bending moments at the points E and D and also at the mid-point of AC.

Ans. $M_D = 186.3$ kNm, $M_E = 150.0$ kNm, $M_{\text{mid AC}} = 3.95$ kNm.

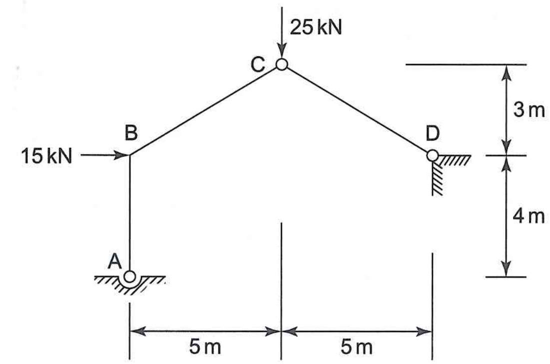


FIGURE P.6.7

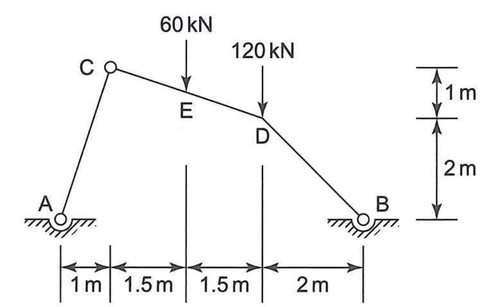


FIGURE P.6.8

CHAPTER

Stress and Strain

7

We are now in a position to calculate internal force distributions in a variety of structural systems, i.e. normal forces, shear forces and bending moments in beams and arches, axial forces in truss members, the tensions in suspension cables and torque distributions in beams. These internal force systems are distributed throughout the cross section of a structural member in the form of stresses. However, although there are four basic types of internal force, there are only two types of stress: one which acts perpendicularly to the cross section of a member and one which acts tangentially. The former is known as a *direct stress*, the latter as a *shear stress.*

The distribution of these stresses over the cross section of a structural member depends upon the internal force system at the section and also upon the geometry of the cross section. In some cases, as we shall see later, these distributions are complex, particularly those produced by the bending and shear of unsymmetrical sections. We can, however, examine the nature of each of these stresses by considering simple loading systems acting on structural members whose cross sections have some degree of symmetry. At the same time we shall define the corresponding strains and investigate the relationships between the two.

7.1 Direct stress in tension and compression

The simplest form of direct stress system is that produced by an axial load. Suppose that a structural member has a uniform 'I' cross section of area A and is subjected to an axial tensile load, P, as shown in Fig. 7.1(a). At any section 'mm' the internal force is a normal force which, from the arguments

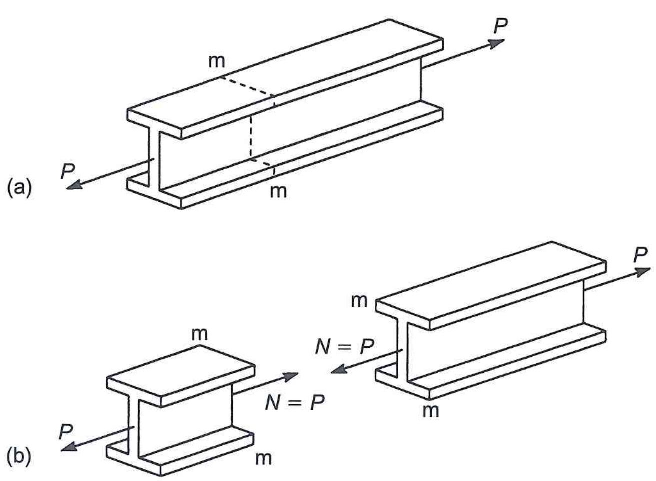


FIGURE 7.1

Structural member with axial load.

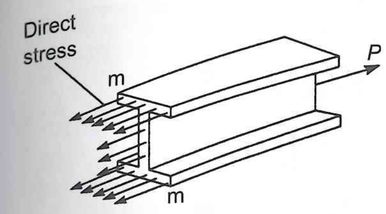


FIGURE 7.2

Internal force distribution in a beam section.

presented in Chapter 3, is equal to P (Fig. 7.1(b)). It is clear that this normal force is not resisted at just one point on each face of the section as Fig. 7.1(b) indicates but at every point as shown in Fig. 7.2. We assume in fact that P is distributed uniformly over the complete face of the section so that at any point in the cross section there is an intensity of force, i.e. stress, to which we give the symbol σ and which we define as

$$\sigma = \frac{P}{A} \tag{7.1}$$

This direct stress acts in the direction shown in Fig. 7.2 when P is tensile and in the reverse direction when P is compressive. The sign convention for direct stress is identical to that for normal force; a tensile stress is therefore positive while a compressive stress is negative. The SI unit of stress is the pascal (Pa) where 1 Pa is 1 N/m^2. However this is a rather small quantity in many cases so generally we shall use mega-pascals (MPa) where 1 MPa = 1 N/mm^2.

In Fig. 7.1 the section mm is some distance from the point of application of the load. At sections in the proximity of the applied load the distribution of direct stress will depend upon the method of application of the load, and only in the case where the applied load is distributed uniformly over the cross section will the direct stress be uniform over sections in this region. In other cases *stress concentrations* arise which require specialized analysis; this topic is covered in more advanced texts on strength of materials and stress analysis.

We shall see in Chapter 8 that it is the level of stress that governs the behaviour of structural materials. For a given material, failure, or breakdown of the crystalline structure of the material under load, occurs at a constant value of stress. For example, in the case of steel subjected to simple tension failure begins at a stress of about 300 N/mm^2, although variations occur in steels manufactured to different specifications. This stress is independent of size or shape and may therefore be used as the basis for the design of structures fabricated from steel. Failure stress varies considerably from material to material and in some cases depends upon whether the material is subjected to tension or compression.

A knowledge of the failure stress of a material is essential in structural design where, generally, a designer wishes to determine a minimum size for a structural member carrying a given load. For example, for a member fabricated from a given material and subjected to axial load, we would use Eq. (7.1) either to determine a minimum area of cross section for a given load or to check the stress level in a given member carrying a given load.

EXAMPLE 7.1

A short column has a rectangular cross section with sides in the ratio 1:2 (Fig. 7.3). Determine the minimum dimensions of the column section if the column carries an axial load of 800 kN and the failure stress of the material of the column is 400 N/mm^2.

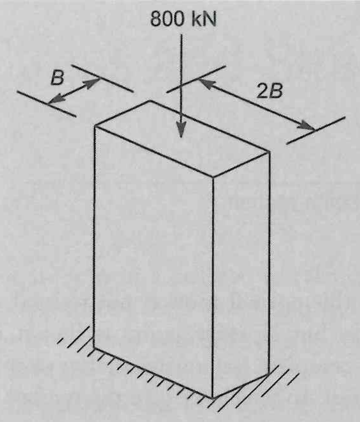


FIGURE 7.3
Column of Ex. 7.1.

From Eq. (7.1) the minimum area of the cross section is given by

$$A_{min} = \frac{P}{\sigma_{max}} = \frac{800 \times 10^3}{400} = 2000 \text{ mm}^2$$

But

$$A_{min} = 2B^2 = 2000 \text{ mm}^2$$

from which

$$B = 31.6 \text{ mm}$$

Therefore the minimum dimensions of the column cross section are 31.6 mm × 63.2 mm. In practice these dimensions would be rounded up to 32 mm × 64 mm or, if the column were of some standard section, the next section having a cross-sectional area greater than 2000 mm^2 would be chosen. Also the column would not be designed to the limit of its failure stress but to a working or design stress which would incorporate some safety factor (see Section 8.7).

7.2 Shear stress in shear and torsion

An externally applied shear load induces an internal shear force which is tangential to the faces of a beam cross section. Figure 7.4(a) illustrates such a situation for a cantilever beam carrying a shear load W at its free end. We have seen in Chapter 3 that the action of W is to cause sliding of one face of the cross section relative to the other; W also induces internal bending moments which produce internal direct stress systems; these are considered in a later chapter. The internal shear force S ($= W$) required to maintain the vertical equilibrium of the portions of the beam is distributed over each face of the cross section. Thus at any point in the cross section there is a tangential intensity of force which is termed *shear stress*. This shear stress is not distributed uniformly over the faces of the cross section as we shall see in Chapter 10. For the moment, however, we shall define the average shear stress over the faces of the cross section as

$$\tau_{av} = \frac{W}{A} \tag{7.2}$$

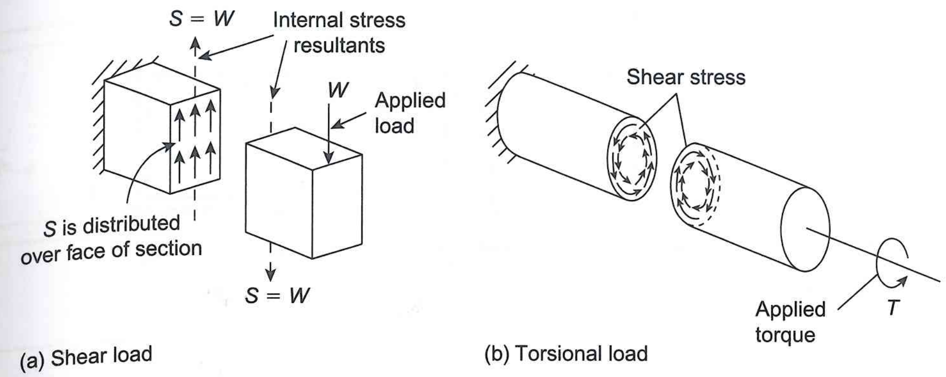


FIGURE 7.4
Generation of shear stresses in beam sections.

Note that the internal shear force S shown in Fig. 7.4(a) is, according to the sign convention adopted in Chapter 3, positive. However, the applied load W would produce an internal shear force in the opposite direction on the positive face of the section so that S would actually be negative.

A system of shear stresses is induced in a different way in the circular-section bar shown in Fig. 7.4 (b) where the internal torque (T) tends to produce a relative rotational sliding of the two faces of the cross section. The shear stresses are tangential to concentric circular paths in the faces of the cross section. We shall examine the shear stress due to torsion in various cross sections in Chapter 11.

7.3 Complementary shear stress

Consider the cantilever beam shown in Fig. 7.5(a). Let us suppose that the beam is of rectangular cross section having a depth h and unit thickness; it carries a vertical shear load W at its free end. The internal shear forces on the opposite faces mm and nn of an elemental length δx of the beam are distributed as shear stresses in some manner over each face as shown in Fig. 7.5(b). Suppose now that we isolate a small rectangular element ABCD of depth δh of this elemental length of beam (Fig. 7.5(c)) and consider its equilibrium. Since the element is small, the shear stresses τ on the faces AD and BC may be regarded as constant. The shear force resultants of these shear stresses clearly satisfy vertical equilibrium of the element but rotationally produce an anticlockwise couple. This must be equilibrated by a clockwise couple which can only be produced by shear forces on the horizontal faces AB and CD of the element. Let τ' be the shear stresses induced by these shear forces. The equilibrium of the element is satisfied in both horizontal and vertical directions since the resultant force in either direction is zero. However, the shear forces on the faces BC and AD form a couple which would cause rotation of the element in an anticlockwise sense. We need, therefore, a clockwise balancing couple and this can only be produced by shear forces on the faces AB and CD of the element; the shear stresses corresponding to these shear forces are τ' as shown. Then for rotational equilibrium of the element about the corner D

$$\tau' \times \delta x \times 1 \times \delta h = \tau \times \delta h \times 1 \times \delta x$$

which gives

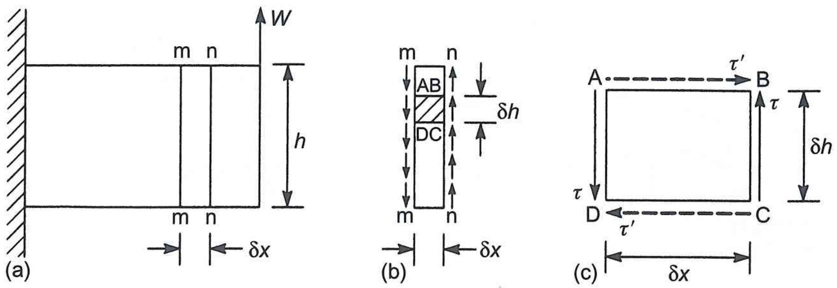


FIGURE 7.5

Complementary shear stress.

We see, therefore, that a shear stress acting on a given plane is always accompanied by an equal *complementary shear stress* acting on planes perpendicular to the given plane and in the opposite sense.

7.4 Direct strain

Since no material is completely rigid, the application of loads produces distortion. An axial tensile load, for example, will cause a structural member to increase in length, whereas a compressive load would cause it to shorten.

Suppose that δ is the change in length produced by either a tensile or compressive axial load. We now define the *direct strain,* ε, in the member in non-dimensional form as the change in length per unit length of the member. Hence

$$\varepsilon = \frac{\delta}{L_0} \tag{7.4}$$

where L_0 is the length of the member in its unloaded state. Clearly ε may be either a tensile (positive) strain or a compressive (negative) strain. Equation (7.4) is applicable only when distortions are relatively small and can be used for values of strain up to and around 0.001, which is adequate for most structural problems. For larger values, load–displacement relationships become complex and are therefore left for more advanced texts.

We shall see in Section 7.7 that it is convenient to measure distortion in this non-dimensional form since there is a direct relationship between the stress in a member and the accompanying strain. The strain in an axially loaded member therefore depends solely upon the level of stress in the member and is independent of its length or cross-sectional geometry.

7.5 Shear strain

In Section 7.3 we established that shear loads applied to a structural member induce a system of shear and complementary shear stresses on any small rectangular element. The distortion in such an element due to these shear stresses does not involve a change in length but a change in shape as shown in Fig. 7.6. We define the *shear strain,* γ, in the element as the change in angle between two originally mutually perpendicular edges. Thus in Fig. 7.6

$$\gamma = \phi \text{ radians} \tag{7.5}$$

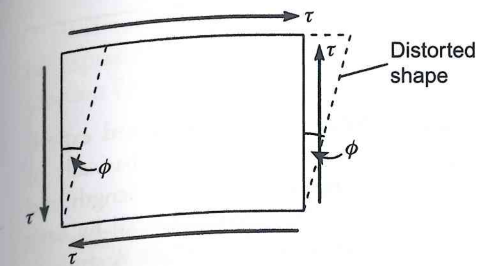


FIGURE 7.6

Shear strain in an element.

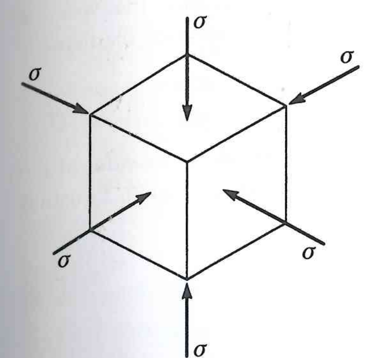


FIGURE 7.7

Cube subjected to hydrostatic pressure.

7.6 Volumetric strain due to hydrostatic pressure

A rather special case of strain which we shall find useful later occurs when a cube of material is subjected to equal compressive stresses, σ, on all six faces as shown in Fig. 7.7. This state of stress is that which would be experienced by the cube if it were immersed at some depth in a fluid, hence the term hydrostatic pressure. The analysis would, in fact, be equally valid if σ were a tensile stress.

Suppose that the original length of each side of the cube is L_0 and that δ is the decrease in length of each side due to the stress. Then, defining the ***volumetric strain*** as the change in volume per unit volume, we have

$$\text{volumetric strain} = \frac{L_0^3 - (L_0 - \delta)^3}{L_0^3}$$

Expanding the bracketed term and neglecting second- and higher-order powers of δ gives

$$\text{volumetric strain} = \frac{3L_0^2\delta}{L_0^3}$$

from which

$$\text{volumetric strain} = \frac{3\delta}{L_0} \tag{7.6}$$

Thus we see that for this case the volumetric strain is three times the linear strain in any of the three

7.7 Stress–strain relationships

Hooke's law and Young's modulus

The relationship between direct stress and strain for a particular material may be determined experimentally by a *tensile test* which is described in detail in Chapter 8. A tensile test consists basically of applying an axial tensile load in known increments to a specimen of material of a given length and cross-sectional area and measuring the corresponding increases in length. The stress produced by each value of load may be calculated from Eq. (7.1) and the corresponding strain from Eq. (7.4). A stress–strain curve is then drawn which, for some materials, would have a shape similar to that shown in Fig. 7.8. Stress–strain curves for other materials differ in detail but, generally, all have a linear portion such as ab in Fig. 7.8. In this region stress is directly proportional to strain, a relationship that was discovered in 1678 by Robert Hooke and which is known as *Hooke's law*. It may be expressed mathematically as

$$\sigma = E\varepsilon \tag{7.7}$$

where E is the constant of proportionality. E is known as *Young's modulus* or the *elastic modulus* of the material and has the same units as stress. For mild steel E is of the order of 200 kN/mm^2. Equation (7.7) may be written in alternative form as

$$\frac{\sigma}{\varepsilon} = E \tag{7.8}$$

For many materials E has the same value in tension and compression.

Shear modulus

By comparison with Eq. (7.8) we can define the *shear modulus* or *modulus of rigidity, G*, of a material as the ratio of shear stress to shear strain; thus

$$G = \frac{\tau}{\gamma} \tag{7.9}$$

Volume or bulk modulus

Again, the *volume modulus* or *bulk modulus, K*, of a material is defined in a similar manner as the ratio of volumetric stress to volumetric strain, i.e.

$$K = \frac{\text{volumetric stress}}{\text{volumetric strain}} \tag{7.10}$$

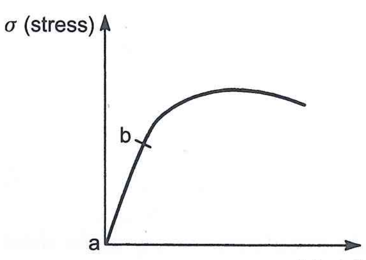


FIGURE 7.8

It is not usual to assign separate symbols to volumetric stress and strain since they may, respectively, be expressed in terms of direct stress and linear strain. Thus in the case of hydrostatic pressure (Section 7.6)

$$K = \frac{\sigma}{3\varepsilon} \tag{7.11}$$

EXAMPLE 7.2

A mild steel column is hollow and circular in cross section with an external diameter of 350 mm and an internal diameter of 300 mm. It carries a compressive axial load of 2000 kN. Determine the direct stress in the column and also the shortening of the column if its initial height is 5 m. Take $E = 200\,000$ N/mm^2.

The cross-sectional area A of the column is given by

$$A = \frac{\pi}{4}(350^2 - 300^2) = 25\,525.4\ \text{mm}^2$$

The direct stress σ in the column is, therefore, from Eq. (7.1)

$$\sigma = -\frac{2000 \times 10^3}{25\,525.4} = -78.4\ \text{N/mm}^2\text{(compression)}$$

The corresponding strain is obtained from either Eq. (7.7) or Eq. (7.8) and is

$$\varepsilon = \frac{-78.4}{200\,000} = -0.000\,39$$

Finally the shortening, δ, of the column follows from Eq. (7.4), i.e.

$$\delta = 0.000\,39 \times 5 \times 10^3 = 1.95\ \text{mm}$$

EXAMPLE 7.3

A short, deep cantilever beam is 500 mm long by 200 mm deep and is 2 mm thick. It carries a vertically downward load of 10 kN at its free end. Assuming that the shear stress is uniformly distributed over the cross section of the beam, calculate the deflection due to shear at the free end. Take $G = 25\,000$ N/mm^2.

The internal shear force is constant along the length of the beam and equal to 10 kN. Since the shear stress is uniform over the cross section of the beam, we may use Eq. (7.2) to determine its value, i.e.

$$\tau_{av} = \frac{W}{A} = \frac{10 \times 10^3}{200 \times 2} = 25\ \text{N/mm}^2$$

This shear stress is constant along the length of the beam; it follows from Eq. (7.9) that the shear strain is also constant along the length of the beam and is given by

$$\gamma = \frac{\tau_{av}}{G} = \frac{25}{25\,000} = 0.001\ \text{rad}$$

This value is in fact the angle that the beam makes with the horizontal. The deflection, Δ_s, due to shear at the free end is therefore

$$\Delta_s = 0.001 \times 500 = 0.5 \text{ mm}$$

In practice, the solution of this particular problem would be a great deal more complex than this since the shear stress distribution is not uniform. Deflections due to shear are investigated in Chapter 13.

7.8 Poisson effect

It is common experience that a material such as rubber suffers a reduction in cross-sectional area when stretched under a tensile load. This effect, known as the *Poisson effect*, also occurs in structural materials subjected to tensile and compressive loads, although in the latter case the cross-sectional area increases. In the region where the stress–strain curve of a material is linear, the ratio of lateral strain to longitudinal strain is a constant which is known as *Poisson's ratio* and is given the symbol v. The effect is illustrated in Fig. 7.9.

Consider now the action of different direct stress systems acting on an elemental cube of material (Fig. 7.10). The stresses are all tensile stresses and are given suffixes which designate their directions in relation to the system of axes specified in Section 3.2. In Fig. 7.10(a) the direct strain, ε_x, in the x direction is obtained directly from either Eq. (7.7) or Eq. (7.8) and is

$$\varepsilon_x = \frac{\sigma_x}{E}$$

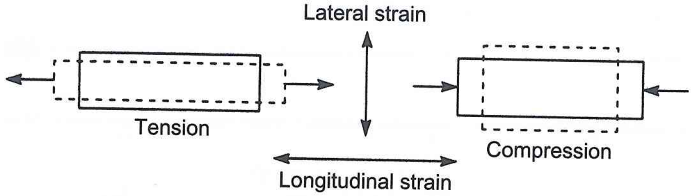


FIGURE 7.9

The Poisson effect.

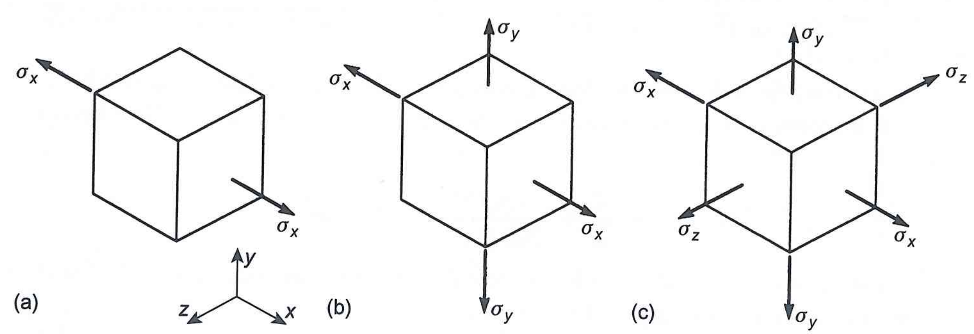


FIGURE 7.10

The Poisson effect in a cube of material.

Due to the Poisson effect there are accompanying strains in the y and z directions given by

$$\varepsilon_y = -v\varepsilon_x \quad \varepsilon_z = -v\varepsilon_x$$

or, substituting for ε_x in terms of σ_x

$$\varepsilon_y = -v\frac{\sigma_x}{E} \quad \varepsilon_z = -v\frac{\sigma_x}{E} \tag{7.12}$$

These strains are negative since they are associated with contractions as opposed to positive strains produced by extensions.

In Fig. 7.10(b) the direct stress σ_y has an effect on the direct strain ε_x as does σ_x on ε_y. Thus

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} \quad \varepsilon_y = \frac{\sigma_y}{E} - \frac{v\sigma_x}{E} \quad \varepsilon_z = -\frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} \tag{7.13}$$

By a similar argument, the strains in the x, y and z directions for the cube of Fig. 7.10(c) are

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{v\sigma_y}{E} - \frac{v\sigma_z}{E} \quad \varepsilon_y = \frac{\sigma_y}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_z}{E} \quad \varepsilon_z = \frac{\sigma_z}{E} - \frac{v\sigma_x}{E} - \frac{v\sigma_y}{E} \tag{7.14}$$

Let us now suppose that the cube of material in Fig. 7.10(c) is subjected to a uniform stress on each face such that $\sigma_x = \sigma_y = \sigma_z = \sigma$. The strain in each of the axial directions is therefore the same and is, from any one of Eq. (7.14)

$$\varepsilon = \frac{\sigma}{E}(1 - 2v)$$

In Section 7.6 we showed that the volumetric strain in a cube of material subjected to equal stresses on all faces is three times the linear strain. Thus in this case

$$\text{volumetric strain} = \frac{3\sigma}{E}(1 - 2v) \tag{7.15}$$

It would be unreasonable to suppose that the volume of a cube of material subjected to tensile stresses on all faces could decrease. It follows that Eq. (7.15) cannot have a negative value. We conclude, therefore, that v must always be less than 0.5. For most metals v has a value in the region of 0.3 while for concrete v can be as low as 0.1.

Collectively E, G, K and v are known as the *elastic constants* of a material.

EXAMPLE 7.4

A cube of material is subjected to the following direct stress system: $\sigma_x = +120$ N/mm^2, $\sigma_y = +80$ N/mm^2 and $\sigma_z = -100$ N/mm^2. If Young's modulus, E, is 200 000 N/mm^2 and Poisson's ratio, ν, is 0.3 calculate the direct strain in the x, y and z directions and hence the volumetric strain in the cube.

The strain in the x direction is given by the first of Eqs (7.14) and is

$$\varepsilon_x = (120 - 0.3 \times 80 + 0.3 \times 100)/200\,000 = 6.3 \times 10^{-4}$$

Similarly, from the second of Eqs (7.14)

$$\varepsilon_y = (80 - 0.3 \times 120 + 0.3 \times 100)/200\,000 = 3.7 \times 10^{-4}$$

and from the third of Eqs (7.14)

$$\varepsilon_z = (-100 - 0.3 \times 120 - 0.3 \times 80)/200\,000 = -8.0 \times 10^{-4}$$

If L_0 is the initial length of each side of the cube then the final lengths are

$$L_x = L_0 + 6.3 \times 10^{-4} L_0 = 1.00063 L_0$$
$$L_y = L_0 + 3.7 \times 10^{-4} L_0 = 1.00037 L_0$$
$$L_z = L_0 - 8.0 \times 10^{-4} L_0 = 0.9992 L_0$$

The volumetric strain in the cube is then

$$\text{Vol. Strain} = [L_0^3 - (1.00063)(1.00037)(0.9992)L_0^3]/L_0^3$$

i.e

$$\text{Vol. Strain} = -1.99 \times 10^{-4}$$

7.9 Relationships between the elastic constants

There are different methods for determining the relationships between the elastic constants. The one presented here is relatively simple in approach and does not require a knowledge of topics other than those already covered.

In Fig. 7.11(a), ABCD is a square element of material of unit thickness and is in equilibrium under a shear and complementary shear stress system τ. Imagine now that the element is 'cut' along the diagonal AC as shown in Fig. 7.11(b). In order to maintain the equilibrium of the triangular portion ABC it is possible that a direct force and a shear force are required on the face AC. These forces, if they exist, will be distributed over the face of the element in the form of direct and shear stress systems, respectively. Since the element is small, these stresses may be assumed to be constant along the face AC. Let the direct stress on AC in the direction BD be σ_{BD} and the shear stress on AC be τ_{AC}. Then resolving forces on the element in the direction BD we have

$$\sigma_{BD} AC \times 1 - \tau AB \times 1 \times \cos 45^\circ - \tau BC \times 1 \times \cos 45^\circ = 0$$

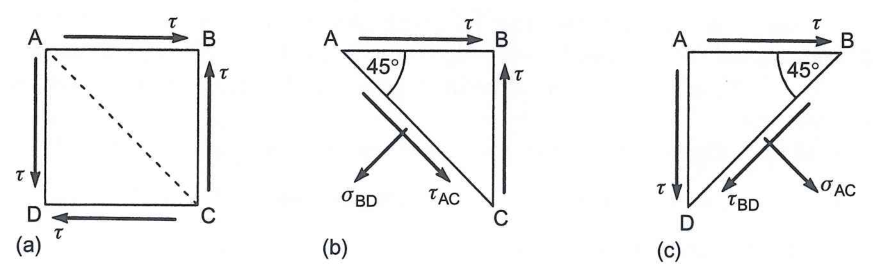


FIGURE 7.11

Determination of the relationships between the elastic constants.

Dividing through by AC

$$\sigma_{BD} = \tau \frac{AB}{AC} \cos 45^\circ + \tau \frac{BC}{AC} \cos 45^\circ$$

or

$$\sigma_{BD} = \tau \cos^2 45^\circ + \tau \cos^2 45^\circ$$

from which

$$\sigma_{BD} = \tau \qquad (7.16)$$

The positive sign indicates that σ_{BD} is a tensile stress. Similarly, resolving forces in the direction AC

$$\tau_{AC} AC \times 1 + \tau AB \times 1 \times \cos 45^\circ - \tau BC \times 1 \times \cos 45^\circ = 0$$

Again dividing through by AC we obtain

$$\tau_{AC} = -\tau \cos^2 45^\circ + \tau \cos^2 45^\circ = 0$$

A similar analysis of the triangular element ABD in Fig. 7.11(c) shows that

$$\sigma_{AC} = -\tau \qquad (7.17)$$

and

$$\tau_{BD} = 0$$

Hence we see that on planes parallel to the diagonals of the element there are direct stresses σ_{BD} (tensile) and σ_{AC} (compressive) both numerically equal to τ as shown in Fig. 7.12. It follows from Section 7.8 that the direct strain in the direction BD is given by

$$\varepsilon_{BD} = \frac{\sigma_{BD}}{E} + \frac{v\sigma_{AC}}{E} = \frac{\tau}{E}(1 + v) \qquad (7.18)$$

Note that the compressive stress σ_{AC} makes a positive contribution to the strain ε_{BD}.

In Section 7.5 we defined shear strain and saw that under pure shear, only a change of shape is involved. Thus the element ABCD of Fig. 7.11(a) distorts into the shape A′B′CD shown in Fig. 7.13. The shear strain γ produced by the shear stress τ is then given by

$$\gamma = \phi \text{ radians} = \frac{B'B}{BC} \qquad (7.19)$$

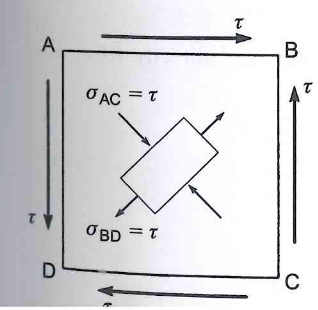


FIGURE 7.12

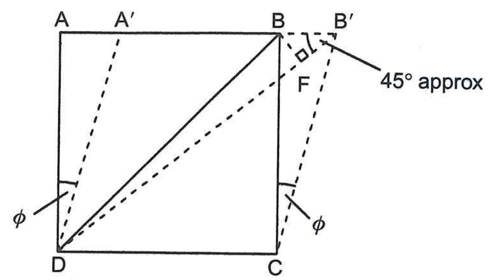


FIGURE 7.13

Distortion due to shear in element.

since ϕ is a small angle. The increase in length of the diagonal DB to DB′ is approximately equal to FB′ where BF is perpendicular to DB′. Thus

$$\varepsilon_{DB} = \frac{DB' - DB}{DB} = \frac{FB'}{DB}$$

Again, since ϕ is a small angle, $B\hat{B}'F \simeq 45°$ so that

$$FB' = BB' \cos 45°$$

Also

$$DB = \frac{BC}{\cos 45°}$$

Hence

$$\varepsilon_{DB} = \frac{B'B \cos^2 45°}{BC} = \frac{1}{2}\frac{B'B}{BC}$$

Therefore, from Eq. (7.19)

$$\varepsilon_{DB} = \frac{1}{2}\gamma \tag{7.20}$$

Substituting for ε_{DB} in Eq. (7.18) we obtain

$$\frac{1}{2}\gamma = \frac{\tau}{E}(1 + v)$$

or, since $\tau/\gamma = G$ from Eq. (7.9)

$$G = \frac{E}{2(1 + v)} \quad \text{or} \quad E = 2G(1 + v) \tag{7.21}$$

The relationship between Young's modulus E and bulk modulus K is obtained directly from Eqs (7.10) and (7.15). Thus, from Eq. (7.10)

$$\text{volumetric strain} = \frac{\sigma}{K}$$

where σ is the volumetric stress. Substituting in Eq. (7.15)

$$\frac{\sigma}{K} = \frac{3\sigma}{E}(1 - 2v)$$

from which

$$K = \frac{E}{3(1 - 2v)} \tag{7.22}$$

Eliminating E from Eqs (7.21) and (7.22) gives

$$K = \frac{2G(1 + v)}{3(1 - 2v)} \tag{7.23}$$

EXAMPLE 7.5

If the cube in Ex. 7.4 carries shear and complementary shear stresses of 60 N/mm^2 in the xy plane calculate the corresponding shear strains.

From Eqs (7.21)

$$G = 200\,000/2(1 + 0.3) = 76923\ \text{N/mm}^2$$

Then, from Eq. (7.9)

$$\gamma_{xy} = 60/76923 = 7.8 \times 10^{-4}$$

EXAMPLE 7.6

A cube of material is subjected to a compressive stress σ on each of its faces. If $v = 0.3$ and $E = 200\,000$ N/mm^2, calculate the value of this stress if the volume of the cube is reduced by 0.1%. Calculate also the percentage reduction in length of one of the sides.

From Eq. (7.22)

$$K = \frac{200\,000}{3(1 - 2 \times 0.3)} = 167\,000\ \text{N/mm}^2$$

The volumetric strain is 0.001 since the volume of the block is reduced by 0.1%.

Therefore, from Eq. (7.10)

$$0.001 = \frac{\sigma}{K}$$

or

$$\sigma = 0.001 \times 167\,000 = 167\ \text{N/mm}^2$$

In Section 7.6 we established that the volumetric strain in a cube subjected to a uniform stress on all six faces is three times the linear strain. Thus in this case

$$\text{linear strain} = \frac{1}{3} \times 0.001 = 0.00033$$

The length of one side of the cube is therefore reduced by 0.033%.

EXAMPLE 7.7

A mild steel column of height 5 m is hollow and circular in cross section with an external diameter of 400 mm and a wall thickness of 20 mm. If the column carries a compressive load of 2500 kN calculate the direct stress in the column and also the shortening of the column. Take $E = 200\ 000\ \text{N/mm}^2$. If the ends of the column are then fixed so that no further axial movement is possible calculate the total direct stress in the column when it is subjected to a temperature rise of 20 *K*. The coefficient of linear expansion of the material of the column is 0.00005/*K* and with the usual notation $L = L_0(1 + \alpha T)$.

The cross sectional area, *A*, of the column is given by

$$A = \pi(400^2 - 360^2)/4 = 23876.1\ \text{mm}^2$$

Then, from Eq. (7.1)

$$\sigma = 2500 \times 10^3/23876.1 = 104.7\ \text{N/mm}^2(\text{compression})$$

The direct strain in the column is obtained from either of Eqs (7.7) or (7.8) and is

$$\varepsilon = \sigma/E = 104.7/200\ 000 = 0.00052$$

so that the shortening, δ, of the column is, from Eq. (7.4), given by

$$\delta = 0.00052 \times 5 \times 10^3 = 2.62\ \text{mm}$$

Due to the temperature rise the column would, if not prevented, increase in height with no corresponding change in stress. However, this change in height is prevented thereby causing, in effect, a strain and an accompanying stress. Now $L = L_0(1 + \alpha T)$ so that the change in height due to a temperature increase would be $L - L_0$ which is equal to $L_0\alpha T$. The strain corresponding to the suppression of this change in height is then $(L - L_0)/L_0 = \alpha T$. The accompanying direct stress is then, from Eq. (7.7), given by

$$\sigma = 0.00005 \times 20 \times 200\ 000 = 200\ \text{N/mm}^2$$

which is compressive since the increase in height is prevented. The total stress in the column produced by the load and the temperature rise is then

$$\sigma(\text{total}) = 200 + 104.7 = 304.7\ \text{N/mm}^2$$

7.10 Strain energy in simple tension or compression

An important concept in the analysis of structures is that of *strain energy*. The total strain energy of a structural member may comprise the separate strain energies due to axial load, bending moment, shear and torsion. In this section we shall concentrate on the strain energy due to tensile or compressive loads; the strain energy produced by each of the other loading systems is considered in the relevant, later chapters.

A structural member subjected to a gradually increasing tensile load *P* gradually increases in length (Fig. 7.14(a)). The load–extension curve for the member is linear until the limit of proportionality is exceeded, as shown in Fig. 7.14(b). The geometry of the non-linear portion of the curve depends upon the properties of the material of the member (see Chapter 8). Clearly the load *P* moves through small

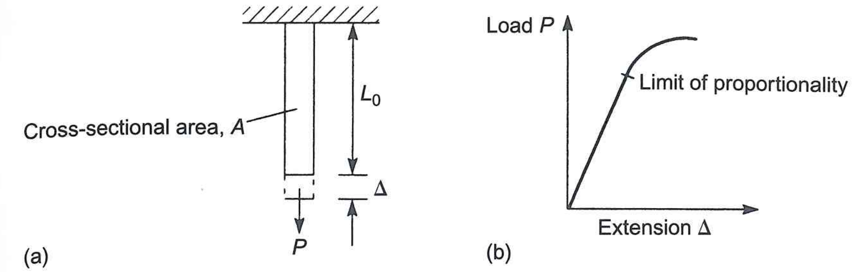


FIGURE 7.14

Load–extension curve for an axially loaded member.

extend, is stored in the member as strain energy. If the value of *P* is restricted so that the limit of proportionality is not exceeded, the gradual removal of *P* results in the member returning to its original length and the strain energy stored in the member may be recovered in the form of work. When the limit of proportionality is exceeded, not all of the work done by *P* is recoverable; some is used in producing a permanent distortion of the member (see Chapter 8), the related energy appearing largely as heat.

Suppose the structural member of Fig. 7.14(a) is gradually loaded to some value of *P* within the limit of proportionality of the material of the member, the corresponding elongation being Δ. Let the elongation corresponding to some intermediate value of load, say P_1, be Δ_1 (Fig. 7.15). Then a small increase in load of δP_1 will produce a small increase, $\delta\Delta_1$, in elongation. The incremental work done in producing this increment in elongation may be taken as equal to the average load between P_1 and $P_1 + \delta P_1$ multiplied by $\delta\Delta_1$. Thus

$$\text{incremental work done} = \left[\frac{P_1 + (P_1 + \delta P_1)}{2}\right]\delta\Delta_1$$

which, neglecting second-order terms, becomes

$$\text{incremental work done} = P_1\delta\Delta_1$$

The total work done on the member by the load *P* in producing the elongation Δ is therefore given by

$$\text{total work done} = \int_0^{\Delta} P_1\ \text{d}\Delta_1 \tag{7.24}$$

Since the load–extension relationship is linear, then

$$P_1 = K\Delta_1 \tag{7.25}$$

where *K* is some constant whose value depends upon the material properties of the member. Substituting the particular values of *P* and Δ in Eq. (7.25), we obtain

$$K = \frac{P}{\Delta}$$

so that Eq. (7.25) becomes

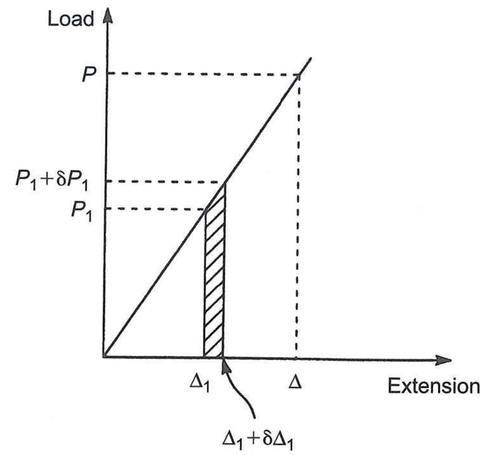


FIGURE 7.15

Work done by a gradually applied load.

Now substituting for P_1 in Eq. (7.24) we have

$$\text{total work done} = \int_0^{\Delta} \frac{P}{\Delta}\Delta_1 \mathrm{d}\Delta_1$$

Integration of this equation yields

$$\text{total work done} = \frac{1}{2}P\Delta \tag{7.26}$$

Alternatively, we see that the right-hand side of Eq. (7.24) represents the area under the load–extension curve, so that again we obtain

$$\text{total work done} = \frac{1}{2}P\Delta$$

By the law of conservation of energy, the total work done is equal to the strain energy, U, stored in the member. Thus

$$U = \frac{1}{2}P\Delta \tag{7.27}$$

The direct stress, σ, in the member of Fig. 7.14(a) corresponding to the load P is given by Eq. (7.1), i.e.

$$\sigma = \frac{P}{A}$$

Also the direct strain, ε, corresponding to the elongation Δ is, from Eq. (7.4)

$$\varepsilon = \frac{\Delta}{L_0}$$

Furthermore, since the load–extension curve is linear, the direct stress and strain are related by Eq. (7.7), so that

$$\frac{P}{A} = E\frac{\Delta}{L_0}$$

from which

$$\Delta = \frac{PL_0}{AE} \tag{7.28}$$

In Eq. (7.28) the quantity L_0/AE determines the magnitude of the displacement produced by a given load; it is therefore known as the *flexibility* of the member. Conversely, by transposing Eq. (7.28) we see that

$$P = \frac{AE}{L_0}\Delta$$

in which the quantity AE/L_0 determines the magnitude of the load required to produce a given displacement. The term AE/L_0 is then the *stiffness* of the member.

Substituting for Δ in Eq. (7.27) gives

$$U = \frac{P^2L_0}{2AE} \tag{7.29}$$

It is often convenient to express strain energy in terms of the direct stress σ. Rewriting Eq. (7.29) in the form

$$U = \frac{1}{2}\frac{P^2}{A^2}\frac{AL_0}{E}$$

we obtain

$$U = \frac{\sigma^2}{2E} \times AL_0 \tag{7.30}$$

in which we see that AL_0 is the volume of the member. The strain energy per unit volume of the member is then

$$\frac{\sigma^2}{2E}$$

The greatest amount of strain energy per unit volume that can be stored in a member without exceeding the limit of proportionality is known as the *modulus of resilience* and is reached when the direct stress in the member is equal to the direct stress corresponding to the elastic limit of the material of the member.

The strain energy, U, may also be expressed in terms of the elongation, Δ, or the direct strain, ε. Thus, substituting for P in Eq. (7.29)

$$U = \frac{EA\Delta^2}{2L_0} \tag{7.31}$$

or, substituting for σ in Eq. (7.30)

$$U = \frac{1}{2}E\varepsilon^2 \times AL_0 \tag{7.32}$$

The above expressions for strain energy also apply to structural members subjected to compressive loads since the work done by P in Fig. 7.14(a) is independent of the direction of movement of P. It fol-

The concept of strain energy has numerous and wide ranging applications in structural analysis particularly in the solution of statically indeterminate structures. We shall examine in detail some of the uses of strain energy later but here we shall illustrate its use by applying it to some relatively simple structural problems.

Deflection of a simple truss

The truss shown in Fig. 7.16 carries a gradually applied load W at the joint A. Considering the vertical equilibrium of joint A

$$P_{AB} \cos 45° - W = 0$$

so that

$$P_{AB} = 1.41\, W \text{ (tension)}$$

Now resolving forces horizontally at A

$$P_{AC} + P_{AB} \cos 45° = 0$$

which gives

$$P_{AC} = -W \text{ (compression)}$$

It is obvious from inspection that P_{AC} is a compressive force but, for consistency, we continue with the convention adopted in Chapter 4 for solving trusses where all members are assumed, initially, to be in tension.

The strain energy of each member is then, from Eq. (7.29)

$$U_{AB} = \frac{(1.41W)^2 \times 1.41L}{2AE} = \frac{1.41W^2L}{AE}$$

$$U_{AC} = \frac{W^2L}{2AE}$$

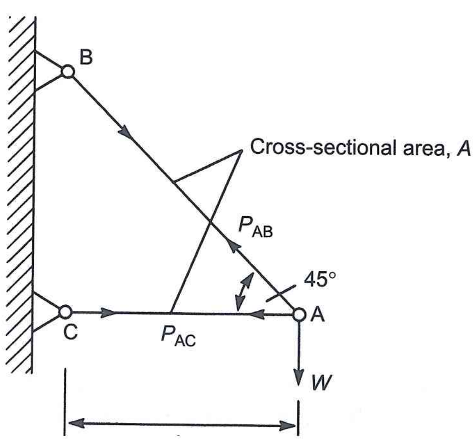


FIGURE 7.16

If the *vertical* deflection of A is Δ_v, the work done by the gradually applied load, W, is

$$\frac{1}{2} W\Delta_v$$

Then equating the work done to the total strain energy of the truss we have

$$\frac{1}{2} W\Delta_v = \frac{1.41W^2L}{AE} + \frac{W^2L}{2AE}$$

so that

$$\Delta_v = \frac{3.82WL}{AE}$$

Using strain energy to calculate deflections in this way has limitations. In the above example Δ_V, is, in fact, only the vertical component of the actual deflection of the joint A since A moves horizontally as well as vertically. Therefore we can only find the deflection of a load *in its own line of action* by this method. Furthermore, the method cannot be applied to structures subjected to more than one applied load as each load would contribute to the total work done by moving through an unknown displacement in its own line of action. There would, therefore, be as many unknown displacements as loads in the work–energy equation. We shall return to examine energy methods in much greater detail in Chapter 15.

EXAMPLE 7.8

Calculate the vertical displacement of the joint C in the truss shown in Fig. 7.17. All members have a cross sectional area of 500 mm² and a Young's modulus of 200 000 N/mm².

The forces in the members of the truss may be found using the method of joints and are as follows:

$$F_{BC} = +282.8 \text{ kN},\ F_{CD} = -200.0 \text{ kN},\ F_{BD} = 0,\ F_{DE} = -200.0 \text{ kN},$$
$$F_{BE} = -282.8 \text{ kN},\ F_{AB} = +400 \text{ kN}.$$

The lengths of BE and BC are each $= \sqrt{(1.0^2 + 1.0^2)} = 1.41$ m. Then, from Eq. (7.29) the total strain energy of the truss is given by

$$U = (282.8^2 \times 1.41 + 200.0^2 \times 1.0 + 200.0^2 \times 1.0 + 282.8^2 \times 1.41 + 400.0^2 \times 1.0) \times 10^9/2 \times 500 \times 200\,000$$

which gives

$$U = 2.33 \times 10^6 \text{ Nmm.}$$

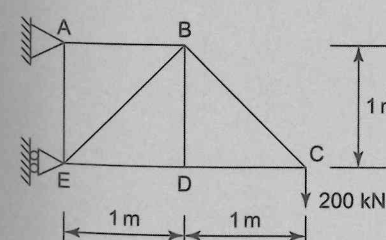


FIGURE 7.17
Truss of Ex. 7.8

This is equal to the work done by the load as it moves through the vertical displacement, Δ_V. Then

$$200 \times 10^3 \Delta_V/2 = 2.33 \times 10^6$$

so that

$$\Delta_V = 23.3 \text{ mm.}$$

Composite structural members

Axially loaded composite members are of direct interest in civil engineering where concrete columns are reinforced by steel bars and steel columns are frequently embedded in concrete as a fire precaution.

In Fig. 7.18 a concrete column of cross-sectional area A_C is reinforced by two steel bars having a combined cross-sectional area A_S. The modulus of elasticity of the concrete is E_C and that of the steel E_S. A load P is transmitted to the column through a plate which we shall assume is rigid so that the deflection of the concrete is equal to that of the steel. It follows that their respective strains are equal since both have the same original length. Since E_C is not equal to E_S we see from Eq. (7.7) that the compressive stresses, σ_C and σ_S, in the concrete and steel, respectively, must have different values. This also means that unless A_C and A_S have particular values, the compressive loads, P_C and P_S, in the concrete and steel are also different. The problem is therefore statically indeterminate since we can write down only one equilibrium equation, i.e.

$$P_C + P_S = P \tag{7.33}$$

The second required equation derives from the fact that the displacements of the steel and concrete are identical since, as noted above, they are connected by the rigid plate; this is a *compatibility of displacement* condition. Then, from Eq. (7.28)

$$\frac{P_C L}{A_C E_C} = \frac{P_S L}{A_S E_S} \tag{7.34}$$

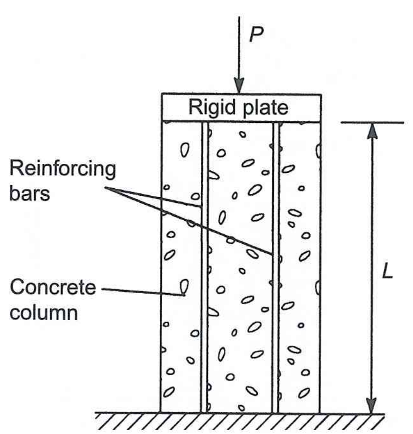


FIGURE 7.18
Composite concrete column

Substituting for P_C from Eq. (7.34) in Eq. (7.33) gives

$$P_S\left(\frac{A_C E_C}{A_S E_S} + 1\right) = P$$

from which

$$P_S = \frac{A_S E_S}{A_C E_C + A_S E_S} P \tag{7.35}$$

P_C follows directly from Eqs (7.34) and (7.35), i.e.

$$P_C = \frac{A_C E_C}{A_C E_C + A_S E_S} P \tag{7.36}$$

The vertical displacement, δ, of the column is obtained using either side of Eq. (7.34) and the appropriate compressive load, P_C or P_S. Thus

$$\delta = \frac{PL}{A_C E_C + A_S E_S} \tag{7.37}$$

The direct stresses in the steel and concrete are obtained from Eqs (7.35) and (7.36), thus

$$\sigma_S = \frac{E_S}{A_C E_C + A_S E_S} P \quad \sigma_C = \frac{E_C}{A_C E_C + A_S E_S} P \tag{7.38}$$

We could, in fact, have solved directly for the stresses by writing Eqs (7.33) and (7.34) as

$$\sigma_C A_C + \sigma_S A_S = P \tag{7.39}$$

and

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \tag{7.40}$$

respectively.

EXAMPLE 7.9

A reinforced concrete column, 5 m high, has the cross section shown in Fig. 7.19. It is reinforced by four steel bars each 20 mm in diameter and carries a load of 1000 kN. If Young's modulus for steel is 200 000 N/mm^2 and that for concrete is 15 000 N/mm^2, calculate the stress in the steel and in the concrete and also the shortening of the column.

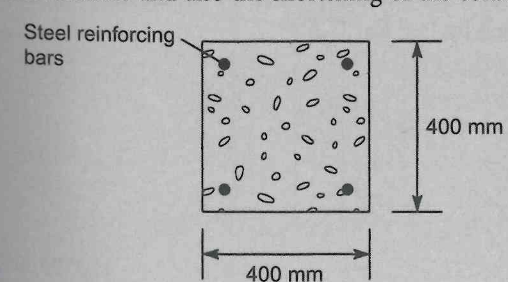


FIGURE 7.19
Reinforced concrete column of Ex. 7.9.

The total cross-sectional area, A_S, of the steel reinforcement is

$$A_S = 4 \times \frac{\pi}{4} \times 20^2 = 1257 \text{ mm}^2$$

The cross-sectional area, A_C, of the concrete is reduced due to the presence of the steel and is given by

$$A_C = 400^2 - 1257 = 158\,743 \text{ mm}^2$$

Equations (7.38) then give

$$\sigma_S = \frac{200\,000 \times 1000 \times 10^3}{158\,743 \times 15\,000 + 1257 \times 200\,000} = 76.0 \text{ N/mm}^2$$

$$\sigma_C = \frac{15\,000 \times 1000 \times 10^3}{158\,743 \times 15\,000 + 1257 \times 200\,000} = 5.7 \text{ N/mm}^2$$

The deflection, δ, of the column is obtained using either side of Eq. (7.40). Thus

$$\delta = \frac{\sigma_C L}{E_C} = \frac{5.7 \times 5 \times 10^3}{15\,000} = 1.9 \text{ mm}$$

Thermal effects

It is possible for stresses to be induced by temperature changes in composite members which are additional to those produced by applied loads. These stresses arise when the components of a composite member have different rates of thermal expansion and contraction.

First, let us consider a member subjected to a uniform temperature rise, ΔT, along its length. The member expands from its original length, L_0, to a length, L_T, given by

$$L_T = L_0(1 + \alpha \Delta T)$$

where α is the coefficient of linear expansion of the material of the member. In the condition shown in Fig. 7.20 the member has been allowed to expand freely so that no stresses are induced. The increase in the length of the member is then

$$L_T - L_0 = L_0 \alpha \Delta T$$

Suppose now that expansion is completely prevented so that the final length of the member after the temperature rise is still L_0. The member has, in effect, been compressed by an amount $L_0\alpha\Delta T$, thereby producing a compressive strain, ε, which is given by (see Eq. (7.4))

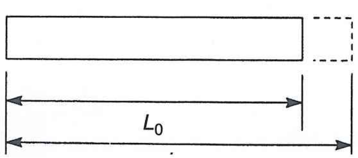


FIGURE 7.20

$$\varepsilon = \frac{L_0 \alpha \Delta T}{L_0} = \alpha \Delta T \tag{7.41}$$

The corresponding compressive stress, σ, is from Eq. (7.7)

$$\sigma = E \alpha \Delta T \tag{7.42}$$

In composite members the restriction on expansion or contraction is usually imposed by the attachment of one component to another. For example, in a reinforced concrete column, the bond between the reinforcing steel and the concrete prevents the free expansion or contraction of either.

Consider the reinforced concrete column shown in Fig. 7.21(a) which is subjected to a temperature rise, ΔT. For simplicity we shall suppose that the reinforcement consists of a single steel bar of cross-sectional area, A_S, located along the axis of the column; the actual cross-sectional area of concrete is A_C. Young's modulus and the coefficient of linear expansion of the concrete are E_C and α_C, respectively, while the corresponding values for the steel are E_S and α_S. We shall assume that $\alpha_S > \alpha_C$.

Figure 7.21(b) shows the positions the concrete and steel would attain if they were allowed to expand freely; in this situation neither material is stressed. The displacements $L_0\alpha_C\Delta T$ and $L_0\alpha_S\Delta T$ are obtained directly from Eq. (7.41). However, since they are attached to each other, the concrete prevents the steel from expanding this full amount while the steel forces the concrete to expand further than it otherwise would; their final positions are shown in Fig. 7.21(c). It can be seen that δ_C is the effective elongation of the concrete which induces a direct tensile load, P_C. Similarly δ_S is the effective contraction of the steel which induces a compressive load, P_S. There is no externally applied load so that the resultant axial load at any section of the column is zero so that

$$P_C \text{ (tension)} = P_S \text{ (compression)} \tag{7.43}$$

Also, from Fig. 7.21(b) and (c) we see that

$$\delta_C + \delta_S = L_0\alpha_S\Delta T - L_0\alpha_C\Delta T$$

or

$$\delta_C + \delta_S = L_0\Delta T(\alpha_S - \alpha_C) \tag{7.44}$$

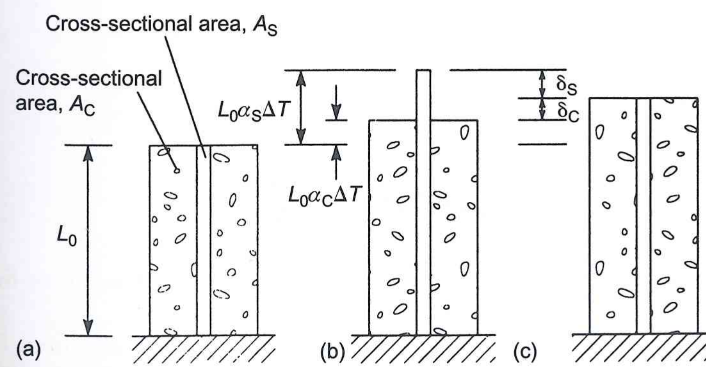


FIGURE 7.21

From Eq. (7.28)

$$\delta_C = \frac{P_C L_0}{A_C E_C} \qquad \delta_S = \frac{P_S L_0}{A_S E_S} \tag{7.45}$$

Substituting for δ_C and δ_S in Eq. (7.44) we obtain

$$\frac{P_C}{A_C E_C} + \frac{P_S}{A_S E_S} = \Delta T(\alpha_S - \alpha_C) \tag{7.46}$$

Simultaneous solution of Eqs (7.43) and (7.46) gives

$$P_C\,(\text{tension}) = P_S\,(\text{compression}) = \frac{\Delta T(\alpha_S - \alpha_C)}{\left(\dfrac{1}{A_C E_C} + \dfrac{1}{A_S E_S}\right)} \tag{7.47}$$

or

$$P_C\,(\text{tension}) = P_S\,(\text{compression}) = \frac{\Delta T(\alpha_S - \alpha_C) A_C E_C A_S E_S}{A_C E_C + A_S E_S} \tag{7.48}$$

The tensile stress, σ_C, in the concrete and the compressive stress, σ_S, in the steel follow directly from Eq. (7.48).

$$\sigma_C = \frac{P_C}{A_C} = \frac{\Delta T(\alpha_S - \alpha_C) E_C A_S E_S}{A_C E_C + A_S E_S}$$

$$\sigma_S = \frac{P_S}{A_S} = \frac{\Delta T(\alpha_S - \alpha_C) A_C E_C E_S}{A_C E_C + A_S E_S} \tag{7.49}$$

From Fig. 7.21(b) and (c) it can be seen that the actual elongation, δ, of the column is given by either

$$\delta = L_0 \alpha_C \Delta T + \delta_C \quad \text{or} \quad \delta = L_0 \alpha_S \Delta T - \delta_S \tag{7.50}$$

Using the first of Eq. (7.50) and substituting for δ_C from Eq. (7.45) then P_C from Eq. (7.48) we have

$$\delta = L_0 \alpha_C \Delta T + \frac{\Delta T(\alpha_S - \alpha_C) A_C E_C A_S E_S L_0}{A_C E_C (A_C E_C + A_S E_S)}$$

which simplifies to

$$\delta = L_0 \Delta T \left(\frac{\alpha_C A_C E_C + \alpha_S A_S E_S}{A_C E_C + A_S E_S} \right) \tag{7.51}$$

Clearly when $\alpha_C = \alpha_S = \alpha$, say, $P_C = P_S = 0$, $\sigma_C = \sigma_S = 0$ and $\delta = L_0 \alpha \Delta T$ as for unrestrained expansion.

The above analysis also applies to the case, $\alpha_C > \alpha_S$, when, as can be seen from Eqs (7.48) and (7.49) the signs of P_C, P_S, σ_C and σ_S are reversed. Thus the load and stress in the concrete become compressive, while those in the steel become tensile. A similar argument applies when ΔT specifies a

Equation (7.44) is an expression of the compatibility of displacement of the concrete and steel. Also note that the stresses could have been obtained directly by writing Eqs (7.43) and (7.44) as

$$\sigma_C A_C = \sigma_S A_S$$

and

$$\frac{\sigma_C L_0}{E_C} + \frac{\sigma_S L_0}{E_S} = L_0 \Delta T(\alpha_S - \alpha_C)$$

respectively.

EXAMPLE 7.10

A rigid slab of weight 100 kN is supported on three columns each of height 4 m and cross-sectional area 300 mm^2 arranged in line. The two outer columns are fabricated from material having a Young's modulus of 80 000 N/mm^2 and a coefficient of linear expansion of 1.85×10^{-5}/°C; the corresponding values for the inner column are 200 000 N/mm^2 and 1.2×10^{-5}/°C. If the slab remains firmly attached to each column, determine the stress in each column and the displacement of the slab if the temperature is increased by 100°C.

The problem may be solved by determining separately the stresses and displacements produced by the applied load and the temperature rise; the two sets of results are then superimposed. Let subscripts o and i refer to the outer and inner columns, respectively. Using Eq. (7.38) we have

$$\sigma_i\,(\text{load}) = \frac{E_i}{A_o E_o + A_i E_i} P \qquad \sigma_o\,(\text{load}) = \frac{E_o}{A_o E_o + A_i E_i} P \tag{i}$$

In Eq. (i)

$$A_o E_o + A_i E_i = 2 \times 300 \times 80\,000 + 300 \times 200\,000 = 108.0 \times 10^6$$

Then

$$\sigma_i\,(\text{load}) = \frac{200\,000 \times 100 \times 10^3}{108.0 \times 10^6} = 185.2\ \text{N/mm}^2\ (\text{compression})$$

$$\sigma_o\,(\text{load}) = \frac{80\,000 \times 100 \times 10^3}{108.0 \times 10^6} = 74.1\ \text{N/mm}^2\ (\text{compression})$$

Equation (7.49) give the values of σ_i (temp.) and σ_o (temp.) produced by the temperature rise, i.e.

$$\sigma_o\,(\text{temp.}) = \frac{\Delta T(\alpha_i - \alpha_o) E_o A_i E_i}{A_o E_o + A_i E_i}$$

$$\sigma_i\,(\text{temp.}) = \frac{\Delta T(\alpha_i - \alpha_o) A_o E_o E_i}{A_o E_o + A_i E_i} \tag{ii}$$

In Eq. (ii) $\alpha_o > \alpha_i$ so that σ_o (temp.) is a compressive stress while σ_i (temp.) is a tensile stress. Hence

$$\sigma_o\ (\text{temp.}) = \frac{100(1.2 - 1.85) \times 10^{-5} \times 80\,000 \times 300 \times 200\,000}{105.0 \times 10^6}$$

$$= -28.9\ \text{N/mm}^2\ (\text{i.e. compression})$$

$$\sigma_i\ (\text{temp.}) = \frac{100(1.2 - 1.85) \times 10^{-5} \times 2 \times 300 \times 80\,000 \times 200\,000}{108.0 \times 10^6}$$

$$= -57.8\ \text{N/mm}^2\ (\text{i.e. tension})$$

Superimposing the sets of stresses, we obtain the final values of stress, σ_i and σ_o, due to load and temperature change combined. Hence

$$\sigma_i = 185.2 - 57.8 = 127.4\ \text{N/mm}^2\ (\text{compression})$$

$$\sigma_o = 74.1 + 28.9 = 103.0\ \text{N/mm}^2\ (\text{compression})$$

The displacements due to the load and temperature change are found using Eqs (7.37) and (7.51), respectively. Hence

$$\delta\ (\text{load}) = \frac{100 \times 10^3 \times 4 \times 10^3}{108.0 \times 10^6} = 3.7\ \text{mm (contraction)}$$

$$\delta\ (\text{temp.}) = 4 \times 10^3 \times 100 \times \left(\frac{1.85 \times 10^{-5} \times 2 \times 300 \times 80\,000 + 1.2 \times 10^{-5} \times 300 \times 200\,000}{108.0 \times 10^6}\right)$$

$$= 6.0\ \text{mm (elongation)}$$

The final displacement of the slab involves an overall elongation of the columns of $6.0 - 3.7 = 2.3$ mm.

Initial stresses and prestressing

The terms initial stress and prestressing refer to structural situations in which some or all of the components of a structure are in a state of stress *before* external loads are applied. In some cases, for example welded connections, this is an unavoidable by-product of fabrication and unless the whole connection is stress-relieved by suitable heat treatment the initial stresses are not known with any real accuracy. On the other hand, the initial stress in a component may be controlled as in a bolted connection; the subsequent applied load may or may not affect the initial stress in the bolt.

Initial stresses may be deliberately induced in a structural member so that the adverse effects of an applied load are minimized. In this the category is the prestressing of beams fabricated from concrete which is particularly weak in tension. An overall state of compression is induced in the concrete so that tensile stresses due to applied loads merely reduce the level of compressive stress in the concrete rather than cause tension. Two methods of prestressing are employed, pre- and post-tensioning. In the former the prestressing tendons are positioned in the mould before the concrete is poured and loaded to the required level of tensile stress. After the concrete has set, the tendons are released and the tensile load in the tendons is transmitted, as a compressive load, to the concrete. In a post-tensioned beam, metal tubes

or conduits are located in the mould at points where reinforcement is required, the concrete is poured and allowed to set. The reinforcing tendons are then passed through the conduits, tensioned and finally attached to end plates which transmit the tendon tensile load, as a compressive load, to the concrete.

Usually the reinforcement in a concrete beam supporting vertical shear loads is placed closer to either the upper or the lower surface since such a loading system induces tension in one part of the beam and compression in the other; clearly the reinforcement is placed in the tension zone. To demonstrate the basic principle, however, we shall investigate the case of a post-tensioned beam containing one axially loaded prestressing tendon.

Suppose that the initial load in the prestressing tendon in the concrete beam shown in Fig. 7.22 is F. In the absence of an applied load the resultant load at any section of the beam is zero so that the load in the concrete is also F but compressive. If now a tensile load, P, is applied to the beam, the tensile load in the prestressing tendon will increase by an amount ΔP_T while the compressive load in the concrete will decrease by an amount ΔP_C. From a consideration of equilibrium

$$\Delta P_T + \Delta P_C = P \tag{7.52}$$

Furthermore, the total tensile load in the tendon is $F + \Delta P_T$ while the total compressive load in the concrete is $F - \Delta P_C$.

The tendon and concrete beam are interconnected through the end plates so that they both suffer the same elongation, δ, due to P. Then, from Eq. (7.28)

$$\delta = \frac{\Delta P_T L}{A_T E_T} = \frac{\Delta P_C L}{A_C E_C} \tag{7.53}$$

where E_T and E_C are Young's modulus for the tendon and the concrete, respectively. From Eq. (7.53)

$$\Delta P_T = \frac{A_T E_T}{A_C E_C} \Delta P_C \tag{7.54}$$

Substituting in Eq. (7.52) for ΔP_T we obtain

$$\Delta P_C \left(\frac{A_T E_T}{A_C E_C} + 1\right) = P$$

which gives

$$\Delta P_C = \frac{A_C E_C}{A_C E_C + A_T E_T} P \tag{7.55}$$

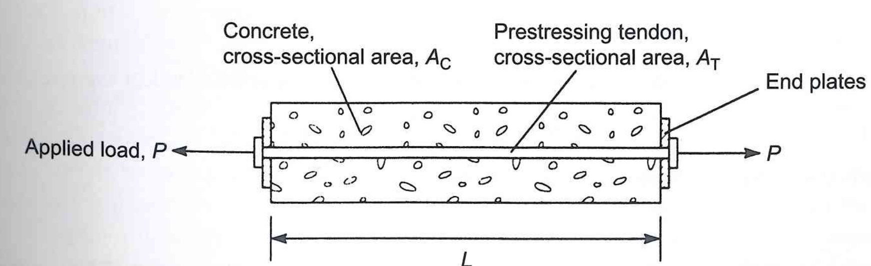


FIGURE 7.22
Prestress

Substituting now for ΔP_C in Eq. (7.54) from Eq. (7.55) gives

$$\Delta P_T = \frac{A_T E_T}{A_C E_C + A_T E_T} P \tag{7.56}$$

The final loads, P_C and P_T, in the concrete and tendon, respectively, are then

$$P_C = F - \frac{A_C E_C}{A_C E_C + A_T E_T} P \quad \text{(compression)} \tag{7.57}$$

and

$$P_T = F + \frac{A_T E_T}{A_C E_C + A_T E_T} P \quad \text{(tension)} \tag{7.58}$$

The corresponding final stresses, σ_C and σ_T, follow directly and are given by

$$\sigma_C = \frac{P_C}{A_C} = \frac{1}{A_C}\left(F - \frac{A_C E_C}{A_C E_C + A_T E_T} P\right) \quad \text{(compression)} \tag{7.59}$$

and

$$\sigma_T = \frac{P_T}{A_T} = \frac{1}{A_T}\left(F + \frac{A_T E_T}{A_C E_C + A_T E_T} P\right) \quad \text{(tension)} \tag{7.60}$$

Obviously if the bracketed term in Eq. (7.59) is negative then σ_C will be a tensile stress.

Finally the elongation, δ, of the beam due to P is obtained from either of Eq. (7.53) and is

$$\delta = \frac{L}{A_C E_C + A_T E_T} P \tag{7.61}$$

EXAMPLE 7.11

A concrete beam of rectangular cross section, 120 mm × 300 mm, is to be reinforced by six high-tensile steel prestressing tendons each having a cross-sectional area of 300 mm². If the level of prestress in the tendons is 150 N/mm², determine the corresponding compressive stress in the concrete. If the reinforced beam is subjected to an axial tensile load of 150 kN, determine the final stress in the steel and in the concrete assuming that the ratio of the elastic modulus of steel to that of concrete is 15.

The cross-sectional area, A_C, of the concrete in the beam is given by

$$A_C = 120 \times 300 - 6 \times 300 = 342\,000 \text{ mm}^2$$

The initial compressive load in the concrete is equal to the initial tensile load in the steel; thus

$$\sigma_{Ci} \times 34\,200 = 150 \times 6 \times 300 \tag{i}$$

where σ_{Ci} is the initial compressive stress in the concrete. Hence

$$\sigma_{Ci} = 7.9 \text{ N/mm}^2$$

The final stress in the concrete and in the steel are given by Eqs (7.59) and (7.60), respectively. From Eq. (7.59)

$$\sigma_C = \frac{F}{A_C} - \frac{E_C}{A_C E_C + A_T E_T} P \tag{ii}$$

in which $F/A_C = \sigma_{Ci} = 7.9$ N/mm². Rearranging Eq. (ii) we have

$$\sigma_C = 7.9 - \frac{1}{A_C + \left(\frac{E_T}{E_C}\right) A_T} P$$

or

$$\sigma_C = 7.9 - \frac{150 \times 10^3}{34\,200 + 15 \times 6 \times 300} = 5.4 \text{ N/mm}^2 \quad \text{(compression)}$$

Similarly, from Eq. (7.60)

$$\sigma_T = 150 + \frac{1}{\left(\frac{E_C}{E_T}\right) A_C + A_T} P$$

from which

$$\sigma_T = 150 + \frac{150 \times 10^3}{\frac{1}{15} \times 34\,200 + 6 \times 300} = 186.8 \text{ N/mm}^2 \quad \text{(tension)}$$

7.11 Plane stress

In some situations the behaviour of a structure, or part of it, can be regarded as two-dimensional. For example, the stresses produced in a flat plate which is subjected to loads solely in its own plane would form a two-dimensional stress system; in other words, a *plane stress* system. These stresses would, however, produce strains perpendicular to the surfaces of the plate due to the Poisson effect (Section 7.8).

An example of a plane stress system is that produced in the walls of a thin cylindrical shell by internal pressure. Figure 7.23 shows a long, thin-walled cylindrical shell subjected to an internal pressure p. This internal pressure has a dual effect; it acts on the sealed ends of the shell thereby producing a *longitudinal* direct stress in cross sections of the shell and it also tends to separate one-half of the shell from the other along a diametral plane causing *circumferential* or *hoop* stresses. These two situations are shown in Figs. 7.24 and 7.25, respectively.

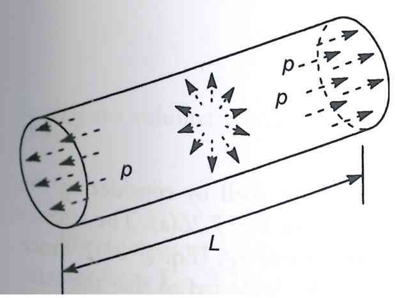


FIGURE 7.23

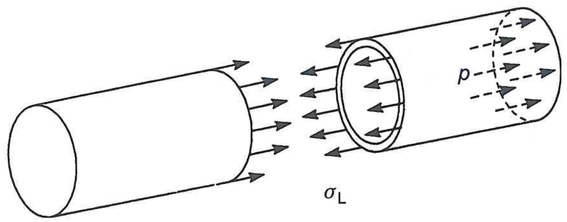


FIGURE 7.24

Longitudinal stresses due to internal pressure.

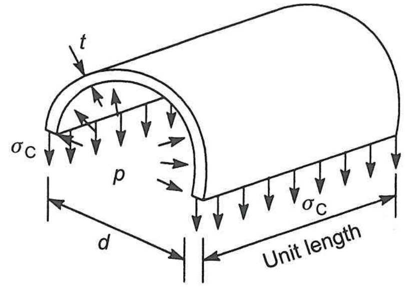


FIGURE 7.25

Circumferential stress due to internal pressure.

Suppose that d is the internal diameter of the shell and t the thickness of its walls. In Fig. 7.24 the axial load on each end of the shell due to the pressure p is

$$p \times \frac{\pi d^2}{4}$$

This load is equilibrated by an internal force corresponding to the longitudinal direct stress, σ_L, so that

$$\sigma_L \pi\, dt = p\frac{\pi\, d^2}{4}$$

which gives

$$\sigma_L = \frac{pd}{4t} \qquad (7.62)$$

Now consider a unit length of the half shell formed by a diametral plane (Fig. 7.25). The force on the shell, produced by p, in the opposite direction to the circumferential stress, σ_C, is given by

$$p \times \text{projected area of the shell in the direction of } \sigma_C$$

Therefore for equilibrium of the unit length of shell

$$2\sigma_C \times (1 \times t) = p \times (1 \times d)$$

which gives

$$\sigma_C = \frac{pd}{2t} \qquad (7.63)$$

We can now represent the state of stress at any point in the wall of the shell by considering the stress acting on the edges of a very small element of the shell wall as shown in Fig. 7.26(a). The stresses comprise the longitudinal stress, σ_L, (Eq. (7.62)) and the circumferential stress, σ_C, (Eq. (7.63)). Since the element is very small, the effect of the curvature of the shell wall can be neglected so that the state

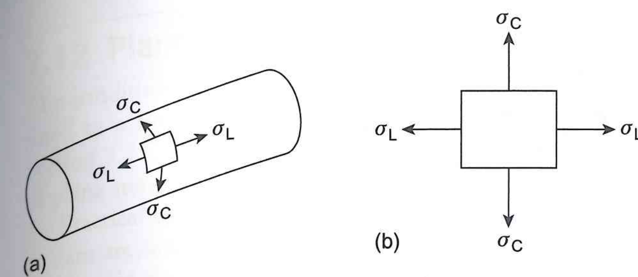


FIGURE 7.26

Two-dimensional stress system.

of stress may be represented as a *two-dimensional* or *plane* stress system acting on a plane element of thickness, t (Fig. 7.26(b)).

In addition to stresses, the internal pressure produces corresponding strains in the walls of the shell which lead to a change in volume. Consider the element of Fig. 7.26(b). The longitudinal strain, ε_L, is, from Eq. (7.13)

$$\varepsilon_L = \frac{\sigma_L}{E} - v\frac{\sigma_C}{E}$$

or, substituting for σ_L and σ_C from Eqs (7.62) and (7.63), respectively

$$\varepsilon_L = \frac{pd}{2tE}\left(\frac{1}{2} - v\right) \qquad (7.64)$$

Similarly, the circumferential strain, ε_C, is given by

$$\varepsilon_C = \frac{pd}{2tE}\left(1 - \frac{1}{2}v\right) \qquad (7.65)$$

The increase in length of the shell is $\varepsilon_L L$ while the increase in circumference is $\varepsilon_C \pi d$. We see from the latter expression that the increase in circumference of the shell corresponds to an increase in diameter, $\varepsilon_C d$, so that the circumferential strain is equal to diametral strain (and also radial strain). The increase in volume, ΔV, of the shell is then given by

$$\Delta V = \frac{\pi}{4}(d + \varepsilon_C d)^2 (L + \varepsilon_L L) - \frac{\pi}{4} d^2 L$$

which, when second-order terms are neglected, simplifies to

$$\Delta V = \frac{\pi d^2 L}{4}(2\varepsilon_C + \varepsilon_L) \qquad (7.66)$$

Substituting for ε_L and ε_C in Eq. (7.66) from Eqs (7.64) and (7.65) we obtain

$$\Delta V = \frac{\pi d^2 L}{4}\frac{pd}{tE}\left(\frac{5}{4} - v\right)$$

so that the volumetric strain is

$$\frac{\Delta V}{(\pi d^2 L/4)} = \frac{pd}{tE}\left(\frac{5}{4} - v\right) \qquad (7.67)$$

The analysis of a spherical shell is somewhat simpler since only one direct stress is involved. It can be seen from Fig. 7.27(a) and (b) that no matter which diametral plane is chosen, the tensile stress, σ,

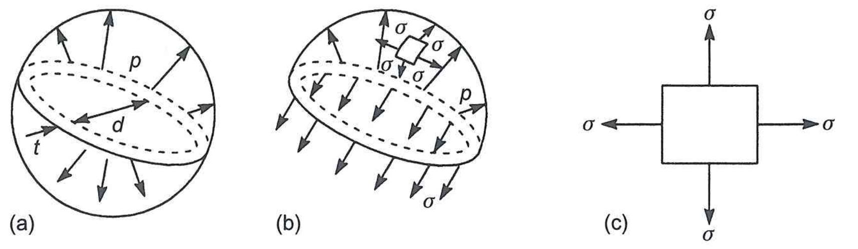


FIGURE 7.27

Stress in a spherical shell.

in the walls of the shell is constant. Thus for the equilibrium of the hemispherical portion shown in Fig. 7.27(b)

$$\sigma \times \pi \, dt = p \times \frac{\pi d^2}{4}$$

from which

$$\sigma = \frac{pd}{4t} \quad (7.68)$$

Again we have a two-dimensional state of stress acting on a small element of the shell wall (Fig. 7.27(c)) but in this case the direct stresses in the two directions are equal. Also the volumetric strain is determined in an identical manner to that for the cylindrical shell and is

$$\frac{3pd}{4tE}(1 - v) \quad (7.69)$$

EXAMPLE 7.12

A thin-walled, cylindrical shell has an internal diameter of 2 m and is fabricated from plates 20 mm thick. Calculate the safe pressure in the shell if the tensile strength of the plates is 400 N/mm^2 and the factor of safety is 6. Determine also the percentage increase in the volume of the shell when it is subjected to this pressure. Take Young's modulus $E = 200\ 000\ N/mm^2$ and Poisson's ratio $v = 0.3$.

The maximum tensile stress in the walls of the shell is the circumferential stress, σ_C, given by Eq. (7.63). Then

$$\frac{400}{6} = \frac{p \times 2 \times 10^3}{2 \times 20}$$

from which

$$p = 1.33\ N/mm^2$$

The volumetric strain is obtained from Eq. (7.67) and is

$$\frac{1.33 \times 2 \times 10^3}{20 \times 200\ 000}\left(\frac{5}{4} - 0.3\right) = 0.00063$$

Hence the percentage increase in volume is 0.063%.

7.12 Plane strain

The condition of *plane strain* occurs when all the strains in a structure, or part of a structure, are confined to a single plane. This does not necessarily coincide with a plane stress system as we noted in Section 7.11. Conversely, it generally requires a three-dimensional stress system to produce a condition of plane strain.

Practical examples of plane strain situations are retaining walls or dams where the ends of the wall or dam are constrained against movement and the loading is constant along its length. All cross sections are then in the same condition so that any thin slice of the wall or dam taken perpendicularly to its length would only be subjected to strains in its own plane.

We shall examine more complex cases of plane stress and plane strain in Chapter 14.

PROBLEMS

P.7.1 A column 3 m high has a hollow circular cross section of external diameter 300 mm and carries an axial load of 5000 kN. If the stress in the column is limited to 150 N/mm^2 and the shortening of the column under load must not exceed 2 mm calculate the maximum allowable internal diameter. Take $E = 200\ 000\ N/mm^2$.

Ans. 205.6 mm.

P.7.2 A steel girder is firmly attached to a wall at each end so that changes in its length are prevented. If the girder is initially unstressed, calculate the stress induced in the girder when it is subjected to a uniform temperature rise of 30 K. The coefficient of linear expansion of the steel is 0.000 05/K and Young's modulus $E = 180\ 000\ N/mm^2$. (Note $L = L_0(1 + \alpha T)$.)

Ans. 270 N/mm^2 (compression).

P.7.3 A column 3 m high has a solid circular cross section and carries an axial load of 10 000 kN. If the direct stress in the column is limited to 150 N/mm^2 determine the minimum allowable diameter. Calculate also the shortening of the column due to this load and the increase in its diameter. Take $E = 200\ 000\ N/mm^2$ and $v = 0.3$.

Ans. 291.3 mm, 2.25 mm, 0.066 mm.

P.7.4 A structural member has a rectangular cross section of side 100 × 300 mm and carries an axial tensile load of 5000 kN. Calculate the direct stress in the member, its increase in length over a span of 5 m and the percentage change in its cross sectional area under the load, Take $E = 200\ 000\ N/mm^2$ and $\nu = 0.3$.

Ans. 166.7 N/mm^2, 4.17 mm, 0.043%."

P.7.5 The block of material shown in Fig. P.7.5 is subjected to the stress system shown. If Young's modulus, E, is 200 000 N/mm^2 and Poisson's ratio, ν, is 0.3 calculate the percentage change in volume in the block.

Ans. 0.049%.

P.7.6 A structural member, 2 m long, is found to be 1.5 mm short when positioned in a framework. To enable the member to be fitted it is heated uniformly along its length. Determine the necessary temperature rise. Calculate also the residual stress in the member when it cools to its

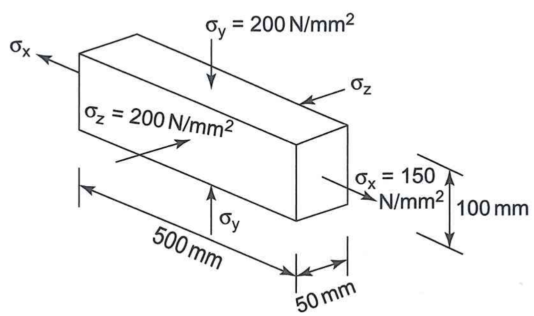


FIGURE P.7.5

If the member has a rectangular cross section, determine the percentage change in cross-sectional area when the member is fixed in position and at its original temperature.

Young's modulus $E = 200\ 000$ N/mm^2, Poisson's ratio $\nu = 0.3$ and the coefficient of linear expansion of the material of the member is 0.000 012/K.

Ans. 62.5 K, 150 N/mm^2 (tension), 0.045% (reduction).

P.7.7 A member of a framework is required to carry an axial tensile load of 100 kN. It is proposed that the member be comprised of two angle sections back to back in which one 18 mm diameter hole is allowed per angle for connections. If the allowable stress is 155 N/mm^2, suggest suitable angles.

Ans. Required minimum area of cross section = 645.2 mm^2. From steel tables, two equal angles 50 × 50 × 5 mm are satisfactory.

P.7.8 A vertical hanger supporting the deck of a suspension bridge is formed from a steel cable 25 m long and having a diameter of 7.5 mm. If the density of the steel is 7850 kg/m^3 and the load at the lower end of the hanger is 5 kN, determine the maximum stress in the cable and its elongation. Young's modulus $E = 200\ 000$ N/mm^2.

Ans. 115.1 N/mm^2, 14.3 mm.

P.7.9 A concrete chimney 40 m high has a cross-sectional area (of concrete) of 0.15 m^2 and is stayed by three groups of four cables attached to the chimney at heights of 15, 25 and 35 m respectively. If each cable is anchored to the ground at a distance of 20 m from the base of the chimney and tensioned to a force of 15 kN, calculate the maximum stress in the chimney and the shortening of the chimney including the effect of its own weight. The density of concrete is 2500 kg/m^3 and Young's modulus $E = 20\ 000$ N/mm^2.

Ans. 1.9 N/mm^2, 2.2 mm.

P.7.10 A column of height h has a rectangular cross section which tapers linearly in width from b_1 at the base of the column to b_2 at the top. The breadth of the cross section is constant and equal to a. Determine the shortening of the column due to an axial load P.

Ans. $(Ph/[aE(b_1 - b_2)])\ \log_e(b_1/b_2)$.

P.7.11 Determine the vertical deflection of the 20 kN load in the truss shown in Fig. P.7.11. The cross-sectional area of the tension members is 100 mm^2 while that of the compression members is 200 mm^2. Young's modulus $E = 205\ 000$ N/mm^2.

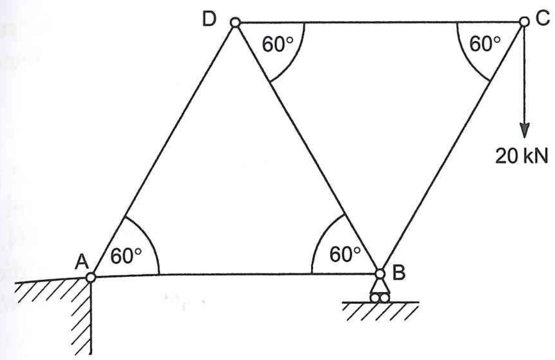


FIGURE P.7.11

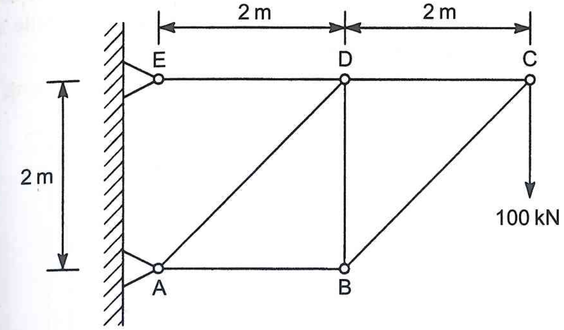


FIGURE P.7.12

P.7.12 The truss shown in Fig. P.7.12 has members of cross-sectional area 1200 mm^2 and Young's modulus 205 000 N/mm^2. Determine the vertical deflection of the load.

Ans. 10.3 mm.

P.7.13 Three identical bars of length L are hung in a vertical position as shown in Fig. P.7.13. A rigid, weightless beam is attached to their lower ends and this in turn carries a load P. Calculate the load in each bar.

Ans. $P_1 = P/12$, $P_2 = P/3$, $P_3 = 7P/12$.

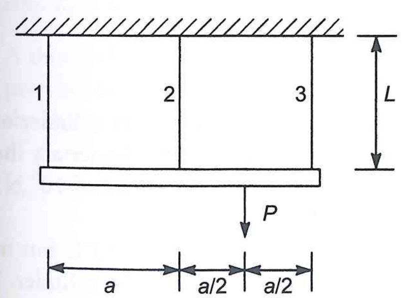


FIGURE P.7.13

P.7.14 A composite column is formed by placing a steel bar, 20 mm in diameter and 200 mm long, 25 mm, respectively. The column is then subjected to an axial load of 50 kN. If E for steel is 200 000 N/mm^2 and E for the alloy is 70 000 N/mm^2, calculate the stress in the cylinder and in the bar, the shortening of the column and the strain energy stored in the column.

Ans. 46.5 N/mm^2 (cylinder), 132.9 N/mm^2 (bar), 0.13 mm, 3.3 Nm.

P.7.15 A timber column, 3 m high, has a rectangular cross section, 100 mm × 200 mm, and is reinforced over its complete length by two steel plates each 200 mm wide and 10 mm thick attached to its 200 mm wide faces. The column is designed to carry a load of 100 kN. If the failure stress of the timber is 55 N/mm^2 and that of the steel is 380 N/mm^2, check the design using a factor of safety of 3 for the timber and 2 for the steel. E (timber) = 15 000 N/mm^2, E (steel) = 200 000 N/mm^2.

Ans. σ (timber) = 13.6 N/mm^2 (allowable stress = 18.3 N/mm^2),
σ (steel) = 181.8 N/mm^2 (allowable stress = 190 N/mm^2).

P.7.16 The composite bar shown in Fig. P.7.16 is initially unstressed. If the temperature of the bar is reduced by an amount T uniformly along its length, find an expression for the tensile stress induced. The coefficients of linear expansion of steel and aluminium are α_S and α_A per unit temperature change, respectively, while the corresponding values of Young's modulus are E_S and E_A.

Ans. $T(\alpha_S L_1 + \alpha_A L_2)/(L_1/E_S + L_2/E_A)$.

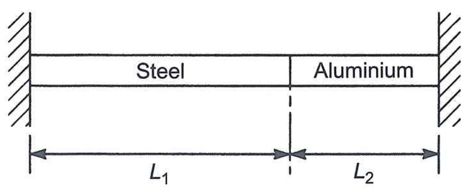


FIGURE P.7.16

P.7.17 A short bar of copper, 25 mm in diameter, is enclosed centrally within a steel tube of external diameter 36 mm and thickness 3 mm. At 0°C the ends of the bar and tube are rigidly fastened together and the complete assembly heated to 80°C. Calculate the stress in the bar and in the tube if E for copper is 100 000 N/mm^2, E for steel is 200 000 N/mm^2 and the coefficients of linear expansion of copper and steel are 0.000 01/°C and 0.000 006/°C, respectively.

Ans. σ (steel) = 28.3 N/mm^2 (tension),
σ (copper) = 17.9 N/mm^2 (compression).

P.7.18 A bar of mild steel of diameter 75 mm is placed inside a hollow aluminium cylinder of internal diameter 75 mm and external diameter 100 mm; both bar and cylinder are the same length. The resulting composite bar is subjected to an axial compressive load of 10^6 N. If the bar and cylinder contract by the same amount, calculate the stress in each.

The temperature of the compressed composite bar is then reduced by 150°C but no change in length is permitted. Calculate the final stress in the bar and in the cylinder. Take E (steel) = 200 000 N/mm^2, E (aluminium) = 80 000 N/mm^2, α (steel) = 0.000 012/°C, α (aluminium) = 0.000 005/°C.

σ (aluminium) = 69.1 N/mm^2 (compression).
Final stress: σ (steel) = 187.4 N/mm^2 (tension),
σ (aluminium) = −9.1 N/mm^2 (compression).

P.7.19 Two structural members are connected together by a hinge which is formed as shown in Fig. P.7.19. The bolt is tightened up onto the sleeve through rigid end plates until the tensile force in the bolt is 10 kN. The distance between the head of the bolt and the nut is then 100 mm and the sleeve is 80 mm in length. If the diameter of the bolt is 15 mm and the internal and outside diameters of the sleeve are 20 and 30 mm, respectively, calculate the final stresses in the bolt and sleeve when an external tensile load of 5 kN is applied to the bolt.

Ans. σ (bolt) = 65.4 N/mm^2 (tension),
σ (sleeve) = 16.7 N/mm^2 (compression).

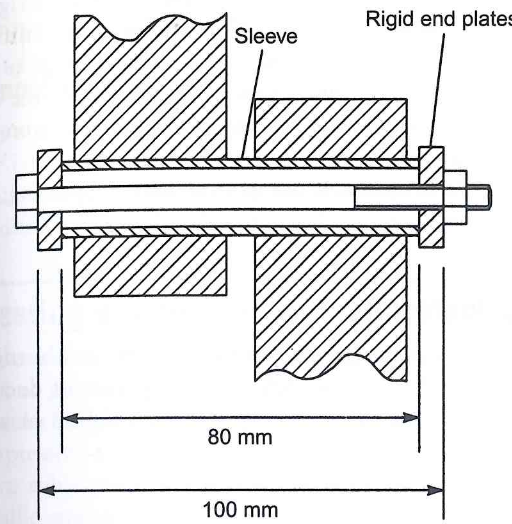


FIGURE P.7.19

P.7.20 Calculate the minimum wall thickness of a cast iron water pipe having an internal diameter of 1 m under a head of 120 m. The limiting tensile strength of cast iron is 20 N/mm^2 and the density of water is 1000 kg/m^3.

Ans. 29.4 mm.

P.7.21 A thin-walled spherical shell is fabricated from steel plates and has to withstand an internal pressure of 0.75 N/mm^2. The internal diameter is 3 m and the joint efficiency 80%. Calculate the thickness of plates required using a working stress of 80 N/mm^2. (Note, effective thickness of plates = 0.8 × actual thickness).

Ans. 8.8 mm.